PHYSICAL REVIEW D VOLUME 12, NUMBER 2 15 JULY 1975

Gravitational Born amplitudes and kinematical constraints

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(Received 18 February 1975)

It is shown that in the Born approximation the helicity amplitudes for graviton-graviton scattering are completely determined by kinematical considerations alone, without detailed Feynman graph calculations, reflecting thus the uniqueness of pure Einstein theory. We show that the nonuniqueness of the matter-graviton amplitudes is related to the existence of nonminimal interactions. In the presence of such interactions, helicity is not conserved in elastic scattering of massless particles. A simplified three-graviton vertex with two on-shell gravitons is presented.

I. INTRODUCTION

DeWitt in his 1967 paper[1] comments: "The tediousness of the algebra involved in obtaining the graviton-graviton cross section may be inferred from the complexity of the vertex functions. . .the fact that the final results are ridiculously simple leads one to believe that there must be an easier way." Both he and Cooke[2] and more recently Berends and Gastmans,[3] the latter authors having recourse to Veltman's algebraic manipulation computer program,[4] have calculated the Born helicity amplitudes for graviton-graviton, graviton-photon, and graviton-scalar scattering, using the complicated three- and four-graviton vertices that are obtained from the Einstein Lagrangian. As a side result all these authors have noted that in the Born amplitudes for elastic scattering a strong form of helicity conservation holds if all particles are massless, namely each particle conserves its helicity.

In this paper we show that the simplicity of the Born amplitudes reflects the strong kinematical constraints that these amplitudes satisfy. In fact we show that because of the high spin of the graviton the kinematical constraints are so restrictive that with knowledge of the dynamical singularities (poles from single-particle exchange) we are able to determine these amplitudes almost uniquely. In particular, the graviton-graviton Born amplitudes are completely determined. Since our only input (besides location of the pole singularities) is kinematical and follows from Lorentz invariance, we may regard our results as the S-matrix counterpart of the usual proof[5] that Einstein theory is the unique theory of spin-two massless particles. For other gravitational processes, the amplitudes are not completely determined kinematically. We conjecture and show in some cases that whatever freedom is left in these amplitudes reflects the existence of other interactions. We find that in the presence of these new, nonminimal interactions helicity is no longer conserved.

Having obtained the general structure of the helicity amplitudes it is a simple matter to determine their actual values in conventional Einstein theory by calculating these amplitudes at suitably chosen points. We find that in general the free parameters can be determined by computing residues at poles. It is thus only at this stage that we need the explicit form of the vertices. Calculations of gravitational processes are usually rather opaque owing to the complicated algebra. Specifically the three- and four-graviton vertices look rather discouraging. Fortunately, simple vertex expressions can be obtained when some gravitons are on shell by employing DeWitt's background field method.[1] Using these simplified vertices we find it trivial to complete the determination of the amplitudes.

The plan of our paper is as follows: After a section which summarizes the various properties of helicity amplitudes we present our three-graviton vertex in the form we need, with two gravitons on the mass shell. Next we show how Born helicity amplitudes for graviton-graviton scattering can be determined from knowledge of their dynamical and kinematical structure. We then apply our method to other processes involving gravitons, scalars, and photons. We find that the lower spin of some of the particles leads to less-stringent kinematical constraints, but the helicity amplitudes are still almost uniquely determined. We examine some new interactions which may take advantage of the freedom left in the amplitudes. We show that the "improved" theory[6] of scalar-graviton interactions does not give contributions different from those of the conventional theory and point out an error in the graviton-photon amplitudes of Ref. 3. Using the results of our paper, we discuss in a separate publication[7] the Reggeization of gravitons.

Our conventions are as follows: The Mandelstam invariants for the process $1+2\rightarrow 3+4$ are defined by $s=-(P_1+P_2)^2$, $t=-(P_1-P_3)^2$ and $u=-(P_1-P_4)^2$. Our metric is such that $P_\mu P_\mu=\vec{P}^2-P_0^2=\vec{P}^2+P_4^2$. The Ricci tensor $R_{\mu\nu}$ is related to the Riemann curvature tensor $R^\mu_{\nu\alpha\beta}$ by

$$R_{\mu\nu}=R^\lambda_{\mu\nu\lambda}, \quad (1)$$

and the latter is defined by

$$R^\rho_{\mu\nu\sigma}=(\partial_\nu\Gamma^\rho_{\mu\sigma}+\Gamma^\rho_{\nu\alpha}\Gamma^\alpha_{\mu\sigma})-(\nu\leftrightarrow\sigma), \quad (2)$$

where Γ is the Christoffel symbol.

The graviton field $h_{\mu\nu}$ is defined as the deviation from flat space, $g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu}$; since a boson field has dimensions of mass we have multiplied $h_{\mu\nu}$ by the gravitational coupling constant κ. The actual value $\kappa^2=32\pi G$, where G is Newton's constant, follows from the requirement that one-graviton exchange between two scalar particles at rest reproduces Newton's law.

The Einstein Lagrangian for pure gravity is

$$\mathcal{L}^E=2g^{1/2}(-\kappa^{-2}R). \quad (3)$$

The kinetic term obtained from $\mathcal{L}^E$ is

$$\mathcal{L}^{(2)}=-\tfrac{1}{2}h_{\mu\nu,\lambda}{}^2+h_\mu{}^2-h_\mu h_{,\mu}+\tfrac{1}{2}h_{,\mu}{}^2 \quad (4)$$

and leads to a graviton propagator

$$D^F_{\mu\nu,\rho\sigma}(k)=-\tfrac{1}{2}i(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\nu}\eta_{\rho\sigma})k^{-2}. \quad (5)$$

(Momentum factors in the numerator may be omitted; see Sec. III.) We have used the notation $h_\mu=h_{\mu\nu,\nu}$, $h=h_{\nu\nu}$.

The three- and four-graviton vertices are given by terms in $\mathcal{L}^E$ cubic and quartic in the graviton field $h_{\mu\nu}$. We are working in the gauge[8] where the polarization tensors for helicity ±2 are related to those of photons with helicity ±1 by

$$\epsilon^{\pm2}_{\mu\nu}(k)=\epsilon^{\pm1}_\mu(k)\epsilon^{\pm1}_\nu(k). \quad (6)$$

It follows that they are transverse and traceless.

Our conventions are the same as those of Refs. 1–3 and of Gross and Jackiw[8] and Chester,[9] who have also calculated some graviton processes.

II. HELICITY FORMALISM

We denote by $F_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}$ the helicity amplitudes for the scattering process $1+2\rightarrow3+4$ of particles with helicities λ_i. Their relation to the S matrix is

$$S=1+(2\pi)^4\delta^4(P_i-P_f)\prod_{i=1}^4(2E_i)^{-1/2}F, \quad (7)$$

where E_i is the center-of-mass energy of the ith particle. These helicity amplitudes satisfy a number of relations, and we summarize below the relevant ones.

As a consequence of parity conservation, time-reversal invariance for a reaction $a+b\rightarrow a+b$, Bose symmetry for a reaction $a+a\rightarrow b+c$, and invariance under particle-antiparticle conjugation for a reaction $a+\bar{a}\rightarrow b+\bar{b}$, we have

$$F_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}=(-1)^{\lambda-\mu}F_{-\lambda_3,-\lambda_4;-\lambda_1,-\lambda_2}, \quad (8)$$

$$F_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}=(-1)^{\lambda-\mu}F_{\lambda_1,\lambda_2;\lambda_3,\lambda_4}, \quad (9)$$

$$F_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}(s,t,u)=(-1)^{\lambda-2s_1}F_{\lambda_3,\lambda_4;\lambda_2,\lambda_1}(s,u,t), \quad (10)$$

$$F_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}=(-1)^{\lambda-\mu}F_{\lambda_4,\lambda_3;\lambda_2,\lambda_1}, \quad (11)$$

where $\lambda=\lambda_1-\lambda_2$, $\mu=\lambda_3-\lambda_4$.

We require that helicity amplitudes be Lorentz invariant when all spinning particles are massless. As is well known,[10] the theory is then gauge invariant. In such cases the helicity amplitudes are functions of the Mandelstam invariants s, t, and u and the masses of scalar particles. Moreover, the crossing matrix becomes diagonal:

$$\begin{aligned}F^s_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}(s,t,u)&=(-1)^{|\lambda_1|-\lambda_1+|\lambda_4|-\lambda_4}\\&\quad\times F^t_{\lambda_3,-\lambda_1;-\lambda_4,\lambda_2}(t,s,u),\\F^s_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}(s,t,u)&=(-1)^{\lambda_2-\lambda_4}\\&\quad\times F^u_{-\lambda_1,\lambda_4;-\lambda_3,\lambda_2}(u,t,s).\end{aligned} \quad (12)$$

Ader *et al.*[11] have pointed out that the derivation of crossing relations may fail unless the usual crossing path is valid for processes involving massless particles. However, no problems arise if we restrict ourselves to the Born approximation.

The authors of Ref. 11 have investigated the kinematical singularities of helicity amplitudes for processes involving some massless particles. We shall need the following results.

(i) If we write

$$F_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}(s,t,u)=(\sqrt{2}\cos\tfrac{1}{2}\theta)^{|\lambda+\mu|}(\sqrt{2}\sin\tfrac{1}{2}\theta)^{|\lambda-\mu|}\times\hat{F}_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}(s,t,u), \quad (13)$$

then $\hat{F}$ has no kinematical singularities in t and u. The symbol θ denotes the scattering angle in the center-of-mass system.

In the s variable there are kinematical singularities at (pseudo-) thresholds and at $s=0$. The following cases are of interest to us:

(ii) For four massless particles

$$\hat{F}_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}(s,t,u)\sim(\sqrt{s})^{|\lambda-\mu|+|\lambda+\mu|+|\Sigma\lambda_i|} \quad (14)$$

near $s=0$ for fixed t or u.

(iii) If $m_1=m_3=0$ and $m_2=m_4=m$, and particles 2 and 4 are spinless, then

$$\hat{F}_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}(s,t,u)\sim(s-m^2)^{2\lambda_m},\quad \lambda_m=\max(|\lambda|,|\mu|) \quad (15)$$

near $s = m^2$ for fixed t or u, and

$$\hat{F}_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}(s,t,u) = (\sqrt{s})^{-|\lambda-\mu|} \tag{16}$$

near $s=0$ for fixed t or u.

III. THE THREE-GRAVITON VERTEX

The three-graviton vertex is usually obtained by collecting all terms in the Einstein action which are cubic in the graviton field $h_{\mu\nu}$. In the pole diagrams of Fig. 1, one of these three graviton fields will be contracted while the other two will be physical. Denoting by $H_{\mu\nu}$ the field which is to be contracted, and by $h_{\mu\nu}$ the physical fields, an alternative way to find the three-graviton vertex is to replace in the Einstein action the field $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ by $g_{\mu\nu} + \kappa H_{\mu\nu} = \eta_{\mu\nu} + \kappa(h_{\mu\nu} + H_{\mu\nu})$, and to expand to first order in $H_{\mu\nu}$ and to second order in $h_{\mu\nu}$. The terms linear in $H_{\mu\nu}$ in the Einstein action determine of course the Euler-Lagrange equations, i.e., the Einstein equations, since $H_{\mu\nu}$ can be considered as a small variation on $g_{\mu\nu}$,

$$\mathcal{L}^E(h+H) = \mathcal{L}^E(h) + 2\kappa^{-1}H_{\mu\nu}[g^{1/2}(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R)] + \cdots. \tag{17}$$

To obtain the three-graviton vertex we must expand the expression in square brackets to second order in $h_{\mu\nu}$. We use the expansions[12]

$$g^{1/2} = \exp[\tfrac{1}{2}\operatorname{tr}\ln(\eta_{\mu\nu} + \kappa h_{\mu\nu})] = 1 + \tfrac{1}{2}\kappa h + \cdots, \tag{18}$$

$$g^{\mu\nu} = \eta_{\mu\nu} - \kappa h_{\mu\nu} + \cdots, \tag{19}$$

$$R_{\mu\nu} = \kappa R^{(1)}_{\mu\nu} + \kappa^2 R^{(2)}_{\mu\nu}, \tag{20}$$

$$R = g^{\mu\nu}R_{\mu\nu} = \kappa R^{(1)} + \kappa^2 R^{(2)}, \tag{21}$$

$$R^{(1)}_{\mu\nu} = \tfrac{1}{2}(h_{,\mu\nu} - h_{\mu,\nu} - h_{\nu,\mu} + \Box h_{\mu\nu}),$$
$$R^{(1)} = \Box h - h_{\alpha,\alpha}, \tag{22}$$

$$R^{(2)} = -\tfrac{3}{4}h_{\alpha\beta,\gamma}{}^2 + \tfrac{1}{2}h_{\alpha\beta,\gamma}h_{\alpha\gamma,\beta} - h_{\alpha\beta}\Box h_{\alpha\beta} + (h_\alpha - \tfrac{1}{2}h_{,\alpha})^2 + 2h_{\nu\beta}(h_{\nu,\beta} - \tfrac{1}{2}h_{,\nu\beta}), \tag{23}$$

$$R^{(2)}_{\mu\nu} = -\tfrac{1}{4}h_{\alpha\beta,\mu}h_{\alpha\beta,\nu} + \tfrac{1}{2}h_{\alpha\beta}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\beta,\mu\nu} - h_{\mu\nu,\alpha\beta}) + \tfrac{1}{2}h_{\alpha\mu,\beta}(h_{\beta\nu,\alpha} - h_{\alpha\nu,\beta}) + \tfrac{1}{2}(h_\alpha - \tfrac{1}{2}h_{,\alpha})(h_{\mu\alpha,\nu} + h_{\nu\alpha,\mu} - h_{\mu\nu,\alpha}). \tag{24}$$

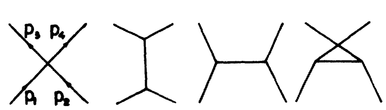

FIG. 1. Born diagrams for graviton-graviton scattering.

We recall the notations $h = h_{\lambda\lambda}$, $h_\mu = h_{\mu\nu,\nu}$, $g = \det g_{\mu\nu}$, and $\Box = \partial_\lambda\partial_\lambda = \vec{\nabla}^2 - (\partial/\partial t)^2$.

Since we have already decided which fields will be physical, we can use the mass-shell equations $h = h_\mu = 0$ for these physical gravitons. Since $R^{(1)}_{\mu\nu}$ and $R^{(1)}$ vanish in this case, we use only $R^{(2)}_{\mu\nu}$ and $R^{(2)}$ and set $g^{1/2} = 1$ and $g_{\mu\nu} = \eta_{\mu\nu}$ in Eq. (17). Furthermore, we can take $R^{(2)} = R^{(2)}_{\lambda\lambda}$. Therefore, the three-graviton vertex becomes

$$V^3(H,h,h) = 2\kappa H_{\mu\nu}(R^{(2)}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}R^{(2)}_{\lambda\lambda}). \tag{25}$$

Using Eq. (24) we finally obtain

$$V^3(H,h,h) = 2\kappa(H_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}H)[h_{\beta\gamma}R^{(1)}_{\beta\mu\gamma\nu} - \tfrac{1}{4}h_{\beta\gamma,\mu}h_{\beta\gamma,\nu} + \tfrac{1}{2}h_{\beta\mu,\gamma}(h_{\gamma\nu,\beta} - h_{\beta\nu,\gamma})], \tag{26}$$

where we have written the Riemann tensor to first order in h

$$R^{(1)}_{\beta\mu\gamma\nu} = \tfrac{1}{2}(h_{\nu\beta,\mu\gamma} + h_{\mu\gamma,\nu\beta} - h_{\mu\nu,\beta\gamma} - h_{\beta\gamma,\mu\nu}). \tag{27}$$

We have verified that this vertex agrees with the usual expression for the three-graviton vertex.

This expression is manageable because we have used the mass-shell equations for the physical gravitons in the vertex itself rather than first derive the general vertex, then contract it with another vertex in all possible ways, and finally use mass-shell equations. The technique of deciding from the beginning which role the gravitons are going to play is also the basis of the background field method invented by DeWitt in order to deal with quantum effects in gravitation.[1] Recently the background field method has been used profitably in one-loop calculations in gravity[12,13]; we see that it also has its merits for Born amplitudes. One can derive in a similar manner a four-graviton vertex with two or more gravitons on shell. However, we shall not need this vertex.

Our three-graviton vertex has some properties which are relevant for our work.

(i) Helicity conservation for collinear physical gravitons. This is obvious from Eqs. (26), (27) since at least one index of each graviton field, hence the polarization tensor, is contracted either with a momentum or an index of the other graviton field. We note that for two incoming collinear gravitons $\epsilon_{\mu\nu}{}^2\epsilon_{\mu\lambda}{}^{-2} = 0$, while if one is incoming and the other one is outgoing $(\epsilon_{\mu\nu}{}^2)^*\epsilon_{\mu\lambda}{}^{-2} = 0$ for parallel momenta.

(ii) Conservation in the sense that replacing in Eq. (25) the field $H_{\mu\nu}$ with momentum Q by Q_μ yields zero. Since the (off-shell-) graviton-(on-shell-) matter vertex is $H_{\mu\nu}T_{\mu\nu}$, where $T_{\mu\nu}$ is the (conserved) matter energy-momentum tensor, we see that any momentum factors in the numerator of the propagator in Eq. (5) may be omitted.

(iii) Gauge dependence: Under external gauge transformations $h_{\mu\nu}(k)\to h_{\mu\nu}+\chi_\mu k_\nu+\chi_\nu k_\mu$ with χ arbitrary and k off-shell, the three-graviton vertex changes in the following manner:

$$\delta V^3(H,p,k)=2\kappa H_{\mu\nu}(\delta R^{(2)}_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\delta R^{(2)}), \tag{28}$$

where

$$\begin{aligned}\delta R^{(2)}_{\mu\nu}=&\tfrac{1}{2}p\cdot\chi[k_\mu h_{\nu\beta}k_\beta+k_\nu h_{\mu\beta}k_\beta-(k^2+2p\cdot k)h_{\mu\nu}\\&+p_\mu h_{\nu\beta}k_\beta+p_\nu h_{\mu\beta}k_\beta]\\&-\tfrac{1}{2}p\cdot k(k_\mu h_{\nu\beta}\chi_\beta+k_\nu h_{\mu\beta}\chi_\beta)\\&+\tfrac{1}{2}(k^2+p\cdot k)(p_\mu h_{\nu\beta}\chi_\beta+p_\nu h_{\mu\beta}\chi_\beta)\\&-\tfrac{1}{2}(2p_\mu p_\nu+k_\mu p_\nu+k_\nu p_\mu)(k_\beta h_{\beta\gamma}\chi_\gamma)\end{aligned} \tag{29}$$

and

$$\delta R^{(2)}=-2p\cdot k(\chi_\mu h_{\mu\nu}k_\nu)+p\cdot\chi(k_\mu h_{\mu\nu}k_\nu). \tag{30}$$

We note that

$$\delta R^{(2)}=\eta_{\mu\nu}\delta R^{(2)}_{\mu\nu}, \tag{31}$$

as should be the case.

In the above expression the field $h_{\mu\nu}$ is on shell and carries (incoming) momentum p. The gauge transformation has been performed on the second graviton with (incoming) momentum k. We observe that

$$Q_\mu Q_\nu(\delta R^{(2)}_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\delta R^{(2)})=0. \tag{32}$$

These results are the outcome of rather lengthy calculations where our simplified three-graviton vertex may not be used. They are needed if one wants to check the gauge invariance of processes where the three-graviton vertex enters.

IV. GRAVITON-GRAVITON SCATTERING

Using Eqs. (8)–(11) we can restrict ourselves to four independent helicity amplitudes

$$\begin{aligned}&A=F_{2,2;2,2}\,,\quad B=F_{2,-2;2,-2}\,,\\&C=F_{2,2;2-2}\,,\quad D=F_{2,2;-2,-2}.\end{aligned} \tag{33}$$

Since all spinning particles are massless, these amplitudes are required to be invariant functions of s, t, and u alone. In Born approximation we have s, t, and u channel graviton-exchange diagrams and contact terms with contributions quadratic in the (dimensional) coupling constant κ, so that our amplitudes have the generic form

$$A=\kappa^2\phi_A(s,t,u)/(stu),\ \text{etc.} \tag{34}$$

Expressing the angle factors in Eq. (13) in terms of invariants we deduce

$$\begin{aligned}&\phi_A=\hat\phi_A,\quad \phi_B=\left(\frac{u}{s}\right)^4\hat\phi_B,\\&\phi_C=\left(\frac{ut}{s^2}\right)^2\hat\phi_C,\quad \phi_D=\hat\phi_D,\end{aligned} \tag{35}$$

where the $\hat\phi$ have no kinematical singularities in t and u. Finally, from Eq. (14) it follows that near $s=0$

$$\begin{aligned}&\hat\phi_A\sim s^4,\quad \hat\phi_B\sim s^4,\\&\hat\phi_C\sim s^6,\quad \hat\phi_D\sim s^0.\end{aligned} \tag{36}$$

[The s pole in the amplitudes of Eq. (34) is dynamical; we distinguish it from kinematical factors.]

Having exhibited the dynamical and kinematical singularities of our amplitudes, we can conclude that the $\hat\phi$ must be analytic functions of s,t,u, hence polynomials. On *dimensional grounds* (the amplitudes are dimensionless, κ^2 has dimensions of s^{-1}) they must be fourth-degree homogeneous polynomials having the behavior of Eq. (36) near $s=0$. Therefore,

$$\begin{aligned}&\hat\phi_A=c_A s^4,\quad \hat\phi_B=c_B s^4,\\&\hat\phi_C=0,\quad \hat\phi_D=P_4(s,t,u),\end{aligned} \tag{37}$$

where P_4 is a fourth-degree homogeneous polynomial in s,t,u. From the crossing relations (12) we obtain

$$c_A=c_B=c, \tag{38}$$

and $\hat\phi_D$ must be a symmetric function of s, t, and u:

$$\hat\phi_D=c'(s^4+t^4+u^4). \tag{39}$$

(Since $s+t+u=0$ any fourth-order symmetric polynomial can be reduced to the above form.)

We have thus obtained, without considering the detailed dynamics,

$$\begin{aligned}&F_{2,2;2,2}=c\kappa^2\frac{s^4}{stu}\,,\\&F_{2,2;2,-2}=0,\\&F_{2,-2;2,-2}=c\kappa^2\frac{u^4}{stu}\,,\\&F_{2,2;-2,-2}=c'\kappa^2\frac{s^4+t^4+u^4}{stu}\,.\end{aligned} \tag{40}$$

We note that from kinematics alone we have deduced the vanishing of the single-helicity-flip amplitude $F_{2,2;2-2}$. Furthermore, the double-helicity-flip amplitude vanishes as well ($c'=0$) since, as we shall show in Sec. VI, any three-graviton vertex conserves helicity when the on-shell gravitons are collinear. This has the consequence that the t-channel graviton-exchange contribution has vanishing residue at $t=0$ (forward scattering). Hence, $F_{2,2;-2,-2}$ cannot have a pole at $t=0$, so that $c'=0$.

Finally, we can determine in Einstein theory the value of the constant c by examining again the

t-pole residue of $F_{2,2;2,2}$. At $t=0$ only two terms contribute in the three-graviton vertex of Eq. (26),

$$V^3(H,h,h) \sim 2\kappa H_{\mu\nu}(-\tfrac{1}{4}h_{\beta\gamma,\mu}h_{\beta\gamma,\nu} - \tfrac{1}{2}h_{\beta\gamma}h_{\beta\gamma,\mu\nu}), \tag{41}$$

and from the t-channel exchange diagram we compute a residue at $t=0$

$$r = i\frac{\kappa^2}{8}(s^2+u^2). \tag{42}$$

Therefore, $c=\frac{1}{4}i$. Our results agree with those of Refs. 1–3. We note that at no stage of our considerations did we need the four-graviton vertex, while the three-graviton vertex is needed only to determine the over-all constant c.

V. GRAVITON-SCALAR AND GRAVITON-PHOTON SYSTEMS

For (massive) scalar-graviton scattering we have two independent helicity amplitudes and Eqs. (8), (12), (13) give

$$\begin{aligned} F_{2,0;2,0} &= \kappa^2\left[\frac{su-m^4}{(s-m^2)^2}\right]^2 \frac{\hat{\phi}_{2,0;2,0}}{(s-m^2)(u-m^2)t}, \\ F_{2,0;-2,0} &= \kappa^2\left[\frac{st}{(s-m^2)^2}\right]^2 \frac{\hat{\phi}_{2,0;-2,0}}{(s-m^2)(u-m^2)t}. \end{aligned} \tag{43}$$

From Eqs. (15), (16) we find near $s=m^2$ and $s=0$

$$\begin{aligned} &\hat{\phi}_{2,0;2,0} \sim (s-m^2)^4, \quad \hat{\phi}_{2,0;2,0} \sim s^0, \\ &\hat{\phi}_{2,0;-2,0} \sim (s-m^2)^4, \quad \hat{\phi}_{2,0;-2,0} \sim s^{-2}. \end{aligned} \tag{44}$$

Hence

$$\begin{aligned} F_{2,0;2,0} &= c\kappa^2 \frac{(su-m^4)^2}{(s-m^2)(u-m^2)t}, \\ F_{2,0;-2,0} &= \kappa^2 t\frac{P_2(s,t,u,m^2)}{(s-m^2)(u-m^2)}, \end{aligned} \tag{45}$$

where P_2 is a second-order homogeneous polynomial in s, t, u, and m, symmetric (from crossing symmetry) in s and u. We have used the fact that in the Born approximation no $1/m^2$ terms can appear.

P_2 is further restricted by the fact that the residue at $s=m^2$ (where $u-m^2=-t$) should be independent of t since only the s-channel diagram (with a scalar intermediate particle) contributes. Therefore P_2 can be written in the form

$$P_2 = \alpha(s-m^2)(u-m^2) + c'm^4, \tag{46}$$

so that

$$F_{2,0;-2,0} = c'\kappa^2\frac{m^4 t}{(s-m^2)(u-m^2)} + \alpha\kappa^2 t. \tag{47}$$

In conventional Einstein theory the scalar is minimally coupled to gravitation, so that the Lagrangian is

$$\mathcal{L} = g^{1/2}(-2\kappa^{-2}R - \tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} - \tfrac{1}{2}m^2\phi^2). \tag{48}$$

By calculating t and u pole residues it is trivial to show that $c=c'=\frac{1}{4}i$. In the conventional theory $\alpha=0$. This fact does not follow from general considerations and points to the possible existence of other scalar-graviton interactions for which $\alpha \neq 0$. We will come back to this question in the next section.

Graviton-photon scattering (or graviton–massless-Yang-Mills scattering since the diagrams are the same in Born approximation) is described by six independent amplitudes. We deduce from Eqs. (13), (14) and crossing symmetry

$$\begin{aligned} F_{2,1;2,1} &= c\kappa^2\frac{s^2}{t}, \\ F_{2,-1;2,-1} &= c\kappa^2\frac{u^2}{t}, \\ F_{2,-1;-2,1} &= c'\kappa^2\frac{t^3}{su}, \\ F_{2,1;-2,-1} &= c''\kappa^2\frac{s^3+u^3}{su} + \beta'\kappa^2 t, \\ F_{2,1;-2,1} &= \beta\kappa^2 t, \\ F_{2,1;2,-1} &= 0. \end{aligned} \tag{49}$$

We note that the amplitudes in Eq. (21) of Ref. 3 have to be interchanged.

In conventional theory the one-graviton–two-photon vertex is

$$V^{(3)}(h,F,F) = \tfrac{1}{2}\kappa h_{\mu\nu}[F_{\mu\lambda}F_{\nu\lambda} - \tfrac{1}{4}\eta_{\mu\nu}(F_{\kappa\lambda})^2]. \tag{50}$$

As before, by looking at pole residues we easily deduce $c=\frac{1}{4}i$, $c'=c''=0$. The β, β' terms vanish as well but we interpret their presence as an indication of freedom of interactions permitted by the low spin of the photon.

Finally, we quote photon-photon gravitational scattering amplitudes:

$$\begin{aligned} F_{1,1;1,1} &= c\kappa^2\frac{s^3}{tu} + \gamma\kappa^2 s, \\ F_{1,1;-1,-1} &= c'\kappa^2\frac{s^4+t^4+u^4}{stu}, \\ F_{1,-1;1,-1} &= c\kappa^2\frac{u^3}{st} + \gamma\kappa^2 u, \\ F_{1,1;1,-1} &= 0. \end{aligned} \tag{51}$$

Since the vertex in Eq. (50) conserves helicity for collinear photons, the argument that led to the vanishing of $F_{2,2;-2,-2}$ shows that $F_{1,1;-1,-1}=0$. We then find $c=-\frac{1}{4}i$ and $\gamma=0$.

VI. FREE PARAMETERS AND HELICITY NONCONSERVATION

In the preceding sections we have seen that our kinematical and dynamical considerations do not completely determine all amplitudes. We discuss in this section to what extent this reflects freedom in the choice of interactions. We emphasize that we are only looking at interactions which lead to helicity amplitudes proportional to κ^2. This restricts us to interactions having rather definite dimensions.

We have seen that for graviton-graviton scattering three of the helicity amplitudes are completely determined. The vanishing of the double-helicity-flip amplitude $F_{2,2;-2,-2}$ followed only as a consequence of helicity conservation of the three-graviton vertex for two collinear physical gravitons. While this seems to be a property of Einstein theory it is in fact quite general: *Any three-graviton vertex with coupling constant having the dimensions of κ conserves helicity when the on-shell gravitons are collinear.* This is a trivial consequence of the fact that the vertex must contain two derivatives. Any scalar made up of three graviton fields and two derivatives leads to helicity conservation. This is also the case if we consider a vertex made up of two graviton fields and some other (spin-zero or -one) field. We can therefore conclude from our S-matrix considerations that for massless spin-2 particles any helicity amplitudes proportional to κ^2 must be those of conventional Einstein theory.

For photon-photon scattering the same helicity-conservation argument holds for any vertex made up of two photon fields $F_{\mu\nu}$ and some other field. On general grounds we can conclude therefore that $F_{1,1;-1,-1}=0$ in Eq. (51). However, the monomial terms in the remaining amplitudes correspond to some real possible interactions. For instance the interaction Lagrangian

$$\mathcal{L}_I=\kappa(F_{\mu\nu})^2\sigma, \tag{52}$$

where σ is a massless scalar field, will give precisely these monomial terms.

We turn now to the more interesting case of graviton-matter scattering. For graviton-photon scattering the vanishing of the pole terms in the helicity-flip amplitudes $F_{2,-1;-2,1}$ and $F_{2,1;-2,-1}$ again follows from general helicity-conservation properties of the vertices (this time we would study the problem for backward scattering at $u=0$). There remain monomial terms $\beta\kappa^2 t$. Similarly, for scalar-graviton scattering we find a monomial term $\alpha\kappa^2 t$. All such terms appear in graviton-helicity-flip amplitudes and lead one to suspect that interactions similar to those of Eq. (52) are responsible for them. For instance for graviton-scalar scattering the following interaction terms lead to such monomial terms in the helicity amplitude

$$\mathcal{L}_I=g^{1/2}\left[\kappa(R_{\mu\nu\rho\sigma})^2\phi+\frac{1}{\kappa}\,\phi^3\right], \tag{53}$$

where $R_{\mu\nu\rho\sigma}$ is the full curvature tensor (covariant summation understood).

We come now to the question of helicity conservation in elastic scattering processes of massless particles. The reader will easily convince himself that the nonminimal couplings in Eqs. (52), (53) yield nonvanishing contributions (the monomial terms) only to the helicity-flip amplitudes. Therefore, helicity conservation is not a general property but rather a specific characteristic of the models considered in Refs. 1–3.

To conclude this section we discuss briefly the "improved" theory of scalar-graviton interactions.[6] In this theory one adds to the Lagrangian of Eq. (48) the term $\lambda g^{1/2}\phi^2 R$. This term produces new three- and four-point scalar-graviton vertices from the linear and quadratic terms in R. Since the linear term vanishes on shell it can only contribute to a t-channel graviton-exchange diagram. Thus the "improving" term generates only a contact term and a t-pole term and leads to contributions (proportional to κ^2) of the form

$$\begin{aligned}F'_{2,0;2,0}&=\kappa^2\lambda\left(\frac{u}{s}\right)^2\frac{\hat{\phi}'_{2,0;2,0}}{t}\,,\\ F'_{2,0;-2,0}&=\kappa^2\lambda\left(\frac{t}{s}\right)^2\frac{\hat{\phi}'_{2,0;-2,0}}{t}\,,\end{aligned} \tag{54}$$

with

$$\begin{aligned}&\hat{\phi}'_{2,0;2,0}\sim s^4\,,\\ &\hat{\phi}'_{2,0;-2,0}\sim s^2\end{aligned} \tag{55}$$

near $s=0$. We deduce on dimensional grounds

$$\begin{aligned}&F'_{2,0;2,0}=0,\\ &F'_{2,0;-2,0}=c\kappa^2\lambda t\,.\end{aligned} \tag{56}$$

It would appear that the improved theory can produce the monomial term in Eq. (47) without recourse to the artificial interaction of Eq. (53). However, a detailed calculation of the actual value of c gives zero.

To see this we write the new contributions to scalar-graviton scattering in the form

$$F'=i\lambda\kappa^2\phi^2R^{(2)}+i\lambda\kappa^2\phi^2(\Box\delta_{\rho\sigma}+2\partial_\rho\partial_\sigma)\times\frac{1}{\Box}\left(R^{(2)}_{\rho\sigma}-\tfrac{1}{2}\eta_{\rho\sigma}R^{(2)}\right). \tag{57}$$

Since $\partial_\rho(R^{(2)}_{\rho\sigma}-\frac{1}{2}\eta_{\rho\sigma}R^{(2)})=0$ (see Sec. III) and the remaining part of the second term cancels the contact contribution, we see that the improved theory gives no additional contributions to scalar-graviton scattering (nor, for that matter, to massless scalar-scalar scattering).

VII. CONCLUSIONS

We have shown in this paper that the high spin of the graviton leads to kinematical constraints on graviton helicity amplitudes strong enough to overcome the extra freedom generated by the dimensional character of the coupling constant and to determine them to a large extent (in Born approximation) with no reference to the detailed dynamics. The graviton-graviton amplitudes are completely determined. The amplitudes for graviton-scalar, graviton-photon, and photon-photon (gravitational) scattering are not completely determined by kinematics. In a few cases there is the freedom of extra monomial terms, and we have shown that this freedom expresses the existence of alternative, nonminimal interactions between gravitons and matter.

In conventional Einstein theory these monomial terms are absent. All amplitudes are therefore determined up to multiplicative constants. These constants are fixed by calculating the residues of the amplitudes at a particular pole. For their computation we need consider only the three-graviton vertex with collinear external graviton momenta. We have found simplified expressions for the three-graviton vertex with two on-shell gravitons, using the background field method. Using this vertex, the computation of the multiplicative constants becomes trivial.

We have given examples of nonminimal interactions which give rise to monomial terms in the helicity amplitudes. These interactions violate helicity conservation in elastic scattering processes of massless particles. We conclude that helicity conservation, although valid in Einstein theory, is not a general property of massless systems. In Ref. 14 it is shown that helicity is conserved in conformally invariant theories; it is interesting that helicity is conserved in Einstein theory although it is not conformally invariant.

The basic ingredient in our considerations is Lorentz invariance. It determines all the general properties of helicity amplitudes that we use. It also fixes the other property that we need, namely the vanishing of any three-graviton vertex when two of the gravitons are on shell, collinear, and with opposite helicities. The only dynamical input is the existence of single-particle poles and the proportionality of the Born amplitudes to κ^2. (We note that lower powers of κ in graviton-graviton scattering are excluded on dynamical grounds; our kinematical arguments also show that in this case most amplitudes would vanish. Higher powers of κ are not excluded and are in fact realized by such nonminimal interactions as $\kappa^2 g^{1/2} R^3$.) The fact that we are thus able to determine graviton Born amplitudes can therefore be regarded as the S-matrix counterpart to the usual proofs that Einstein theory is the essentially unique theory of massless spin-two particles.

ACKNOWLEDGMENTS

We would like to thank H. Pendleton and R. Jackiw for discussions.

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*Research supported by the National Science Foundation.

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