New treatment of the bound-state problem in quantum field theory

M. Hossein Partovi
*Department of Physics, Arya-Mehr University, Tehran, Iran**
and Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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A new formalism for the description of the two-body system and well suited to the bound-state problem is developed within the context of quantum electrodynamics. The resulting two-body equation is a two-time, covariant, functional-differential one which, in the center-of-mass frame, reduces to a single-time, Schrödinger-type equation. Application of this equation is illustrated by calculating several known recoil and structure corrections to the energy levels of positronium and hydrogen. Further extensions and applications of the formalism are discussed.

I. INTRODUCTION

The Bethe-Salpeter (BS) equation[1] is the only existing formalism capable of dealing with the bound-state problem in quantum field theory in a general way. In applications to the two-body quantum-electrodynamic bound systems, where rather precise comparisons with experiments are made, and in other applications, the BS equation has been found remarkably awkward to work with. In fact, the difficulty of handling the BS equation has encouraged the development of alternative ways of dealing with particular aspects of the problem.[2] However, these methods are limited in their scope, and cannot be comsidered as satisfactory alternatives to the BS formalism.

In this paper, we shall develop a new and general two-body equation, applicable to the bound as well as scattering states, which will remove some of the undesirable features of the BS equation. The formalism will be conveniently developed from Schwinger's[1] functional formulation and within the context of QED. The resulting equation will be a functional-differential one whose outstanding feature (in contradistinction to the BS equation) is that, while being a two-time equation in a general coordinate system, it reduces to a single-time one in the rest frame of the two-body system, where it assumes a Schrödinger form with the energy eigenvalue appearing linearly.

The plan of the paper is as follows: In Sec. II, we present the derivation of the two-body equation and some general remarks. In Sec. III A, we outline an approximation procedure for the application of our equations to the calculation of non-self-energy corrections to the energy levels of the hydrogen and positronium atoms. Here we obtain the Dirac and Breit equations, for the said atoms, respectively, as natural limits of our equation. Section III B presents the perturbation scheme appropriate to functional-differential equations. In Secs. III C and III D, we present illustrative calculations of some (known) recoil and size effects in positronium and hydrogen. Section IV contains conclusions and remarks on applications and further extension of this work.

II. DERIVATION OF THE TWO-BODY EQUATION

We start by recording some of the necessary definitions and results of the variational formulation.[3] The usual interaction Hamiltonian for the fermion-photon system is augmented by the addition of the external current term $J^\mu \mathcal{A}_\mu$, where $\mathcal{A}_\mu$ denotes the electromagnetic field operator. The fermion Green's function in the presence of J is defined by (all states and operators refer to the Heisenberg picture)

$$S^J_{\alpha\beta}(x,y) = -i\,\frac{\langle J,\text{out}|\,T[\psi_\alpha(x)\bar{\psi}_\beta(y)]\,|J,\text{in}\rangle}{\langle J,\text{out}|\,J,\text{in}\rangle}\,,$$

and it obeys the equation

$$\left[\gamma_\mu\left(p^\mu_x - eA^\mu(x) - ie\,\frac{\delta}{\delta J_\mu(x)}\right) - m\right] S^J(x,y) = \delta^{(4)}(x-y), \quad (1)$$

where $|J,\text{in}\rangle$ ($|J,\text{out}\rangle$) represents the incoming (outgoing) vacuum in the presence of the external source, and A is the vacuum expectation value of the electromagnetic potential (in the presence of the external source):

$$A^\mu = \frac{\langle J,\text{out}|\mathcal{A}^\mu|J,\text{in}\rangle}{\langle J,\text{out}|J,\text{in}\rangle}\,.$$

The photon Green's function is defined by

$$D^J_{\mu\nu}(x,y) = \frac{\delta A_\mu(x)}{\delta J^\nu(y)}\,,$$

and it obeys the equation (here written in the Lorentz gauge)

$$\Box_x^2 D^J_{\mu\nu}(x,y) = g_{\mu\nu}\delta^{(4)}(x-y) - ie\,\mathrm{tr}\left[\gamma_\mu \frac{\delta S^J(x,x)}{\delta J^\nu(y)}\right]. \tag{2}$$

On eliminating J in favor of A, Eqs. (1) and (2) become

$$\mathfrak{L}_x S^A(x,y) = \delta^{(4)}(x-y), \tag{1'}$$

$$\mathfrak{L}_x \equiv \gamma_\mu\left[p_x^\mu - eA^\mu(x) - ie\int d^4z\, D_\nu^{A\mu}(z,x)\frac{\delta}{\delta A_\nu(z)}\right] - m,$$

$$\Box_x^2 D^A_{\mu\nu}(x,y) = g_{\mu\nu}\delta^{(4)}(x-y) - ie\int d^4z\,\mathrm{tr}\left[\gamma_\mu \frac{\delta S^A(x,x)}{\delta A_\lambda(z)}\right] D^A_{\lambda\nu}(z,y). \tag{2'}$$

The two-particle Green's function for the fermion fields (a) and (b) is defined by

$$K^A(x,y|x',y') = \frac{\langle A,\mathrm{out}|T[\psi^{(a)}(x)\psi^{(b)}(y)\bar\psi^{(a)}(x')\bar\psi^{(b)}(y')]|A,\mathrm{in}\rangle}{\langle A,\mathrm{out}|A,\mathrm{in}\rangle}.$$

The BS equation may be obtained by considering the effect of the operator $\mathfrak{L}_x^{(a)}\mathfrak{L}_y^{(b)}$ on K^A.

Thus far we have recorded some known results of the variational formulation preliminary to our derivation. The latter is based on a consideration of the structure resulting from an additive application of the operators $\mathfrak{L}_x^{(a)}$ and $\mathfrak{L}_y^{(b)}$. Thus we consider

$$[\gamma_\mu^{(a)}P^\mu\mathfrak{L}_x^{(a)} + \gamma_\mu^{(b)}P^\mu\mathfrak{L}_y^{(b)}]K^A(x,y|x',y') = \gamma_\mu^{(a)}P^\mu\delta^{(4)}(x-x')S^{(b)A}(y,y') + \gamma_\mu^{(b)}P^\mu\delta^{(4)}(y-y')S^{(a)A}(x,x'), \tag{3}$$

where P^μ is the total momentum of the two-body system. Equation (3) is quite awkward to handle owing to the appearance of the one-particle Green's functions on the right-hand side. However, we actually only need an equation for the two-body *amplitude*, $\chi^A(x,y)$, defined by

$$\chi^A(x,y) = \frac{\langle A,\mathrm{out}|T[\psi^{(a)}(x)\psi^{(b)}(y)]|\,\alpha,A,\mathrm{in}\rangle}{\langle A,\mathrm{out}|A,\mathrm{in}\rangle}, \tag{4}$$

where χ^A represents the amplitude corresponding to the two-body state $|\alpha,A,\mathrm{in}\rangle$ in the presence of the external source A and reduces to the physical amplitude χ upon setting A equal to zero. The prescription for extracting χ from K has been given by Gell-Mann and Low.[1] Upon effecting this prescription, one obtains the homogeneous equation

$$n^\mu[\gamma_\mu^{(a)}\mathfrak{L}_x^{(a)} + \gamma_\mu^{(b)}\mathfrak{L}_y^{(b)}]\chi^A(x,y) = 0, \tag{5}$$

where $n^\mu = P^\mu/(P^2)^{1/2}$ is the (four) velocity of the state $|\alpha,\mathrm{in}\rangle$. Equation (5) is our two-body equation. The supplementary equation for the photon Green's function follows from (2') by the inclusion of the currents generated by the fields (a) and (b). Thus

$$\Box_x^2 D^A_{\mu\nu}(x,x') = g_{\mu\nu}\delta^{(4)}(x-x') - i\int d^4z\,\mathrm{tr}\left[e^{(a)}\gamma_\mu^{(a)}\frac{\delta S^{(a)A}(x,x)}{\delta A_\lambda(z)} + e^{(b)}\gamma_\mu^{(b)}\frac{\delta S^{(b)A}(x,x)}{\delta A_\lambda(z)}\right] D^A_{\lambda\nu}(z,x'). \tag{6}$$

Clearly Eqs. (1'), (5), and (6) constitute a closed system of numerical functional differential equations which serve to determine χ^A and thereby χ. The physical information such as the scattering amplitude for scattering states and energy levels for bound states are then obtained from χ in the usual manner.

We now proceed to comment on the significant features of Eq. (5). First, we note that in the center-of-mass frame where $n^\mu = g_0^\mu$, Eq. (5) reduces to

$$(\gamma_0^{(a)}\mathfrak{L}_x^{(a)} + \gamma_0^{(b)}\mathfrak{L}_y^{(b)})\chi^A(x,y) = 0. \tag{7}$$

Owing to the occurrence of the time derivatives in the combination $\partial/\partial x_0 + \partial/\partial y_0$ in Eq. (7), the relative time $x^0 - y^0$ may be eliminated from the equation, leaving the common time $t = x^0 = y^0$ and an equation of the Schrödinger type,

$$i\frac{\partial}{\partial t}\chi^A = H\chi^A,$$

where H involves neither time derivatives nor the energy of the state represented by χ^A. Evidently, the requirement of covariance allows a Schrödinger-type description only in a particular frame, and in fact in the only physically significant one. Moreover, the assumption of this form in the center-of-mass frame leads to a genuine eigenvalue problem, in contrast to the BS equation.

Other significant properties of Eq. (5) will be demonstrated in the following section where it will be applied to the hydrogen and positronium

atoms. Here we conclude our remarks by noting that the formal equivalence of Eqs. (1′), (5), and (6) to the Feynman-Dyson series may be demonstrated by a perturbation expansion of the former.

III. SELECTED APPLICATIONS

A. Approximation procedure

Here we will describe an approximation procedure which will be applied to the calculation of some (known) recoil and size effects in the positronium and hydrogen atoms. We may therefore eliminate all self-energy effects from Eqs. (1′), (5), and (6). This may be accomplished by dropping the fermion-loop contributions in Eq. (6), thereby replacing D^A by D_F (the Feynman propagator), and preventing the emission and absorption of a photon by the same fermion by means of associating two different external fields A and B with the fields (a) and (b). In this manner Eqs. (5) and (6) reduce to

$$n^\mu(\gamma_\mu^{(a)}L_x^{(a)}+\gamma_\mu^{(b)}L_y^{(b)})\chi^{AB}(x,y)=0\,, \tag{8}$$

where

$$L_x^{(a)}=\gamma_\mu^{(a)}\left[p_x^\mu - e^{(a)}A^\mu(x) - ie^{(a)}\int d^4z\, D_F^{\mu\nu}(z-x)\frac{\delta}{\delta B^\nu(z)}\right]-m^{(a)},$$

$$L_y^{(b)}=\gamma_\mu^{(b)}\left[p_y^\mu - e^{(b)}B^\mu(y) - ie^{(b)}\int d^4z\, D_F^{\mu\nu}(z-y)\frac{\delta}{\delta A^\nu(z)}\right]-m^{(b)}.$$

In terms of Feynman diagrams, Eq. (8) contains all possible graphs where (1) the photon lines are simple (i.e., no polarization effects), and (2) they do not begin and end on the same fermion line. In short, all non-self-energy diagrams are included. We also note here that Eq. (8) is both covariant and gauge invariant.

At this juncture we will adopt the Coulomb gauge

$$D_C^{\mu\nu}(x)=-(2\pi)^{-4}\int d^4k\,\frac{e^{-ik\cdot x}}{k^2+i\epsilon}\left[g^{\mu\nu}+\frac{k^\mu k^\nu - n\cdot k(k^\mu n^\nu+k^\nu n^\mu)}{(k\cdot n)^2-k^2}\right], \tag{9}$$

and proceed in the center-of-mass frame. Equation (8) then takes the form

$$i\frac{\partial}{\partial t}\chi^{AB}(x,y)=(h+U)\chi^{AB}(x,y)\quad(x^0=y^0=t),$$

$$h=h^{(a)}+h^{(b)},\quad U=U^{(a)}+U^{(b)},\quad h^{(a)}=\vec{\alpha}^{(a)}\cdot\vec{p}_x+\beta^{(a)}m^{(a)}, \tag{10}$$

$$U^{(a)}=e^{(a)}\left\{A^0(x)+i\int d^4z\, D_C^{00}(z-x)\frac{\delta}{\delta B^0(z)}-\alpha_i^{(a)}\left[A^i(x)+i\int d^4z\, D_C^{ij}(z-x)\frac{\delta}{\delta B^j(z)}\right]\right\},$$

where $h^{(a)}$ and $U^{(b)}$ are correspondingly defined.

It is now possible to extract the Coulomb potential as the static part of the interaction. We introduce the transformation

$$\chi^{AB}(x,y)=\exp\left\{-i\int_{-\infty}^{t}d\tau\,[e^{(a)}A^0(\tau,\vec{x})+e^{(b)}B^0(\tau,\vec{y})]\right\}\hat{\chi}^{AB}(x,y). \tag{11}$$

Note that in the physical limit of $A,B\to 0$, χ and $\hat{\chi}$ are equivalent. The substitution of (11) in (10) and the consequent functional differentiations yield

$$i\frac{\partial}{\partial t}\hat{\chi}^{AB}=(h+V_C+\hat{U}^{(a)}+\hat{U}^{(b)})\hat{\chi}^{AB}, \tag{12}$$

where

$$V_C=ie^{(a)}\int d^4z\, D_C^{00}(z-x)\frac{\delta}{\delta B^0(z)}\left[-i\int_{-\infty}^{t}d\tau\, e^{(b)}B^0(\tau,\vec{y})\right]+ie^{(b)}\int d^4z\, D_C^{00}(z-y)\frac{\delta}{\delta A^0(z)}\left[-i\int_{-\infty}^{t}d\tau\, e^{(a)}A^0(\tau,\vec{x})\right]$$

$$=\frac{e^{(a)}e^{(b)}}{4\pi}\,\frac{1}{|\vec{x}-\vec{y}|},$$

and

$$\hat{U}^{(a)}=e^{(a)}\left\{i\int d^4z\, D_C^{00}(z-x)\frac{\delta}{\delta B^0(z)}-\alpha_i^{(a)}\left[A^i(x)+\int_{-\infty}^{t}d\tau\,\nabla^i A^0(\tau,\vec{x})+i\int d^4z\, D_C^{ij}(z-x)\frac{\delta}{\delta B^j(z)}\right]\right\},$$

and similarly for $\hat{U}^{(b)}$. Note that the electrostatic interaction expressed in transformation (11) can equivalently be viewed as an infinite series of "static photon" exchanges by considering the exponential as expanded (in powers of e). We shall refer to this fact shortly.

At this point the infinite-proton-mass limit for the hydrogen atom may be derived. We let (a) be the proton and of very large mass. Then $\vec{\alpha}^{(a)}$ may be dropped and $\vec{x}$ set equal to zero (i.e., proton at the origin). Consequently, of all functional variables only B^0 survives, which may be set equal to zero,[4] leaving as the physical limit the desired equation

$$i\frac{\partial}{\partial t}\chi^D(t,\vec{y})=\left(m^{(p)}+h^{(e)}+\frac{e^{(p)}e^{(e)}}{4\pi|\vec{y}|}\right)\chi^D(t,\vec{y}). \tag{13}$$

In the BS treatment, the derivation of the above limit (i.e., $m^{(e)}/m^{(p)}\to 0$) requires the consideration of an infinite number of irreducible diagrams.[5] This circumstance is understood in the light of the remark at the end of the above paragraph.

Another limit of interest applies to the case of comparable masses. It will be noted that after the extraction of the Coulomb potential above, only nonstatic contributions of the interaction have survived in Eq. (12). Thus when $\hat{U}$ is omitted therefrom, we obtain as the static-interaction limit of our two-body equation

$$i\frac{\partial}{\partial t}\chi^B(t,\vec{x},\vec{y})=\left(h^{(a)}+h^{(b)}+\frac{e^{(a)}e^{(b)}}{4\pi|\vec{x}-\vec{y}|}\right)\chi^B(t,\vec{x},\vec{y}), \tag{14}$$

which is (the time-dependent version of) the Breit equation (without the so-called magnetic term).

It will be noted that the above limits (i.e., the Dirac and Breit equations) have been obtained here in a natural and straightforward way, in contrast to the rather contrived procedures of the other methods.

Having obtained the lowest-order equations appropriate to the hydrogen and positronium atoms, we turn in Sec. III B to the development of a suitable perturbation theory.

B. Perturbation theory

Here we shall be concerned with developing a stationary-state perturbation theory for an equation of the form

$$\left(i\frac{\partial}{\partial t}-H_0\right)|\psi\rangle=U|\psi\rangle, \tag{15}$$

where functional operators are entirely contained in U, and the Coulomb potential is included in H_0. The novel feature of (15) is that because of the occurrence of functional derivatives, the calculation of the energy levels of stationary states involves a time-dependent problem, as will be seen in the following.

Let the solutions of the unperturbed equation be $e^{-i\epsilon_n t}|n\rangle$, so that

$$(\epsilon_n-H_0)|n\rangle=0.$$

Let $e^{-iE_nt}|N,A,t\rangle$ be that solution of (15) which in the limit of no external fields (i.e., $A\to 0$) reduces to $e^{-iE_nt}|N\rangle$, which solution in turn reduces to $e^{-i\epsilon_nt}|n\rangle$ in the absence of the perturbation term U. Our aim is the calculation of the energy shift $\Delta_n=E_n-\epsilon_n$. Note that while $|N\rangle$ is time independent, $|N,A,t\rangle$ definitely depends upon t on account of the presence of the external field which effectively destroys time translation invariance until it is set equal to zero.

The equation satisfied by $|N,A,t\rangle$ is

$$\left(i\frac{\partial}{\partial t}+\epsilon_n-H_0\right)|N,A,t\rangle=(U-\Delta_n)|N,A,t\rangle. \tag{16}$$

The Green's function associated with (16) is defined by

$$\left(i\frac{\partial}{\partial t}+\epsilon_n-H_0\right)G(t,t')=\delta(t-t').$$

The question of the boundary conditions satisfied by G is a nontrivial matter, since H_0 involves the sum of two Dirac Hamiltonians (as well as the Coulomb potential). Indeed, one must refer to Eq. (3), whence (16) has originated, and discover the fate of the boundary conditions as they travel from the former (where, of course, they are known) to the latter. Remarkably, and not too surprisingly, they rest on the fact that U is characteristically a sum of $U^{(a)}$ and $U^{(b)}$, each of which involves the coordinates of one of the particles only [cf. Eq. (12)]. Thus the application of G will always involve terms of the form $GU^{(a)}|\phi\rangle$ or $GU^{(b)}|\phi\rangle$. Then G may be written

$$G(t,t')=-ie^{-i(H_0-\epsilon_n)(t-t')}\times[\theta(t-t')\Lambda^+-\theta(t'-t)\Lambda^-], \tag{17}$$

where the projection operators are those of particle (a) [(b)] when G is acting on $U^{(a)}$ [$U^{(b)}$]. In short, G obeys the usual Feynman conditions with respect to the particle whose coordinates appear to the right of it in U.

Equation (16) may be inverted using G:

$$|N,A,t\rangle=|n\rangle+\int dt'\,G(t,t')[U(t')-\Delta_n]|N,A,t'\rangle, \tag{18}$$

with the formal solution

$$|N,A\rangle=[1-G(U-\Delta_n)]^{-1}|n\rangle.$$

Since $|N\rangle$ is time independent, we have, using (16),

$$0=\lim_{A\to 0}\langle n|(i\partial/\partial t+\epsilon_n-H_0-U+\Delta_n)|N,A,t\rangle$$
$$=\lim_{A\to 0}\langle n|(\Delta_n-U)|N,A,t\rangle\,.$$

The last condition, namely

$$\lim_{A\to 0}\langle n|(U-\Delta_n)[1-G(U-\Delta_n)]^{-1}|n\rangle=0\,, \tag{19}$$

is the basis for an order-wise determination of Δ_n by means of an expansion in some (small) parameter, usually the coupling strength.

As an example, we shall consider the usual perturbative expansion where U is considered to be a first-order quantity. Then the first-order energy shift is given by

$$\Delta_n^{(1)}=\lim_{A\to 0}\langle n|U|n\rangle\,, \tag{20}$$

which is the analog of the usual result. The second-order shift is obtained from

$$\lim_{A\to 0}\langle n|(U-\Delta_n^{(1)}-\Delta_n^{(2)})[1+G(U-\Delta_n^{(1)})]\,|n\rangle=0\,,$$

or

$$\Delta_n^{(2)}=\lim_{A\to 0}\langle n|(U-\Delta_n^{(1)})G(U-\Delta_n^{(1)})|n\rangle\,. \tag{21}$$

If $\Delta_n^{(1)}$ happens to vanish, as is the case in some of our applications, then

$$\Delta_n^{(2)}=\lim_{A\to 0}\langle n|UGU|n\rangle\quad(\Delta_n^{(1)}=0)\,. \tag{22}$$

Clearly, higher-order terms are obtainable by means of a straightforward, although lengthier, calculation.

C. Application to positronium

Here we shall apply Eq. (12) to the derivation of the so-called Coulomb correction to the positronium energy levels.[6] This derivation will also demonstrate the application of the boundary conditions stated in the last subsection.

The Coulomb correction Δ_n^C does not involve the transverse part of the electromagnetic interaction. We therefore drop that part and use (12) with

$$U^{C(a)}=ie^{(a)}\Big[\int d^4z\,D_C^{00}(z-x)\frac{\delta}{\delta B^0(z)}$$
$$+i\vec{\alpha}^{(a)}\cdot\int_{-\infty}^{t}d\tau\,\vec{\nabla}A^0(\tau,\vec{\mathbf{x}})\Big]\,,$$

and correspondingly for $U^{C(b)}$.

The desired correction is obtained from the second- and fourth-order contributions of the perturbation expansion of (19);

$$\Delta_n^C=\Delta_n^C(2)+\Delta_n^C(4)\,,$$

$$\Delta_n^C(2)=\lim_{A^0,B^0\to 0}\langle n|U^CGU^C|n\rangle\,,$$

$$\Delta_n^C(4)=\lim_{A^0,B^0\to 0}\langle n|U^CGU^CGU^CGU^C|n\rangle\,.$$

Let us note here that the designation of the order of our perturbation terms refers to the fact that U^C is $O(e)$, and completely ignores the occurrence of the expansion parameter e in the unperturbed wave functions. Thus the leading contribution to Δ_n^C [$=O(e^{10}m^{(e)})$] is contained in the second- and fourth-order terms of our perturbation series. Moreover, we may neglect the Coulomb potential in G and use G_0, the free propagator, in its place. Effecting this replacement and taking account of the functional operations, we may write

$$\Delta_n^C(2)\cong\lim_{A^0,B^0\to 0}\langle n|(U^{C(a)}G_0U^{C(b)}+U^{C(b)}G_0U^{C(a)})|n\rangle\,.$$

The boundary conditions on G_0 may now be specified according to (17):

$$\Delta_n^C(2)\cong\lim_{A^0,B^0\to 0}(-i)\int dt'\langle n|\{U^{C(a)}(t)e^{-i(H_0-\epsilon_n)(t-t')}[\theta(t-t')\Lambda^{+(b)}-\theta(t'-t)\Lambda^{-(b)}]U^{C(b)}(t')$$
$$+U^{C(b)}(t)e^{-i(H_0-\epsilon_n)(t-t')}[\theta(t-t')\Lambda^{+(a)}-\theta(t'-t)\Lambda^{-(a)}]U^{C(a)}(t')\}\,|n\rangle\,. \tag{23}$$

We now perform the functional differentiations, carry out the time integration, and obtain

$$\Delta_n^C(2)\cong\int d(\vec{\mathbf{x}}-\vec{\mathbf{y}})\,d\vec{\mathbf{x}}'\,d\vec{\mathbf{y}}'\,\chi_n^{B\dagger}(\vec{\mathbf{x}}-\vec{\mathbf{y}})[\delta(\vec{\mathbf{x}}-\vec{\mathbf{x}}')\Lambda^{-(b)}(\vec{\mathbf{y}},\vec{\mathbf{y}}')+\delta(\vec{\mathbf{y}}-\vec{\mathbf{y}}')\Lambda^{-(a)}(\vec{\mathbf{x}},\vec{\mathbf{x}}')]\frac{\alpha}{|\vec{\mathbf{x}}'-\vec{\mathbf{y}}'|}\chi_n^B(\vec{\mathbf{x}}'-\vec{\mathbf{y}}')\,,$$

where χ_n^B denotes the Breit wave function corresponding to the state $|n\rangle$ and α is the fine-structure constant. Equation (23) is identical to Eq. (3.3) of Ref. 6 with their ΔE_{Ca} standing for our $\Delta_n^C(2)$.

The calculation of $\Delta_n^C(4)$ does not involve new features not encountered above. We shall not reproduce the calculation here but only mention the fact that the corresponding result of Ref. 6 (their ΔE_{Cb}) is obtained for $\Delta_n^C(4)$, which together with $\Delta_n^C(2)$ add up to the Coulomb correction.

D. Application to hydrogen

Our first task here is the illustration of the use of the Foldy-Wouthuysen (FW) transformation with the present formalism for the derivation of the recoil corrections to the fine structure of the hydrogen atom.

Referring to Eq. (12), we take the particles (a) and (b) to be the proton and the electron, and consider the following FW transformation on the coordinates of the proton:

$$\hat{\chi}^{AB} = e^{iS}\hat{\chi}_s^{AB}, \quad S = \frac{i\beta^{(p)}}{2m^{(p)}}\vec{\alpha}^{(p)}\cdot[\vec{p}_x - e\vec{T}(x)],$$

$$T^i(x) = A^i(x) + \int_{-\infty}^{t} d\tau\, \nabla^i A^0(\tau, \vec{x}) + i\int d^4z\, D_C^{ij}(z-x)\frac{\delta}{\delta B^j(z)}.$$

The resulting equation for $\hat{\chi}_s^{AB}$ correct to $O(1/m^{(p)})$ is

$$i\frac{\partial}{\partial t}\hat{\chi}_s^{AB}(t,\vec{x},\vec{y}) = [h_y^{(e)} + V_C(\vec{x}-\vec{y}) + m^{(p)} + V']\hat{\chi}_s^{AB}(t,\vec{x},\vec{y}), \tag{24}$$

where

$$V' = \hat{U}^{(e)}(y) + ie\int d^4z\, D_C^{00}(z-x)\frac{\delta}{\delta B^0(z)} + \frac{1}{2m^{(p)}}(\vec{p}_x - e\vec{T})^2 - \frac{e}{2m^{(p)}}\vec{\sigma}^{(p)}\cdot(\vec{\nabla}\times\vec{T}).$$

It remains to calculate the lowest-order energy shifts due to the various contributions to V'.

The simplest term in V' is the proton kinetic energy $\vec{p}_x^{\,2}/2m^{(p)}$. To the order under consideration, it is a trivial matter to show that this term gives rise to the so-called reduced-mass corrections.[7]

Next we note that the terms containing A^0 and B^0 will only contribute to the Coulomb correction, which was calculated above for positronium. Thus we drop these, and turn to the transverse contributions

$$\Delta_n^T = \int dt'\, d(\vec{y}-\vec{x})\, d\vec{x}'\, d\vec{y}'\, \chi_n^{D\dagger}(\vec{y}-\vec{x})$$
$$\times\left[\frac{-ie^2}{m^{(p)}}\int d^4z\, D_C^{ij}(z-x)\frac{\delta}{\delta B^j(z)}\vec{p}_x^{\,i}\, G(t,\vec{x},\vec{y}|t',\vec{x}',\vec{y}')\vec{\alpha}^{(e)}\cdot\vec{B}(t',\vec{y}')\right.$$
$$\left. - \frac{ie^2}{m^{(p)}}\int d^4z\, D_C^{ij}(z-y)\frac{\delta}{\delta A^j(z)}\, G(t,\vec{x},\vec{y}|t',\vec{x}',\vec{y}')\vec{p}_{x'}\cdot\vec{A}(t',\vec{x}')\right]\chi_n^D(\vec{y}'-\vec{x}'), \tag{25}$$

$$\Delta_n^{TT} = \int dt'\, dt''\, d(\vec{y}-\vec{x})\, d\vec{x}'\, d\vec{y}'\, d\vec{x}''\, d\vec{y}''\, \chi_n^{D\dagger}(\vec{y}-\vec{x})\left(\frac{-e^4}{2m^{(p)}}\right)$$
$$\times\left[\int d^4z\, d^4z'\, D_C^{ij}(z-x)D_C^{ik}(z'-x)\frac{\delta}{\delta B^j(z)}\frac{\delta}{\delta B^k(z')}\, G(t,\vec{x},\vec{y}|t',\vec{x}',\vec{y}')\vec{\alpha}^{(e)}\cdot\vec{B}(t',\vec{y}')\right.$$
$$\times G(t',\vec{x}',\vec{y}'|t'',\vec{x}'',\vec{y}'')\vec{\alpha}^{(e)}\cdot\vec{B}(t'',\vec{y}'') + \int d^4z\, \alpha_i^{(e)}D_C^{ij}(z-y)\frac{\delta}{\delta A_j(z)}\, G(t,\vec{x},\vec{y}|t',\vec{x}',\vec{y}')$$
$$\left.\times\int d^4z'\, \alpha_k^{(e)}D_C^{kl}(z'-y')\frac{\delta}{\delta A_l(z)}\, G(t',\vec{x}',\vec{y}'|t'',\vec{x}'',\vec{y}'')\vec{A}(x'')\cdot\vec{A}(x'')\right]\chi_n^D(\vec{y}''-\vec{x}''). \tag{26}$$

These are, respectively, the single- and double-transverse contributions. There are other terms contributing to Δ_n^{TT}, but not to the lowest order. Moreover, we may now take the limit of infinite proton mass, and eliminate the proton coordinates from (25) and (26). In this way, (25) becomes

$$\Delta_n^T = \frac{ie^2}{2m^{(p)}}\int dt'\, d\vec{y}\, d\vec{y}'\, \chi_n^{D\dagger}(\vec{y})[\vec{p}_y^{\,i}\, G^D(t,\vec{y}|t',\vec{y}')\alpha_j^{(e)}D_C^{ij}(t'-t,\vec{y}') + \alpha_i^{(e)}D_C^{ij}(t'-t,\vec{y}')G^D(t,\vec{y}|t',\vec{y}')\vec{p}_{y'}^{\,j}]\chi_n^D(\vec{y}'), \tag{27}$$

where now G^D (as well as χ_n^D) refers to the Dirac equation (13). It is now a trivial matter to reduce (27) to (4.12) of Ref. 7, where it is shown to be the usual single-transverse-photon correction.

Next we take Δ_n^{TT} to the infinite-proton-mass limit and obtain

$$\Delta_n^{TT} = \frac{-e^4}{m^{(p)}}\int dt'\, dt''\, d\vec{y}\, d\vec{y}'\, d\vec{y}''\, \chi_n^{D\dagger}(\vec{y})$$
$$\times[G^D(t,\vec{y}|t',\vec{y}')\alpha_j^{(e)}D_C^{ij}(t-t',\vec{y}')G^D(t',\vec{y}'|t'',\vec{y}'')\alpha_k^{(e)}D_C^{ik}(t-t'',\vec{y}'')$$
$$+\alpha_j^{(e)}D_C^{ij}(t''-t,\vec{y})G^D(t,\vec{y}|t',\vec{y}')\alpha_k^{(e)}D_C^{ik}(t''-t',\vec{y}')G^D(t',\vec{y}'|t'',\vec{y}'')]. \tag{28}$$

The dominant part of (28) may be obtained by neglecting the Coulomb interaction in G^D and retaining the spin-independent part of the interaction.[7] This procedure, after some integration, yields

$$\Delta_n^{TT} = \frac{-e^4}{m^{(p)}} \frac{1}{(2\pi)^3} \int d\vec{p}\, d\vec{p}'\, d\vec{k}\, \chi_n^{D\dagger}(\vec{p}') F(\vec{p}' - \vec{k}, \vec{k}, \vec{k} - \vec{p}) \chi_n^D(\vec{p}),$$

where

$$F(\vec{q}', \vec{k}, \vec{q}) = \frac{1}{4qq'(q+q')} \left[\frac{q'}{(q'+m^{(e)})^2 - E^2} - \frac{q}{(q-m^{(e)})^2 - E^2} \right]$$

$$+ \frac{1}{4qq'E} \left\{ \frac{q'(E-m^{(e)})}{[q+E-m^{(e)}][(E-m^{(e)})^2 - q'^2]} - \frac{q(E+m^{(e)})}{[q'+E+m^{(e)}][(E+m^{(e)})^2 - q^2]} \right\} \tag{29}$$

Here $E^2 = (m^{(e)})^2 + \vec{k}^2$. The leading contribution to Δ_n^{TT} may now be extracted according to the procedure of Ref. 7. In fact, applying that procedure to (29) immediately reduces our Δ_n^{TT} to their ΔE_{TT}, the double-transverse-photon contribution.

So far we have considered all spin-independent contributions to V' and have obtained the significant recoil corrections to the hydrogen fine structure. We now turn to a consideration of the effects of the finite size of the proton, and calculate some correction terms to the hyperfine structure of hydrogen.

To account for the finite-size effects, we shall use the usual replacement $\gamma^\mu A_\mu(x) \to \int d^4\xi\, \Gamma^\mu(x-\xi) A_\mu(\xi)$ for the proton interaction term, where

$$\Gamma^\mu(x) = \frac{1}{(2\pi)^4} \int e^{-ik\cdot x} d^4k \left[\gamma^\mu F_1(k^2) + \frac{i\kappa}{2m^{(p)}} \sigma^{\mu\nu} k_\nu F_2(k^2) \right]. \tag{30}$$

Here Γ^μ is the usual vertex function, and F_i are the form factors. If we let

$$f_i(x) = \frac{1}{(2\pi)^4} \int d^4k\, e^{-ik\cdot x} F_i(k^2), \quad i = 1, 2$$

then

$$\Gamma^\mu(x) = \gamma^\mu f_1(x) + \frac{i\kappa}{2m^{(p)}} \sigma^\mu{}_\nu p_x^\nu f_2(x),$$

and if we also define $\rho^\mu(x) = \beta\Gamma^\mu(x)$, we shall have

$$\rho^0(x) = f_1(x) + \frac{\kappa\beta}{2m^{(p)}} \vec{\alpha}\cdot\vec{p}_x f_2(x),$$

$$\vec{\rho}(x) = \left[f_1(x) + \frac{i\kappa\beta}{2m^{(p)}} \frac{\partial}{\partial t} f_2(x) \right] \vec{\alpha} + \frac{i\kappa}{2m^{(p)}} \beta\vec{\sigma}\times\vec{p}_x f_2(x).$$

Note that f_1 and f_2 depend on x^2 only.

With the above inclusion of the vertex function, the development of the two-body equation proceeds as before, and one gets for the analog of Eq. (10)

$$i\frac{\partial}{\partial t} \phi^{AB}(x,y) = [h^{(p)} + h^{(e)} + U^{(p)}(x) + U^{(e)}(y)]\, \phi^{AB}(x,y), \quad x^0 = y^0 = t \tag{31}$$

where

$$U^{(p)}(x) = e\int d^4\xi\, \rho^0(x-\xi) \left[A^0(\xi) + i\int d^4z\, D_C^{00}(z-\xi) \frac{\delta}{\delta B^0(z)} \right]$$

$$- e\int d^4\xi\, \rho_i(x-\xi) \left[A^i(\xi) + i\int d^4z\, D_C^{ij}(z-\xi) \frac{\delta}{\delta B^j(z)} \right],$$

and $U^{(e)}$ is the same as $U^{(b)}$ of (10).

The Coulomb potential of the extended proton is extracted by means of the transformation

$$\phi^{AB}(x,y) = \exp\left\{ -i\int_{-\infty}^{t} d\tau \left[e\int d^4\xi\, f_1(\tau - \xi^0, \vec{x} - \vec{\xi}) A^0(\xi) - eB^0(\tau, \vec{y}) \right] \right\} \hat{\phi}^{AB}(x,y).$$

The resulting equation for $\hat{\phi}^{AB}$ is

$$i\frac{\partial}{\partial t} \hat{\phi}^{AB}(x,y) = [h^{(p)} + h^{(e)} + \tilde{V}_C(\vec{x} - \vec{y}) + \tilde{U}^{(p)}(x) + U^{(e)}(y)]\, \hat{\phi}^{AB}(x,y), \tag{32}$$

where

$$\tilde{V}_C(\vec{x}) = -\alpha \int d^4\xi \, \frac{f_1(\xi^0, \vec{x} - \vec{\xi})}{|\vec{\xi}|} ,$$
$$\tilde{U}^{(p)}(x) = U^{(p)}(x) - e\int d^4\xi \, f_1(x-\xi) A^0(\xi) - e\vec{\alpha}^{(p)} \cdot \vec{\nabla} \int_{-\infty}^{t} d\tau \int d^4\xi \, f_1(\tau - \xi^0, \vec{x} - \vec{\xi}) A^0(\xi) . \tag{33}$$

Equation (32) is now in a proper form for the application of the perturbation methods developed above, and may be used for a systematic derivation of the proton size effects. Here we shall illustrate the method by deriving the Fermi and the so-called nonrelativistic size corrections[7] to the hyperfine structure of hydrogen.

The part of $\tilde{U}^{(p)}(x)$ which contributes to the sought-after corrections is

$$-e\int d^4\xi \left[\alpha_i^{(p)} f_1(x-\xi) + \frac{i\kappa\beta^{(p)}}{2m^{(p)}} \epsilon^{ils}\sigma^l \vec{p}_x^{\,s} f_2(x-\xi)\right]\left[A^i(\xi) + i\int d^4z \, D_C^{ij}(z-\xi)\frac{\delta}{\delta B^j(z)}\right] + U^{(e)}(y) \equiv \tilde{V}' .$$

Therefore, the desired corrections are given by

$$\Delta_n^H = \lim_{\vec{A},\vec{B}\to 0} \langle n|\tilde{V}' G \tilde{V}'|n\rangle .$$

Note that Δ_n^H is the correction to the energy levels of that (modified) Dirac equation which is obtained from (32) when $\tilde{U}^{(p)} + U^{(e)}$ is omitted.

As before, we shall conveniently extract the leading contributions by effecting a FW transformation on the proton coordinates. When this is done, the spin-dependent part of the above expression, after some computation, reduces to

$$\Delta_n^H = -ie^2 \int dt' \, d(\vec{y} - \vec{x}) \, d\vec{y}' \, d\vec{x}' \, \phi_n^\dagger(\vec{y} - \vec{x}) \frac{\beta^{(p)}}{2m^{(p)}}$$
$$\times \int d^4\xi [\vec{\sigma}^{(p)} \times \vec{\nabla} g_M(x-\xi)]_i \, G(t, \vec{x}, \vec{y}|t', \vec{x}', \vec{y}') D_C^{ij}(y' - \xi) \alpha_j^{(e)} \, \phi_n(\vec{y}' - \vec{x}') + \text{c.c.}, \tag{34}$$

where $g_M = f_1 + \kappa f_2$, and c.c. stands for "complex conjugate." It is convenient now to divide Δ_n^H into two parts, Δ_n^F and Δ_n^s, corresponding to a splitting of $g_M(x)$ into $(1+\kappa)\delta^{(4)}(x)$ and $g_M(x) - (1+\kappa)\delta^{(4)}(x)$. Clearly, Δ_n^F represents the correction for a point proton of anomalous magnetic moment κ, and Δ_n^s is the size effect.

Considering Δ_n^F, we calculate it in the infinite-proton-mass limit and then obtain the reduced-mass correction thereto. The first of these is readily obtained from (34) in the limit $m^{(p)}/m^{(e)} \gg 1$. It is (neglecting the difference $\tilde{V}_C - V_C$)

$$(1+\kappa)\frac{\vec{\sigma}^{(p)}}{2m^{(p)}} \cdot \langle n|\vec{\nabla} V_C \times \vec{\alpha}^{(e)}|n\rangle ,$$

where $|n\rangle$ represents a solution of Eq. (13). The above matrix element is easily shown to be $(1 + \frac{3}{2}\alpha^2)E_F$, where E_F is Fermi splitting, and $\frac{3}{2}\alpha^2 E_F$ is the Breit correction.[7] To obtain the reduced-mass correction to E_F, we consider a FW transformation on the electron coordinates in (34), whereupon the latter reduces, after some manipulation, to

$$\frac{1}{2m^{(e)}} \frac{1+\kappa}{2m^{(p)}} \langle n, s|[\vec{\sigma}^{(e)} \cdot \vec{p} \, \vec{\sigma}^{(p)} \cdot \vec{p} - \vec{\sigma}^{(e)} \cdot \vec{\sigma}^{(p)} p^2, V_C]|n, s\rangle .$$

Here $|n, s\rangle$ is a nonrelativistic Coulomb wave function with the reduced mass $m^{(e)}m^{(p)}/(m^{(e)} + m^{(p)})$. The correction to E_F thus obtained is $-3(m^{(e)}/m^{(p)})E_F$, which, together with the Breit correction, lead to the usual result

$$\Delta_n^F = \left(1 + \tfrac{3}{2}\alpha^2 - 3\frac{m^{(e)}}{m^{(p)}}\right) E_F .$$

Finally we consider the size effect Δ_n^s. This contribution is adequately approximated in the infinite-proton-mass and nonrelativistic electron limit, where it is given by

$$\Delta_n^s = \frac{1}{4m^{(e)}m^{(p)}} \frac{e^2}{(2\pi)^4} \int d^4k \, d^4y \, \psi_n^{*s}(\vec{y}) \sigma_i^{(e)} \sigma_j^{(p)} (k_i k_j - k^2\delta_{ij}) \frac{e^{-ik\cdot y}}{k^2 + i\epsilon} [G_M(k^2) - 1] \psi_n^s(\vec{y}) . \tag{35}$$

Here ψ_n^s represents the nonrelativistic, modified, Coulomb wave function (i.e., $\tilde{V}_C$ instead of V_C), and $G_M = F_1 + \kappa F_2$. Upon performing the y^0 integration in (35), Δ_n^s reduces to Eq. (3.15) of Ref. 7, where it is shown to be the nonrelativistic size correction.

The two correction terms obtained above constitute the major recoil and size-effect contributions to the hyperfine splitting of hydrogen.

IV. DISCUSSION AND CONCLUSION

In this work we have developed a new method of dealing with the two-body system in quantum field theory that is well-suited to the bound-state problem. We have demonstrated the capabilities of this method by applying it to the calculation of known effects, providing at the same time a means of comparison with other methods, principally, the BS formalism.

The method as developed so far is incomplete in one important aspect, to wit, renormalization. The development of the renormalization algorithm in the present context is clearly the next step in this program. Since the renormalization procedure hinges on covariance, one should expect an emphasis on the manifestly covariant form of the equations. Of these, only (5) need be considered as a new element, since the renormalization of S^A and D^A is as in the usual theory.

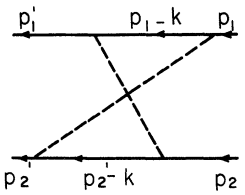

FIG. 1. Example of a two-scalar-meson-exchange diagram.

The applications presented in this paper pertain to bound systems. As mentioned before, the two-body amplitude χ contains the scattering information as well. In fact, a straightforward perturbation expansion of the scattering amplitude obtained from (5) and (6) is the same as the usual Feynman-Dyson series, as expected. The same expansion as obtained from the single-time version of the equation in the center-of-mass frame, on the other hand, while still equivalent to the Feynman-Dyson series, gives rise to diagrams that are neither of the Feynman type, nor of the old-fashioned variety. Figure 1 shows an example in neutral scalar coupling. The contribution of this diagram is, aside from inessential factors,

$$u^{(a)\dagger}(\vec{p}_1')u^{(b)\dagger}(\vec{p}_2')\int d\omega\, d\omega'\, d\vec{k}(\omega^2-\vec{k}^2-\mu^2+i\epsilon)^{-1}[\omega'^2-(\vec{k}+\vec{p}_1'-\vec{p}_1)^2-\mu^2+i\epsilon]^{-1}$$
$$\times(\omega'+W-h^{(a)}_{\vec{p}_1'}-h^{(b)}_{\vec{p}_2'-\vec{k}}+i\epsilon^{(a)})^{-1}(\omega+\omega'+W-h^{(a)}_{\vec{p}_1-\vec{k}}-h^{(b)}_{\vec{p}_2'-\vec{k}}+i\epsilon^{(b)})^{-1}$$
$$\times(\omega+W-h^{(a)}_{\vec{p}_1-\vec{k}}-h^{(b)}_{\vec{p}_2}+i\epsilon^{(a)})^{-1}u^{(a)}(\vec{p}_1)u^{(b)}(\vec{p}_2),$$

where μ represents the mass of the exchanged scalar particle, and W is the total energy. Moreover, $\epsilon^{(a)}$ is a small positive (negative) quantity according as particle (a) is in a positive- (negative-) energy state. In other words,

$$(\omega+W-h^{(a)}-h^{(b)}+i\epsilon^{(a)})^{-1}=(\omega+W-h^{(a)}-h^{(b)}+i\epsilon)^{-1}\Lambda^{+(a)}+(\omega+W-h^{(a)}-h^{(b)}-i\epsilon)^{-1}\Lambda^{-(a)},$$

where ϵ is positive. This "$i\epsilon$ rule" follows from the boundary conditions stated in Sec. III B [cf. Eq. (7)]. As usual, all topologically distinct diagrams, and additionally all different time orderings thereof, must be included. When these are put together appropriately, one recovers the corresponding Feynman-diagram contributions.

As exemplified in the derivations of Sec. III A, the present formalism is an effective method of obtaining various limits and reductions of field-theoretical interactions. Thus application to the derivation of an effective nucleon-nucleon interaction for use with an equation of the Breit type [e.g., Eq. (14)] would be worthwhile. Also, the investigation of the electromagnetic interaction of the nucleons in connection with the isotopic noninvariance of the nuclear force is a particularly suitable problem. Finally, an investigation of the possibilities of nonperturbative applications of Eq. (5) to scattering problems is indicated.

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*Permanent address.

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