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Time-evolution problem in Regge calculus*

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The simplectic approximation to Einstein's equations ("Regge calculus") is derived by considering the net to be actually a (singular) Riemannian manifold. Specific nets for open and closed spaces are introduced in terms of which one can formulate the general time-evolution problem, which thereby reduces to the repeated solution of finite sets of coupled nonlinear (algebraic) equations. The initial-value problem is also formulated in simplectic terms.

I. INTRODUCTION

In order to prepare a problem in continuum physics for machine solution one almost always rewrites the basic partial differential equations (field equations) in discrete form. In other words, one samples the field quantities at finely spaced, selected points, and replaces derivatives by differences. "Discretization" using a net of simplexes is altogether different from this "partial difference" scheme.

Rather than fill spacetime with a grid of *points* one divides it into a *net* of simplexes. Instead of replacing derivatives by differences one seeks the simplectic *analogs* of the fundamental quantities and operations of the continuum theory. (The analogy is even so close as to be a singular instance of the continuum case.) The simplectic version of a field (I will call this a *thatch* for short) may be associated to any of the simplexes of the net, not necessarily just to points, with the tensor character of the thatch expressed by its mode of definition on the simplexes, rather than through many components.

As far as numerical calculations go, the simplectic approach can, when applicable, be expected to be more efficient both because it is more genetically related to the continuum case, and because, for that very reason, it makes sense even as a very crude approximation. It also provides a coordinate-free way to express the solution and in general avoids the problems deriving from the need to work within a particular coordinate or "gauge" condition. What is probably most valuable is that it is no harder to apply to complex topologies than to simple ones.[1]

Even without these "practical" advantages simplectic methods would be of some interest for the insight they furnish into the corresponding continuum equations. And they might even offer a clue to possible discontinuous replacements for field theory that some people see as indicated by the "renormalization" and "quantum gravity" problems.

Regge[2] introduced a simpletic approach to the spacetime metric, and Collins and Williams[1] applied it to the time-symmetric initial-value problem for various topologies. It is interesting that closely related methods have flourished recently in engineering under the name "finite element methods."[3]

After deriving the basic thatch equations in Sec. II we introduce two nets in terms of which the general time-evolution and initial-value problems can be formulated for closed and open spaces, respectively. This leads to a concrete prescription for carrying out actual calculations, although the key questions of existence and stability of solutions are not discussed.

Two appendixes are devoted to clearing up certain technical points (arising from the indefiniteness of the metric) not covered in Regge's treatment.[2] For further details the reader is referred to my thesis,[4] on which this paper is largely based.

II. METRIC NETS

A. The metric thatch

One endows the net Σ with metrical character by assigning to each leg $[ij]$ of the net a "length"—or rather, the square of a length—l_{ij}^2.[5] Consider, then, a particular cell $\sigma \in \Sigma_4$ with vertices 0, 1, 2, 3, 4 (in other words, $\sigma = [01234]$). Just as the three edges of a triangle determine its internal geometry (it is "rigid"), the 10 leg lengths of σ determine its internal geometry. More formally, embed σ linearly in R^4. If, under the embedding, the vertex $[i]$ corresponds to the point x_i, then we seek a (constant) metric $g_{\mu\nu}$ for R^4 such that for all i, j, $l_{ij}{}^2 = g_{\mu\nu}(x_i - x_j)^\mu (x_i - x_j)^\nu$. Since there are ten $l_{ij}{}^2$ and ten independent components of $g_{\mu\nu}$, $g_{\mu\nu}$ must be uniquely determined. An explicit formula for it appears in Ref. 4.

It is not, however, enough that the $l_{ij}{}^2$ define a metric $g_{\mu\nu}(\sigma)$ for the interior of σ. In order that σ can be a "piece" of spacetime the metric must have the signature $-+++$. (This is the analog of the triangle inequality in the Euclidean plane.) In numerical work one must check the signature at each stage.

Having defined a (flat) metric for the interior of each cell we can now "glue" these metrics together at the interface between any two cells, in the obvious way. To be more precise one can introduce a coordinate system in terms of which $g_{\mu\nu}$ is constant throughout the two simplexes σ, ρ and thus provide (the interior of) σ, ρ with a differentiable structure. Doing this for every pair of cells in Σ_4 we define a flat (pseudo-) Riemannian structure for all of the net except the boundaries of the interfaces between cells. At these latter points, the points of $\cup\Sigma_2$ (the set-theoretic union of all 2-simplexes or "bones," which Regge calls the "skeleton"), it may be impossible to find a coordinate system to cover smoothly all the cells which meet there. It is on these bones that the curvature is concentrated.

A two-dimensional example may clarify this. Any two of the four triangular faces of a tetrahedron join smoothly along their common edge. In fact, after removing the other faces, one could flatten them to lie in a plane without at all altering their intrinsic geometry. However, there is no coordinate patch covering a vertex and in which $g_{ij}(x)$ is a smooth function of position. The tetrahedron's intrinsic geometry is everywhere flat except at the four vertices (the "bones") where all the curvature is concentrated. In general the bones are of dimension two less than the manifold itself.

B. The "defect" of a bone

Consider the tetrahedron again. Near any particular vertex it is metrically like a cone and the deviation from flatness at the vertex can be characterized by the "defect angle," were one to cut and flatten the cone. [For a regular tetrahedron this angle is $2\pi - 3(\pi/3) = \pi$.] It is easy to see that this characterization of the "defect" of a bone accords with the usual definition in terms of the nonintegrability of parallel transport (see Fig. 1).

In four dimensions the bones are 2-simplexes, but the notion of defect still applies. Since a net with metric thatch is flat everywhere but the bones, parallel transport around a loop has no effect unless the loop links some bone, and then the result depends only on which bones are linked with what orientation and in what order. In other words, it depends only on the homotopy class of the loop.

Think, now, of a single bone and a loop which circles it once. The space "surrounding" the bone

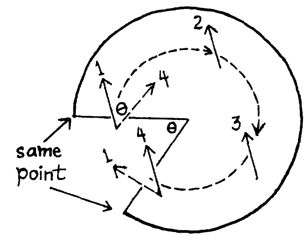


FIG. 1. "Unrolled" cone illustrating the relation of the defect angle θ to the nonintegrability of parallel transport around the bone (vertex).

comprises a "ring" of 4-simplexes (cells) whose mutual intersection is the bone itself. The loop begins in one of these, encounters the others in cyclical order, and returns finally to its point of departure. It is easy to see that a vector parallel to the bone remains unchanged throughout the whole process. Since parallel transport around any loop produces a Lorentz transformation, what we can call the "circulator" of the bone will therefore be a Lorentz transformation which holds fixed the points of a 2-dimensional subspace (that of the bone). There are three cases depending on whether the bone is timelike, spacelike, or null. For a timelike bone the most general circulator is rotation through an angle θ [if $t' = t$ and $z' = z$ then the most general Lorentz transformation is $x' = x \cos\theta - y \sin\theta$, $y' = y\cos\theta + x\sin\theta$ ($\theta = 0$ is *not* the same as $\theta = 2\pi$ though)]. For a spacelike bone the most general circulator is a "boost" with parameter η (if $x' = x$, $y' = y$ then $t' = t\cosh\eta - z\sinh\eta$, $z' = z\cosh - t\sinh\eta$). And for a null bone the most general circulator is also characterized by a single parameter, which, however, is not dimensionless and can be fixed in magnitude only relative to the specific bone.

C. The action

As described above, a metric thatch induces in a net the structure of a (singular) Riemannian manifold. We show now that the continuum expression for the action $S = -\frac{1}{2}\int R\, dV$ makes sense for this manifold, and evaluate it in terms of a sum over the bones.

Since the manifold is flat everywhere outside the bones, the only contribution to $R_{\mu\nu\alpha\beta}$ and *a fortiori* to R is from the neighborhood of a bone. But consider parallel transport around a loop linking some bone, which is a measure of $R_{\mu\nu\alpha\beta}$ there. Since the result is the same no matter where along the bone the loop links it, we see that the bone is homogeneous, and its contribution to S will be proportional to its area. Consider, for example, a timelike bone, infinitely extended and find the action per unit area.

Let the bone be the t-z plane $= \{(txyz)|x = y = 0\}$ and let it have defect θ. If $\theta = 0$, then, replacing the coordinates x, y by r, ϕ, one has for the metric tensor $g_{tt} = -1$, $g_{zz} = 1$, $g_{rr} = 1$, $g_{\phi\phi} = r^2$ with all other components vanishing. We introduce the defect by deleting the "wedge" $2\pi - \theta \le \phi < 2\pi$ from the spacetime and "expanding" ϕ to cover the remainder smoothly, with the result

$$g_{rr} = 1, \quad g_{\phi\phi} = \left(1 - \frac{\theta}{2\pi}\right)^2 r^2 .$$

This metric has a "cusp" at $r = 0$. In order to work with differentiable functions we will "smooth" the cusp temporarily. Thus we evaluate R for the metric

$$g_{rr} = 1, \quad g_{\phi\phi} = e^{2\lambda(r)}$$

with

$$e^{2\lambda} = r^2 \text{ for small } r$$

and

$$e^{2\lambda} = r^2 \left(1 - \frac{\theta}{2\pi}\right)^2 \text{ for large } r .$$

The only nonvanishing Christoffel symbols are

$$\Gamma^r_{\phi\phi} = -\lambda' e^{2\lambda} \text{ and } \Gamma^\phi_{\phi r} = \lambda' \quad \left(\lambda' \equiv \frac{d\lambda}{dr}\right) .$$

Defining $R^\mu_{\nu\alpha\beta} \equiv \Gamma^\mu_{\nu\alpha,\beta} + \Gamma^\lambda_{\nu\alpha}\Gamma_\lambda{}^\mu_\beta - (\alpha \leftrightarrow \beta)$ and $R \equiv g^{\mu\nu} g^{\alpha\beta} R_{\mu\alpha\nu\beta}$, one finds

$$\begin{aligned} R^\phi{}_{r\phi r} &= \lambda'' + (\lambda')^2 , \\ R &= 2[\lambda'' + (\lambda')^2] , \\ (-g)^{1/2} &= (-g_{tt} g_{zz} g_{rr} g_{\phi\phi})^{1/2} \\ &= (g_{rr} g_{\phi\phi})^{1/2} \\ &= e^\lambda . \end{aligned}$$

Thus,

$$\begin{aligned} R(-g)^{1/2} &= 2[\lambda'' + (\lambda')^2] e^\lambda \\ &= 2(e^\lambda)'' , \end{aligned}$$

whence

$$\begin{aligned} -\tfrac{1}{2}\int\int R(-g)^{1/2}\, dr\, d\phi &= -2\pi \int_0^\infty (e^\lambda)'' dr \\ &= -2\pi (e^\lambda)' |_0^\infty . \end{aligned}$$

But near $r = 0$, one has $e^\lambda = r \Rightarrow (e^\lambda)' = 1$, while near $r = \infty$,

$$e^\lambda = r\left(1 - \frac{\theta}{2\pi}\right) \Rightarrow (e^\lambda)' = 1 - \frac{\theta}{2\pi} .$$

Thus

$$\begin{aligned} -\tfrac{1}{2}\int\int R(-g)^{1/2} dr\, d\phi &= -2\pi\left(1 - \frac{\theta}{2\pi} - 1\right) \\ &= \theta , \end{aligned}$$

which is plainly independent of the degree of smoothing in the function $\lambda(r)$.

Extending the integration over the whole bone,

$$\begin{aligned} -\tfrac{1}{2}\iiiint R(-g)^{1/2}\, dr\, d\phi\, dz\, dt &= \theta \int\int dz\, dt \\ &= \theta A . \end{aligned}$$

For a spacelike bone, one finds by a similar analysis $-\frac{1}{2}\int R(-g)^{1/2} d^4x = \eta A$ in which η, the "boost parameter" is defined to be positive for a spacelike defect, which a little thought shows (Fig. 2) to be equivalent to a timelike "infect" or "excess." For a null bone one must work with a three-dimensional metric, but finds without too much more trouble that R, and therefore S, vanishes.

D. The thatch equations

We have just found that each bone contributes to the action in an amount ηA where η stands for the defect (called "η" or "θ" above) and A is the area, considered as a positive number. Summing over all the bones we can write

$$S_g = \sum_{b \in \Sigma_2} \eta(b) A(b) . \tag{1}$$

The "equations of motion" of the metrical thatch require that $\delta S_g = 0$ for all variations of the thatch—in other words, for any variation of the square leg lengths, $l_{ij}{}^2$. Carrying out the variation,

$$\delta S_g = \sum_b \eta(b) \delta A(b) + \sum_b \delta\eta(b) A(b) .$$

But now, just as in the continuum theory, the second term vanishes identically (see Appendix B), and we are left with

$$\delta S_g = \sum_b \eta(b) \delta A(b) .$$

Let us express the variation with respect to a single element $l_{ij}{}^2$ of the thatch. If $[ijk] \in \Sigma_2$ is a bone of the net, then its area is

$$A(ijk) = \tfrac{1}{4} |[[ijk]]|^{1/2} ,$$

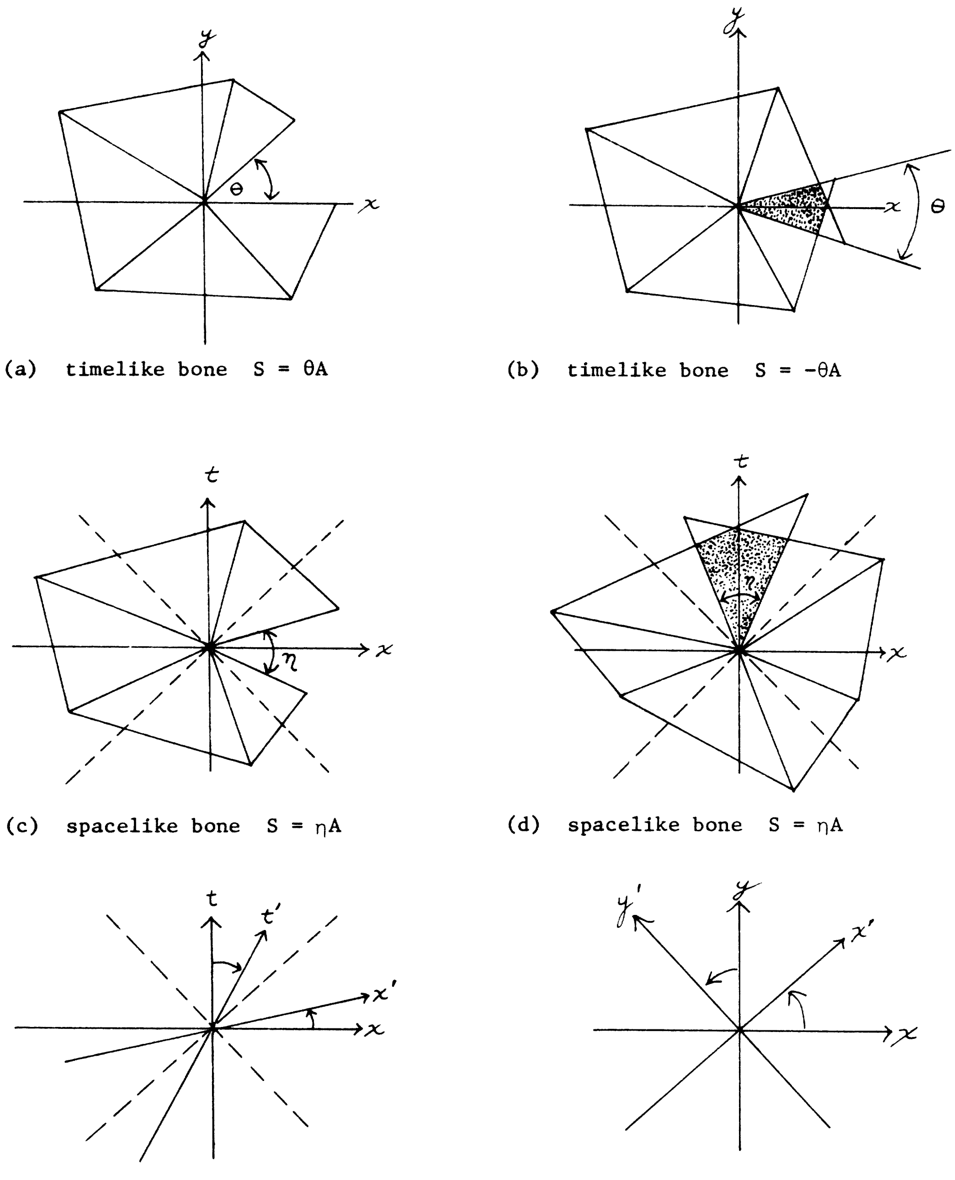
in which, if we set $x = l_{ij}{}^2$, $y = l_{jk}{}^2$, $z = l_{ki}{}^2$, then

$$[[ijk]] \equiv x^2 + y^2 + z^2 - 2(xy + yz + zx) .$$

Thus

$$\frac{\partial A}{\partial l_{ij}{}^2} = \frac{\partial A}{\partial x} = \frac{\pm 1}{16 A} (l_{ij}{}^2 - l_{ik}{}^2 - l_{jk}{}^2) ,$$

where $\pm$ is the sign of $[[ijk]]$.[6] Calling this sign



(e) boost versus rotation

FIG. 2. Typical examples of the action for a bone of area A. Speckled areas indicate overlap. Cases (c) and (d) represent the *same* circulator for the reason indicated in (e), namely that a boost—as opposed to a rotation—turns the two axes in opposite directions.

$\sigma(ijk)$ we get finally

$$G(ij) \equiv \frac{\partial S_g}{\partial l_{ij}^2} = \sum_{b\in\Sigma_2} \eta(b)\, \frac{\partial A(b)}{\partial l_{ij}^2}\,, \tag{2}$$

$$G(ij) = \tfrac{1}{16} \sum_{k\in\Sigma_0} \Theta(ijk)\eta(ijk)\frac{\sigma(ijk)}{A(ijk)} \times (l_{ij}^2 - l_{ik}^2 - l_{jk}^2)\,, \tag{3}$$

in which $\Theta(ijk)$ is 1 if i, j, k are the vertices of some 2-simplex (bone) and 0 otherwise. The "empty-space" thatch equations result from setting $G(ij) = 0$ for all legs $[ij] \in \Sigma_1$.

E. Examples in two and three dimensions

In two dimensions the elements of Σ_{n-2} (the bones) are 0-dimensional and curvature is concentrated entirely on the vertices. Since the "volume" of a point never changes, the variations (2) vanish identically, which implies that S_g is independent of the metric thatch l_{ij}^2. In fact, it is well known that for a two-dimensional manifold (of signature ++) the integral of the curvature depends only on the topology of the manifold (Gauss-Bonnet theorem[7]). In two dimensions, then,

$$G(ij) \equiv 0$$

In three dimensions the bones coincide with the legs and Eq. (2) becomes

$$G(ij) = \eta(ij) .$$

Then the variational equations, $G(ij) = 0$, require that all defects η vanish—the simplectic version of the well-known fact that Einstein's equations have only trivial solutions in three dimensions.

F. The thatch equations with source term

If there are other terms in the action besides S_g then the variational equations will read

$$G(ij) = T(ij) \tag{4}$$

in which, of course, $-T(ij)$ results from varying these other terms with respect to l_{ij}^2. Since $T(ij)$ must represent "matter" we can say that, simplectically, "energy-momentum is concentrated in the legs of the net," even though curvature is diffused throughout Σ_2. And in fact, because of the identity valid in two dimensions, $G(ij) \equiv 0$ (see also previous subsection), it is literally true that $G(ij)$ vanishes outside the 1-simplexes of our singular Riemannian manifold.

G. Coordinate invariance

As pointed out in the Introduction, the simplectic approach provides a coordinate-free method to specify a spacetime geometry. Just for this reason, the well-known coordinate invariance of the continuum formulation finds no analog here. It is *not* true, for example, that, corresponding to a given solution of $G(ij) = 0$ there are an infinite number of others with the same boundary conditions.[8]

A soap-film analogy may serve to clarify this. Aside from the difference in dimension and in signature the "empty-space" problem is very similar to that of approximating a minimal surface (soap film) by a polyhedron. Once the number and connectivity of the faces have been chosen, there will be a unique choice for the vertices which minimizes the total surface area as depicted schematically in Fig. 3. Thus, despite the coordinate ambiguity in an *analytical* solution, the *thatch* solution is unique.

Calculationally this is probably an advantage since it relieves one from the need to choose any "gauge condition" in order to define the time-evolution problem. On the other hand, it leaves one with

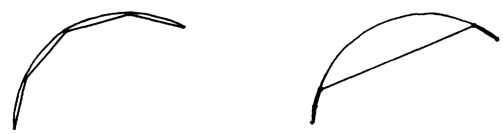


FIG. 3. A good and a bad way to approximate a curve with 4 segments.

less freedom to adjust the net should the thatch begin to go singular during the course of a calculation. What one *can* adjust is the topological character of the net; in fact, it is probably this freedom of topology[9] rather than any numerical freedom of the leg lengths which corresponds to the coordinate or "gauge" freedom in the continuum.

Unfortunately there is one geometry which does possess a full gauge freedom: flat space. Here, as is also clear from the soap-film analogy, the simplectic approximation to the metric is exact (all the defects vanish) and each vertex of Σ has a fourfold freedom to move without affecting the geometry. This means that the time-evolution equations must become underdetermining in the flat-space limit. In other words, both the attempt to produce very accurate solutions with fine nets and the treatment of asymptotically flat metrics can be expected to present extra calculational difficulties.

III. NETS FOR OPEN AND CLOSED SPACETIMES

A. A simplectic net for R^4

The most natural path to arrive at a decomposition of R^4 into 4-simplexes is this: Cover or "tile" R^4 by rectangular regions of which at most five intersect at any point; the *nerve*[10] of this covering will furnish the desired net (Fig. 4). Having taken this path, however, it appears that the answer can be gained directly, and most clearly presented in "affine" coordinates, which moreover are perfectly suited to the symmetries of the net. [Affine coordinates[4] are coordinates in terms of an overcomplete basis ($n+1$ vectors). The vector components as well as the basis vectors are required to sum to zero. If it is more convenient the reader can reduce everything to the usual situation merely by neglecting, e.g., the fifth coordinate.]

We will describe the net by specifying Σ_0 and Σ_1 (that is, the "network" or "graph" formed by the legs). Then the following simple rule (which just expresses that Σ is a nerve) defines Σ_k for $k = 2, 3, 4$:

Any $k+1$ vertices span a k-simplex of the net if and only if they are mutually joined by legs.

Σ_0 comprises all the points of the lattice generated by the 5 basis vectors ($e_j, j = 0, \ldots, 4$), with components $(e_j)^i = \delta^i_j - \frac{1}{5}$. More explicitly, it consists of the following:

(i) all vectors with *integral components* (recall that affine vector components sum to zero);

(ii) vectors differing from those of (i) by one of the following 30 (= 10 + 20) vectors:

$$\pm\tfrac{1}{5}(4\ -1\ -1\ -1\ -1),\quad \pm\tfrac{1}{5}(-1\ 4\ -1\ -1\ -1),\ \ldots,\quad \pm\tfrac{1}{5}(1\ -1\ -1\ -1\ 4),$$

$$\pm\tfrac{1}{5}(3 3\ -2\ -2\ -2),\quad \pm\tfrac{1}{5}(3\ -2\ 3\ -2\ -2),\ldots,\quad \pm\tfrac{1}{5}(-2\ -2\ -2\ 3 3).$$

Finally, any pair of vertices which differ by a vector of the type (ii) (equivalently by a vector all of whose components are less than 1 in absolute value) determine a leg of the net.

This completes the definition of the net. In the rest of this subsection we verify that it is in fact a triangulation of R^4, and we expose some of its properties:

(a) All vertices of the net are equivalent. This follows from the definition of Σ, which is invariant under translation through any lattice vector.

(b) The "isotropic group" of all symmetries of Σ fixing the origin, comprises the 5! permutations of the coordinates with or without an overall sign change. It has therefore 240 elements.

(c) The cells (4-simplexes) of the net fill R^4 without gaps and without any overlap. In other words, the net really is a net. To prove this we note that if any flaw or overlap occurs, it must occur also in the neighborhood of some vertex. Then by (a) it is enough to look near the origin (00000). It is easy to see that the only cells that come near the origin are those related by one of the symmetries (b) to the cell σ_0 with vertices

$$\tfrac{1}{5}(00000),\quad \tfrac{1}{5}(4\ -1\ -1\ -1\ -1),\quad \tfrac{1}{5}(3\ 3\ -2\ -2\ -2),$$
$$\tfrac{1}{5}(2\ 2\ 2\ -3\ -3),\quad \tfrac{1}{5}(1111\ -4).$$

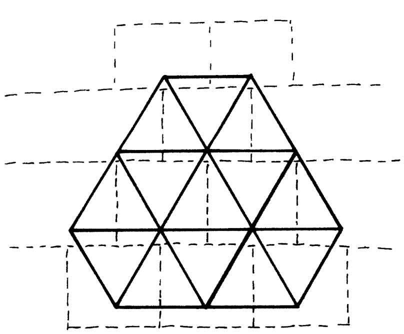


FIG. 4. A familiar simplectic net for R^2 as the "nerve" of a tiling.

But this subset of the vectors of type (ii) (see above) is characterized by the ordering $x^0 \geq x^1 \geq x^2 \geq x^3 \geq x^4$ for its coordinates. Furthermore, any point *interior* to σ_0 is a convex sum (with positive coefficients) of the vertices of σ_0 and thus enjoys the same ordering. On the other hand, *any* point in the neighborhood of the origin has *some* ordering of coordinates and will thus belong to that cell (or those cells if it is on a boundary) whose vertices are those of σ_0 with coordinates permuted to match that ordering. Since there are exactly 5! possible orderings and the same number of cells ("$-\sigma_0$" is the same as σ_0), the assertion is proved.

Finally we introduce some general definitions preparatory to listing some "incidence numbers" for the net.

Definition: $\alpha\,|\,\beta$ (α and β are incident) if and only if α is a subsimplex of β or vice versa.

Definition: $\mathfrak{S}_m(\beta) = m$-star of $\beta = \{\alpha \in \Sigma_m\colon \alpha\,|\,\beta\}$.

Definition: $I(m,k) = \operatorname{card} \mathfrak{S}_m(\beta)$ = number of m-simplexes in the m-star of a k-simplex β. [$I(m,k)$ may have several values if there are k-simplexes in the net inequivalent under any net symmetry.] Thus, for example, it will always be true that

$$m \leq k \Rightarrow I(m,k) = \binom{1+k}{1+m},\ \text{the binomial coefficient.}$$

Here are some easily checked[11] incidence numbers of interest or relevance:

$$I(1,0) = 30,\quad I(2,1) = 8, 14,$$
$$I(4,0) = 120,\quad I(4,2) = 4, 6\ .$$

Notice that legs and bones both come in two inequivalent types. On the other hand, 0-, 3-, and 4-simplexes come in one type only.

B. A simplectic net for $S_3 \times \mathsf{R}$

The spherical character of the net to be described is based on the 4-dimensional analog of the octahedron, a "regular polyhedron" with four pairs of "antipodal" vertices (Fig. 5). Each of these eight vertices implies, for the net, an event which recurs periodically (with period 4) at the same position in space and simultaneous to the antipodal event. The 4 pairs are staggered in phase by 0, 1, 2, 3, respectively. A precise description follows.

Let the vertices be represented as $[t]$ or $[t^*]$ in which t is an integer. Then two vertices $[t_1]$ and $[t_2]$ or $[t_1^*]$ and $[t_2^*]$ determine a leg in Σ_1 if and only if $|t_1 - t_2| \leq 4$, while $[t_1]$ and $[t_2^*]$ determine a leg if and only if $|t_1 - t_2| \leq 3$. (Of course, $[t^*]$ is the vertex "antipodal" to $[t]$, and t is the "time.") As before we complete the description of Σ as the "nerve" determined by Σ_0 and Σ_1, so that, e.g.,

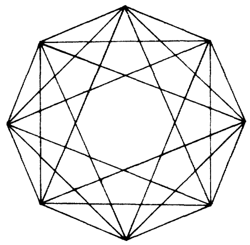


FIG. 5. A regular "polyhedron" in 4 dimensions, representing a constant-time cross section of the net of Sec. III B.

any 5 mutually connected vertices determine a cell of the net.

Let us show that Σ is topologically a 4-dimensional manifold without boundaries. I claim that this is equivalent to the following conditions:

(1) Every simplex belongs to at least one 4-simplex.

(2) Every 3-simplex belongs to exactly two 4-simplexes.

(1) says that every point of Σ has a 4-dimensional "environment," while (2) rules out any boundaries or "bifurcations." Detailed verification of (1) and (2) is an easy, if slightly tedious, matter.

Next we indicate why Σ is homeomorphic to $S_3 \times \mathsf{R}$. In the first place we can embed it in $\mathsf{R}^4 \times \mathsf{R}$ by the scheme

$$[t] \to (x, t),$$

$$[t^*] \to (-x, t)$$

in which x is the vector in R^4:

$$(1\ 0\ 0\ 0) \text{ if } t \equiv 1 \pmod 4,$$

$$(0\ 1\ 0\ 0) \text{ if } t \equiv 2 \pmod 4,$$

$$(0\ 0\ 1\ 0) \text{ if } t \equiv 3 \pmod 4,$$

$$(0\ 0\ 0\ 1) \text{ if } t \equiv 4 \pmod 4.$$

Then it is easy to see that $\Sigma \subset \mathsf{R}^4 \times \mathsf{R}$ is a "cylinder" of the form $B \times \mathsf{R}$ in which B is just the polyhedron pictured in Fig. 5. Since B is thereby homeomorphic to the sphere S_3, Σ is homeomorphic to $S_3 \times \mathsf{R}$.

Here are some incidence numbers for this net:

$$I(1, 0) = 14, \quad I(2, 1) = 6, 8, 10,$$

$$I(4, 0) = 40, \quad I(4, 2) = 4, 6.$$

Finally, we remark that this net for $S_3 \times \mathsf{R}$ can support only a crude approximation to any particular 4-geometry. Unlike the net for R^4 which has no intrinsic scale and can be cast as finely as desired over any continuous spacetime, this one cannot be refined without producing a topologically distinct net.

In the next section we examine the time-evolution and initial-value problems in terms of these nets.

IV. THE TOPOLOGICAL STRUCTURE OF THE THATCH EQUATIONS

A. General considerations

In the case of a purely metric thatch the basic data l_{ij}^2 for the action principle involve one number for each leg of the net. Accordingly there is a single variational equation associated to each leg. What other legs of the net are involved by such an equation?

A single term in the action (1) pertains to all the cells of $\mathfrak{S}_4(b)$. We would thus expect the variational equation of a leg λ to involve all the legs of $\mathfrak{S}_4[\mathfrak{S}_2[\mathfrak{S}_4(\lambda)]]$. But because of the identity discussed in Appendix B the thatch equation (3) involves only $\mathfrak{S}_4(\lambda)$. This is no doubt the simplectic equivalent of Einstein's equations being only second-order despite that the Lagrangian R is itself already second-order in the metric tensor.

If $\mu \in \mathfrak{S}_1[\mathfrak{S}_4(\lambda)]$, then we will say that "λ implicates μ."

B. The time-evolution problem

In this subsection we assume that everything is known up to a given "time" and consider the problem of carrying the solution forward a step. The next subsection will examine the problem of how to "begin" a solution.

Take first the case of the spatially closed net described above. (We discuss this in more detail because there are fewer simplexes to deal with—a closed space has less "space" than an open one.) Suppose known all thatch quantities pertaining to legs previous to $t = 3$ (we will say for short that "all legs previous to $t = 3$ are known") and consider how to extend this knowledge to $t \leq 4$ by means of the thatch equations. In fact, only legs lying wholly after $t = -3$ occur in the same equation with any unknown leg, so it is enough to assume these known. We will call the subnet lying wholly between $t = \pm 3$ an "initial couche."

Consider, for example, the leg [04]. Since $\mathfrak{S}_1([4])$ includes six other legs lying prior to $t = 4$, the most we can really hope for is to find seven equations in terms of which to solve for these seven "new" legs. Happily there are exactly seven legs which implicate both new and couche legs. They are, as is easily checked, the seven extending forward in time from [0]. To express the situation in more detail, we have seven new legs

[4 0], [4 1], [4 1*], [4 2], [4 2*], [4 3], [4 3*]

and for them the equations of the seven legs

[0 1],[0 1*],[0 2],[0 2*],[0 3],[0 3*],[0 4].

In other words, the seven "retarded" legs in $\mathfrak{S}_1([4])$ are determined by the equations of the seven "advanced" legs in $\mathfrak{S}_1([0])$. By symmetry the advanced legs of $\mathfrak{S}_1[0*]$ will similarly determine the retarded legs of $\mathfrak{S}_1[4*]$, and together these include all the unknown legs prior to $t = 4$. Having thus advanced from $t = 3$ to $t = 4$, we can continue indefinitely, and we see that each step requires the solution of two sets of seven equations in seven unknowns. Unfortunately, the equations are nonlinear in the $l_{ij}{}^2$.

Notice, by the way, that in this scheme all equations are utilized (as they must be since there is exactly one for each leg) so that a solution which begins consistent will remain so.

Turn now to the time-evolution problem for the open-space net. We take the first affine coordinate as "time" and assume known all legs prior to $t = \frac{3}{5}$.

In the previous case it could be considered a convenience that the 14 new legs for $t \leqslant 4$ fell into two sets, each of which could be solved for separately. In this case, however, it is crucial that the equations fall into finite clusters in order to avoid solving for an infinite number of unknown legs subject to boundary conditions at spatial infinity, etc. Fortunately the situation turns out to be completely analogous to the previous one with, e.g., the advanced legs of $\mathfrak{S}_1((00000))$ providing exactly enough equations to determine the retarded legs of $\mathfrak{S}_1(\frac{1}{5}(4-1-1-1-1))$. The only difference is that there are 15 new legs in each cluster and an infinite number of clusters rather than only two.

C. The initial-value problem

In contrast to the continuum case the initial-value problem involves thatch equations of exactly the same type as does the time-evolution problem. Where it differs from the latter is in its "topological" structure—in the relation of what is to be found to what is specified.

We begin again with the case of $S_3 \times \mathrm{R}$. As discussed in the last section, the problem is to specify consistently all the legs of the "initial couche" contained between $t = \pm 3$. Of the 66 couche legs there are 18 which implicate only legs of the couche and therefore which imply constraints on the initial-value data. Specifically, they are, as is readily checked,

[−1 3], [−2 2], [−3 1],
[−1*3*], [−2*2*], [−3*1*],
[−1 2], [−1 2*], [−2 1], [−2 1*],
[−1*2*], [−1*2], [−2*1*], [−2*1],
[−1 1], [−1 1*],
[−1*1*], [−1*1].

The scheme which suggests itself is this: to specify freely all the couche legs except for the 18 listed above, and then to solve for the latter by using the 18 constraints which they themselves provide.

As far as $S_3 \times \mathrm{R}$ is concerned then, beginning a solution involves the one-time solution of 18 equations in 18 unknowns, while continuing one begun involves the repeated solution of two sets of seven equations in seven unknowns.

In many respects the initial-value problem for the net of R^4 is similar to that just discussed. On the other hand, the infinity of initial-value data raises whole new problems which may or may not be severe. Only further theoretical investigation or experience with practical application will clarify some of these questions.

At any rate, the initial couche for this net may be taken as the subnet lying wholly between $t = -\frac{3}{5}$ and $t = +\frac{3}{5}$. (Except for the conventional factor of $\frac{1}{5}$ this is just like the previous case.) Observing that a constraint leg is one which implicates only couche legs (in other words, whose 4-star is in the couche) one can count, without too much trouble, 72 couche legs for each vertex at $t = 0$, of which 17 imply constraints. We can therefore specify $\frac{55}{72}$ of the couche legs and solve for the remainder in terms of those specified and of appropriate boundary conditions at "spatial infinity."

And it is easy to see what the boundary conditions should be. Assuming that we pick the "constraint legs" as unspecified, and if we specify all others in a region Ω of the couche, then some of the constraint legs near the boundary of Ω will remain undetermined—namely those implicating legs outside Ω. To specify these in addition to the nonconstraint legs in Ω is to impose boundary conditions at ∞.

Unfortunately there will be, in any practical case, so many initial-value equations (almost 17 for each vertex at $t = 0$) that a direct solution is probably out of the question. Instead one would probably rely on a relaxation method, which, one hopes would be appropriate since the initial-value equations ought, in some sense, to be "elliptic" in analogy with the continuum case.

Alternatively, one might hope to begin somewhere at the "center" of the couche and proceed outward, specifying data until some leg (which must be still free) becomes determined by those already specified. Assuming that such a procedure is possible, there is the further requirement that it be stable in the sense of not leading to some sort of untenable behavior at spatial infinity. Again, these questions need further study.

ACKNOWLEDGMENTS

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APPENDIX A: THE CIRCULATOR OF A NULL BONE, AND HOW IT ENTERS INTO THE THATCH EQUATIONS

Because a null bone is unlikely to arise in the course of an actual calculation we have relegated its discussion to an appendix. We discuss it here, not only for logical completeness, but also for the illumination shed by a "singular case" on a more familiar situation.

1. Parametrization of null rotations

As we have seen in Sec. II B the most general circulator of a bone is a Lorentz transformation fixing some 2-dimensional subspace of spacetime. When the bone is timelike (spacelike) such a transformation is a rotation (boost) characterized by an invariant parameter called the angle (rapidity). Similarly, the possible circulators of a null bone also comprise a one-parameter set. But, unlike the angle and the rapidity this parameter is *not* a Lorentz invariant (it is not "dimensionless"). Nonetheless, there is an invariant implied by the relation of the circulator to its specific bone, as we now show.

Let M be a 4-dimensional vector space, with a basis $\vec{e}_1, \ldots, \vec{e}_4$ in terms of which the scalar product is

$$\langle \xi, \eta \rangle = \xi^1\eta^1 + \xi^2\eta^2 - \xi^3\eta^4 - \xi^4\eta^3 . \tag{A1}$$

Thus $\vec{e}_3$ is null and, together with $\vec{e}_2$, spans a null 2-subspace, $\mathcal{B}$, of M. What is the most general Lorentz transformation fixing $\vec{e}_2$ and $\vec{e}_3$ (and hence $\mathcal{B}$)? It is

$$\Lambda^{\pm}(\lambda) = \begin{bmatrix} \pm 1 & 0 & 0 & \lambda \\ 0 & 1 & 0 & 0 \\ \pm\lambda & 0 & 1 & \lambda^2/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{A2}$$

Since a circulator preserves orientation, we can ignore the minus signs and write simply $\Lambda(\lambda)$. Then

$$\Lambda(\lambda')\Lambda(\lambda'') = \Lambda(\lambda' + \lambda'')$$

so that λ plays the role of an angle. Nevertheless, it is not an invariant because, for example, the Lorentz transformation

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \alpha & \\ & & & \alpha^{-1} \end{bmatrix} \tag{A3}$$

changes λ by a factor of α.

To examine this circumstance let $B^{\nu\beta}$ be any "surface tensor" in the fixed subspace $\mathcal{B}$ and set

$$V_{\mu\nu} = (-g)^{1/2} \epsilon_{\mu\nu\alpha\beta} B^{\alpha\beta} . \tag{A4}$$

In particular, we can imagine that $\vec{e}_2$ and $\vec{e}_3$ span a bone b, that

$$B = \frac{1}{2!} \vec{e}_2 \wedge \vec{e}_3 , \tag{A5}$$

and that $\Lambda(\lambda)$ is the circulator of b. [In application to a net we would refer all tensors to an arbitrarily chosen cell of $\mathfrak{S}_4(b)$.] Then we calculate successively

$$(-g)^{1/2}, \quad B^{23} = -B^{32} = \tfrac{1}{2}, \quad V_{14} = -V_{41} = 1 ,$$

$$[V^{\mu}{}_{\nu}] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ,$$

or comparing with (A2),

$$V = \frac{d}{d\lambda} \Lambda(\lambda) \Big|_{\lambda=0} . \tag{A6}$$

In other words, V $(= V^{\mu}_{\nu})$ is the infinitesimal generator for Λ, whence $\Lambda(\lambda) = e^{\lambda V}$. Noting that

$$V^3 \equiv VVV = 0 \tag{A7}$$

we can express Λ finally in the form

$$\Lambda(\lambda) = e^{\lambda V} = 1 + \lambda V + \tfrac{1}{2}\lambda^2 V^2 . \tag{A8}$$

The previous paragraph shows that, relative to B and Λ, λ is invariantly defined by the relations (A4) and (A8). Furthermore, (A4) in the form

$$V^{\mu}{}_{\nu} = g^{\mu\sigma}(-g)^{1/2} \epsilon_{\sigma\nu\alpha\beta} B^{\alpha\beta} \tag{A9}$$

shows that V has "dimensions" of L^2, from which, with (A8), λ must have "dimensions" of L^{-2}. The case of a non-null bone is precisely analogous except for one thing: One can normalize V through the requirement

$$\hat{V}^{\mu\nu} \hat{V}_{\mu\nu} = \pm 2$$

and define θ or η relative to this dimensionless tensor. In the present case, however, $V \cdot V = -4B \cdot B$

= 0, and one is thrown back on (A9) with its linear dependence on B.

2. An expression for the defect of a spacelike bone

Consider a pure boost Λ and some surface tensor $B^{\mu\nu}$ in the plane fixed by Λ. As before define

$$V^{\mu}{}_{\nu} = g^{\mu\sigma}(-g)^{1/2}\epsilon_{\sigma\nu\alpha\beta}B^{\alpha\beta}, \quad \text{(A10)}$$

$$\Lambda = e^{\lambda V}. \quad \text{(A11)}$$

Calculating as in the previous section one finds

$$\eta = \lambda(2\langle B, B\rangle)^{1/2}, \quad \text{(A12)}$$

where η is the *defect* of Sec. II C if Λ is the circulator corresponding to circulation in the sense indicated by $V^{\mu\nu}$.[12]

The important point now is that λ is a continuous function of the metric thatch $l_{ij}{}^2$ as can be seen from (A10) and (A11) since Λ and g are themselves continuous. Thus as b becomes null, λ goes over into the null parameter λ of the previous subsection.

3. Contribution of a null bone to $G(ij)$

If $b = [ijk] \in \Sigma_2$, then from Eq. (3) its contribution to $G(ij)$ is

$$\frac{\sigma(b)}{16}\,\frac{\eta(b)}{A(b)}(l_{ij}{}^2 - l_{ik}{}^2 - l_{jk}{}^2). \quad \text{(A13)}$$

If b is null, then A and (see Sec. II C) η vanish, leading to 0/0. We can, however, evaluate (A13) as a limit.

As before we refer everything to some cell in $\mathfrak{S}_4(b)$ and work with b spacelike. Then, in the first place, $\sigma(b) = -1$ and

$$A(b) = (\tfrac{1}{2}\langle B, B\rangle)^{1/2}.$$

Combining these with (A12) furnishes for (A13)

$$-\tfrac{1}{8}\lambda(l_{ij}{}^2 - l_{ik}{}^2 - l_{jk}{}^2), \quad \text{(A14)}$$

with λ defined by (A4) and (A8). This is b's contribution to $G(ij)$.

APPENDIX B: PROOF THAT $\Sigma A\delta\eta = 0$

In deriving the metrical thatch equations we used the identity

$$\sum_b A(b)\delta\eta(b) = 0, \quad \text{(B1)}$$

where the sum is over all the bones of the net. Regge[2] has proved this for a positive-definite metric; this appendix extends his proof to the signature $-+++$.

1. Spacetime "Trigonometry"

The notion of defect can be defined either in terms of parallel transport or in terms of angular sums. For a timelike bone ($++$ quotient space) the two are obviously equivalent, but for a spacelike bone ($-+$ quotient space) the definition of "angle" involves some subtlety, it being evident, for example, that θ cannot increase continuously from 0 to 2π during a complete circuit of the bone.

The basic property of angle is additivity on segments of the plane, so that $\theta(x, y)$ in Fig. 6 is independent of where z intervenes. But this additivity is tied up with the relation

$$\cos\theta(x, y) = \frac{x\cdot y}{|x|\,|y|}, \quad \text{(B2)}$$

which might, therefore, be able to define θ in general. However, whenever x or y is timelike there will be an ambiguity in the sign of the right-hand side, not to mention that $\cos\theta$ itself determines θ (which is in general a complex number) only up to a sign. There are two consistent ways (complex conjugates of each other) to resolve these ambiguities, and the following defines one of them.

If x, y are vectors, then define

$$|x| = \langle x, x\rangle^{1/2},$$
$$|x\wedge y| = \langle x\wedge y\,|\,x\wedge y\rangle^{1/2} \equiv (\langle x\wedge y, x\wedge y\rangle/2!\,)^{1/2}, \quad \text{(B3)}$$

in which the root is by definition positive imaginary or positive real for negative or positive argument, respectively. Then we determine θ from the formulas

$$\cos\theta = \frac{\langle x, y\rangle}{|x|\,|y|},$$
$$\sin\theta = \frac{|x\wedge y|}{|x|\,|y|} \quad \text{(B4)}$$

and complete the definition by stipulating

$$0 \le \mathrm{Re}\,\theta \le \pi. \quad \text{(B5)}$$

Let us check, for example, the cosine of the additivity condition illustrated in Fig. 6. To ensure that z is "between" x and y we can conveniently put (since the magnitude of z is irrelevant)

$$z = tx + (1-t)y, \quad 0 \le t \le 1. \quad \text{(B6)}$$

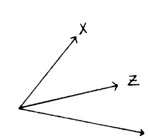

FIG. 6. $\theta(x,y) = \theta(x,z) + \theta(z,y)$.

Then our condition reduces as follows:

$$\cos\theta(x, y) = \cos[\theta(x, z) + \theta(z, y)],$$

$$\cos\theta(x, y) = \cos\theta(x, z)\cos\theta(z, y) - \sin\theta(x, z)\sin\theta(z, y),$$

$$\frac{\langle x, y\rangle}{|x||y|} = \frac{\langle x, z\rangle\langle z, y\rangle}{|x||z|^2|y|} - \frac{|x\wedge z||z\wedge y|}{|x||z|^2|y|},$$

$$|z|^2\langle x, y\rangle = \langle x, z\rangle\langle z, y\rangle - |x\wedge z||z\wedge y|.$$

The final equality is readily verified using (B3) and (B6).

From (B4) it is easy to work out the path of θ in the complex plane during a complete circuit of a spacelike bone (Fig. 7). Taking differences shows that an angle within region I or III is positive pure imaginary, while one in II or IV is negative imaginary.

Finally, what is the formula for the action S in terms of defect angles as defined above? Let θ be the defect, B a tensor representing the bone, and

$$|B| = \langle B|B\rangle^{1/2}, \qquad \text{(B7)}$$

with the same convention that $|B| \sim +i$ or $+1$. In fac , if A is the (real) area of the bone as used in Sec. II D, then

$$A = ||B|| \quad \text{(i.e., absolute value of } |B|),$$

and a detailed check of the sign conventions as indicated, e.g., by Fig. 2, reveals a simple formula for S valid in all cases:

$$iA(b)\eta(b) = |B|\,\theta(b), \qquad \text{(B8)}$$

$$iS = \sum_b |B|\,\theta(b). \qquad \text{(B9)}$$

2. Derivation of the identity

If $\sigma \in \Sigma_4$ is a cell of the net, then let $F(j)$ represent the face opposite vertex j, $B(jk)$ the bone opposite $[jk]$ (geometrically the intersection of F_j and F_k), and $\theta(j, k)$ the angle contained between $F(j)$ and $F(k)$. We will prove the following.

Lemma.

$$\sum_{k,j\in\sigma} |B(jk)|\,\delta\theta(j, k) = 0 \qquad \text{(B10)}$$

if δ denotes a variation in the squared lengths l_{jk}^2 of the legs of σ. Then the desired identity (B1) will result from summing over all the cells of the net:

$$\sum_b |B(b)|\,\delta\theta(b) = 0,$$

or by (B8)

$$i\sum_b A(b)\delta\eta(b) = 0.$$

The following derivation of the identity (B10) rests on three facts. The first follows directly from Stokes's theorem:

$$\sum F(i) = 0. \qquad \text{(B11)}$$

The second involves the volume $|\sigma|$ of σ and the sine of $\theta(i\,j)$:

$$|\sigma| = \tfrac{3}{4}\sin\theta(i\,j)\,|F(i)|\,|F(j)|/|B(ij)|,$$

which follows by elementary geometrical reasoning since formulas such as area $= \frac{1}{2}$ base $\times$ height remain valid under the definitions given in subsection 1 of Appendix B as one can check easily.

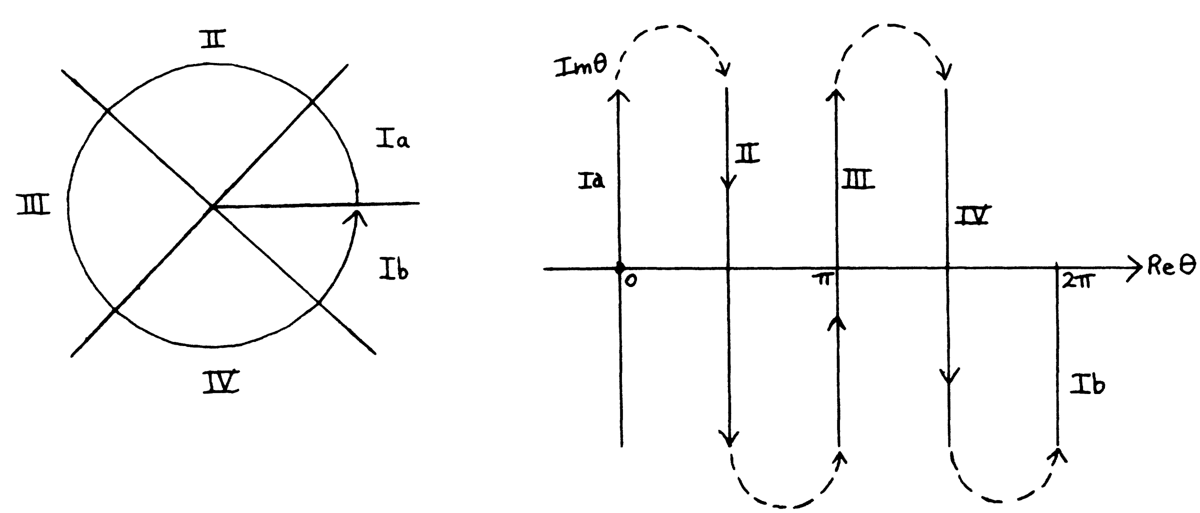

FIG. 7. The dashed lines are "at the point at infinity" in the complex θ plane.

Finally, we need the identity

$$\langle F(i)|F(j)\rangle = -|F(i)|\,|F(j)|\cos\theta(ij),$$

which is similarly easily checked. [The minus sign merely reflects that $F(i)$ and $F(j)$ *face* each other if both are given, say, outward orientation.]

Now we calculate

$$\delta\cos\theta = -\sin\theta\,\delta\theta,$$

$$|B|\delta\theta = -|B|/\sin\theta\,\delta\cos\theta$$

$$= \frac{|B(ij)|}{\sin\theta(ij)}[-\delta\cos\theta(ij)]$$

$$= \tfrac{3}{4}V^{-1}|F(i)|\,|F(j)|\,\delta\frac{\langle F(i)|F(j)\rangle}{|F(i)|\,|F(j)|}$$

$$= \tfrac{3}{4}\frac{1}{V}[\delta\langle F(i)|F(j)\rangle - \langle F(i)|F(j)\rangle\delta\ln|F(i)| - \langle F(i)|F(j)\rangle\delta\ln|F(j)|].$$

Now if we sum on i [j] the third [second] term vanishes by (B11) since $F(i)$ [$F(j)$] enters only linearly. The same argument applies to the first term, completing the proof of the lemma.

APPENDIX C: NOTATIONAL CONVENTIONS

The metric $g_{\mu\nu}$ has signature $-+++$.

If a, b, c are vectors, then

$$a\wedge b = a\otimes b - b\otimes a,$$

$$a\wedge b\wedge c = a\otimes b\otimes c + b\otimes c\otimes a + c\otimes a\otimes b - c\otimes b\otimes a - b\otimes a\otimes c - a\otimes c\otimes b, \text{ etc.}$$

Σ = the net for spacetime.

Σ_m = set of all m simplexes of Σ.

"Vertices," "legs," "bones," "cells" refer, respectively, to Σ_0, Σ_1, Σ_2, Σ_4.

$[jkl]$: an element of Σ_2 with vertices $[j]$, $[k]$, $[l]$.

$\mathfrak{S}_m(\alpha)$ = the m-star of α.

$\mathfrak{S}_m[\mathfrak{S}_p(\alpha)]$ = the m-star of the p-star of α.

$\langle v, \omega\rangle = v^\mu\omega_\mu$, $\langle\omega, \phi\rangle = \omega^{\alpha\beta}\phi_{\alpha\beta}$, etc.

$\langle\omega|\phi\rangle = \langle\omega, \phi\rangle/m!$ for m-forms ω, ϕ.

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[1]See, e.g., P. A. Collins and R. M. Williams, Phys. Rev. D 5, 1908 (1972).

[2]T. Regge, Nuovo Cimento 19, 558 (1961).

[3]C. W. Desai and J. F. Abel, *Introduction to the Finite Element Method* (Van Nostrand Reinhold, New York, 1972).

[4]R. Sorkin, Ph.D. thesis, California Institute of Technology, 1974 (unpublished).

[5]See Appendix C for definitions of Σ_4, $[ij]$, etc.

[6]The unlikely possibility that $A = 0$ is considered in detail in Appendix A.

[7]See Regge (Ref. 2) for a beautiful proof of this from Euler's theorem.

[8]Except for the singular case of zero curvature.

[9]"Topology" here refers to the number and interconnection of the cells chosen to approximate a given manifold, not necessarily to the overall connectivity of the manifold itself.

[10]P. Alexandroff, *Elementary Concepts of Topology* (Dover, New York, 1961). The nerve comprises a vertex for each set in the covering, an edge for each pair of overlapping sets, a 2-simplex for each triplet of mutually overlapping sets, etc.

[11]In deriving such relations it is often convenient to characterize a simplex by its barycenter, for example, σ_0 above by the vector $\frac{1}{5}(2\,1\,0\,{-1}\,{-2})$, which helps clarify the action of the symmetry group.

[12]$\vec{e}\wedge\vec{f}$ "indicates" the sense $\vec{e}\to\vec{f}$.