Some remarks about unquantized non-Abelian gauge fields

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For unquantized non-Abelian gauge fields, remarks are made on the question of whether the field strengths determine the gauge field, on the dual transformations, and on complex gauge fields.

We wish in this paper to make a few simple remarks concerning unquantized non-Abelian gauge fields.[1]

I. DO THE FIELD STRENGTHS $f^k_{\mu\nu}$ DETERMINE THE GAUGE FIELD?

The answer to this question is in general "no" if we consider a multiply connected region, as demonstrated by the Bohm-Aharonov experiment. We shall come back to this topic in a later communication.

In the non-Abelian case, even in a local region, the answer to the question is "no." We shall in fact construct for the case of SU_2 gauge fields[2] two different gauge potentials $(b^k_\mu)_A$ and $(b^k_\mu)_B$ which (a) give the same field strengths, $(f^k_{\mu\nu})_A = (f^k_{\mu\nu})_B$, but (b) cannot be transformed into each other by any gauge transformation. We shall exhibit such an example, for which $(b^k_\mu)_B$ is sourceless and $(b^k_\mu)_A$ is not, so that they cannot possibly be transformed into each other by a gauge transformation.

Take the gauge group to be SU_2. Take the structure constants c^k_{ij} so that $c^3_{12} = c^1_{23} = c^2_{31} = 1$. Consider a region of space that does not include the origin or any point of the negative z axis. Define[3]

$$(b^k_0)_A = 0, \quad (b^k_i)_A = \epsilon_{ikj}\frac{x^j}{r^2}(\Phi - 1) \quad (i = 1, 2, 3), \tag{1}$$

where Φ is a numerical constant. A straightforward evaluation of the field strengths give the following components of $f^k_{\mu\nu}$ as the only nonvanishing ones:

$$(f^l_{ij})_A = \epsilon_{ijk}\frac{x^k x^l}{r^4}(1 - \Phi^2) \quad (i, j, k, l = 1, 2, 3). \tag{2}$$

The field strength is therefore purely "magnetic." The source of the field can be obtained by a straightforward evaluation:

$$(J^k_0)_A = 0, \quad (J^k_i)_A = \epsilon_{ikj}\frac{x^j}{r^4}\Phi(\Phi^2 - 1). \tag{3}$$

If $\Phi \neq 0$ or ± 1, the field is *not* sourceless.

It remains to find $(b^k_\mu)_B$, which will give the same field strength (2) but *is* sourceless. Consider a field with only one isotopic component:

$$(b^1_\mu)_C = (b^2_\mu)_C = (b^3_0)_C = (b^3_r)_C = (b^3_\theta)_C = 0,$$

$$(b^3_\phi)_C = \frac{-(1 - \Phi^2)}{r}\tan\tfrac{1}{2}\theta,$$

where r, θ, ϕ, refer to the standard spherical projections. This is the vector potential along isospin direction 3 of a "magnetic monopole" of strength $-(1 - \Phi^2)$ at the origin. The field strengths are therefore radial magnetic fields:

$$(f^3_{ij})_C = \epsilon_{ijk}\frac{x^k}{r^3}(1 - \Phi^2), \quad (f^1_{ij})_C = (f^2_{ij})_C = 0. \tag{4}$$

Now perform a gauge transformation on this field so that at the point (x^1, x^2, x^3) the isospin direction 3 is rotated to the *isospin direction* $(1/r)(x^1, x^2, x^3)$. Thus the field (4) is rotated into field (2). In other words, if one denotes the field potential and field strength after the gauge transformation by $(b^k_\mu)_B$ and $(f^k_{\mu\nu})_B$, then $(f^k_{ij})_B = (f^k_{ij})_A$. Clearly $(b^k_\mu)_B$ is sourceless because $(b^k_\mu)_C$ is sourceless. Explicitly $(b^k_\mu)_B$ is given by

$$(b^k_0)_B = 0, \quad (b^k_i)_B = \frac{1}{r^2}\left[-\epsilon_{ikj}x^j + \Phi^2\phi^i x^k \tan\tfrac{1}{2}\theta\right], \tag{5}$$

where ϕ^i is the ith component of the unit space-coordinate vector in the ϕ direction.

The above example demonstrates that the field strengths do not determine the gauge field even locally *in any small region.*

II. IS THE DUAL OF A SOURCELESS NON-ABELIAN GAUGE FIELD ALSO A GAUGE FIELD?

The answer to this question is "not always," as shown by explicit examples given in Ref. 4. There one raised this question because in the case of a sourceless electromagnetic field (i.e., a field without electric charge and current density) the dual is also an electromagnetic field, since the sourceless Maxwell equations (which are homogeneous linear) are invariant under the dual transformation[5]

$$f_{\mu\nu} \to f^*_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\xi\eta} f^{\xi\eta}, \tag{6}$$

where $\epsilon_{\mu\nu\xi\eta} = \pm 1$ is antisymmetric in its indices. In the non-Abelian case, this invariance is lost and the dual transformation does not in general lead to a new gauge field. The "symmetry" between "electric" and "magnetic" fields therefore is less useful a concept for non-Abelian gauge fields than for electromagnetism.

III. COMPLEX GAUGE POTENTIALS

Given a gauge group G of dimension[2] m, one usually deals with gauge fields b^k_μ which are real. If now one allows b^k_μ to be complex and writes it as $\underline{b}^k_\mu$, one can define complex gauge field strengths

$$\underline{f}^k_{\mu\nu} = \frac{\partial \underline{b}^k_\mu}{\partial x^\nu} - \frac{\partial \underline{b}^k_\nu}{\partial x^\mu} - c^k_{ij}\underline{b}^i_\mu \underline{b}^j_\nu, \tag{7}$$

which is the same form as in the case when b^k_μ is real. One can then define the source $\underline{J}^k_\mu$ exactly as in Ref. 2 for the case of real b^k_μ (we assume, of course, that the metric $g_{\mu\nu}$ for x_μ is real):

$$\underline{J}^k_\mu = \underline{f}^k_{\mu\nu}{}^{\|\nu}, \tag{8}$$

where the $\|$ derivative contains potential $\underline{b}$. We refer to $\underline{b}^k_\mu$, $\underline{f}^k_{\mu\nu}$, and $\underline{J}^k_\mu$ as the potential, the field strength, and the source for a complex gauge field for the group G.

One could also enlarge the group G into G', which has twice the dimension of G, by taking as its infinitesimal generators X_k and iX_j $(k, j = 1, \ldots, m)$, where X_k are the generators of G. We write X_k and iX_j, respectively, as the first m and second m generators Y_l $(l = 1, \ldots, 2m)$ of G. The (real) structure constants of G' are trivially related to those of G.

Now decompose $\underline{b}$ into real and imaginary parts (we suppress obvious indices):

$$\underline{b} = (b)_r + i(b)_i. \tag{9}$$

Consider $(b^k_\mu)_r$ and $(b^k_\mu)_i$ as, respectively, the first m and second m components of a (real) gauge potential B for the group G'. One can evaluate its field strengths F and its source $\mathcal{J}$. It is not difficult to prove the following theorems.

Theorem: The real and imaginary components of $\underline{f}$ (and $\underline{J}$) are, respectively, the first m and second m components of F (and $\mathcal{J}$). (In other words, a complex gauge field for group G is related to a real gauge field for group G'.)

Theorem: Consider[6] the complex conjugate $(\underline{b}^k_\mu)^*$ of $(\underline{b}^k_\mu)$ as the potential for a complex gauge field. Its field strength and source are the complex conjugates of the field strengths and source of $\underline{b}^k_\mu$.

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[1] Chen Ning Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[2] We follow the notation of Chen Ning Yang, Phys. Rev. Lett. 33, 445 (1974). $\epsilon_{ijk} = \pm 1$ is the antisymmetric tensor. Dummy indices are summed over. Notice that this notation corresponds to the choice of $\epsilon = \frac{1}{2}$ in Ref. 1.

[3] This is closely related to the gauge field discussed by Tai Tsun Wu and Chen Ning Yang, in *Properties of Matter under Unusual Conditions*, edited by H. Mark and S. Fernbach (Interscience, New York, 1969), p. 349.

[4] Gu Chao-hao and Chen Ning Yang, Sci. Sin. 18, 483 (1975).

[5] If the metric is not flat, the right-hand side of (6) should be multiplied by $(-\det g)^{1/2}$.

[6] Notice that here an asterisk denotes complex conjugation, not dual.