Elementary particles in a curved space. IV. Massless particles*

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If physics is stable with respect to a class of perturbations of the spacetime metric, including that of "small" constant four-dimensional curvature, then it may be shown that (1) left-handed and right-handed neutrinos are distinguished by a superselection rule; (2) magnetic monopoles cannot exist; (3) the conformal symmetry associated with the field equations for massless particles with spin 0, $\frac{1}{2}$, and 1 is spontaneously broken—except in the case of neutrinos with fixed chirality.

I. INTRODUCTION

The motivation for this investigation is to seek an understanding of the enigmatic neutrino. The neutrino, like the two other known massless particles, plays a major role in a fundamental interaction that is weak enough to be treated by conventional methods, including perturbative quantum field theory. Yet the only known reason why the neutrino mass should vanish is that it then becomes possible to reduce the number of degrees of freedom—from four to two by means of the chiral projector $\frac{1}{2}(1+\gamma_5)$. Even this is inaccurate since, in fact, four types of neutrinos actually exist: the left-handed ν_e and the right-handed $\bar{\nu}_\mu$ with their respective antiparticles. (We think of ν_e and $\bar{\nu}_\mu$ as the components of a single four-component Dirac field.) It is thus more precise to say that masslessness makes the separation of the left-handed and right-handed neutrino states into invariant superselection subspaces possible. (One of our results is that it also makes it necessary.)

The idea that will be pursued here is to attempt to gain some further insight into the meaning and ramifications of masslessness by generalizing the usual Minkowski-space arena to allow for the possibility of a nonvanishing constant curvature. The result will be directly relevant to flat-space physics if we assume, as we shall do in the discussion that follows, that the mathematical description of physical phenomena is stable with respect to a class of perturbations of the spacetime metric that includes the metric characterized by a "small" constant curvature. In other words, the suggestion is made to invoke a principle of continuity with respect to the curvature (ρ) near $\rho=0$, in order to impose new limitations on physical theories. Our exercise may be helpful in guiding the development of physical theory in another way too, for it is plausible that, when several different notions appear to be basic to a phenomenon in flat space, the more fundamental among them is the one that remains so in a more general context. The main results are the following.

Theorem I. Left-handed and right-handed neutrinos are distinguished by a superselection rule. That is, they are characterized by different values of an exactly conserved quantum number. This follows from the more general conclusions concerning the domain of the Hamiltonian that are obtained in Sec. V.

Theorem II. Free magnetic monopoles cannot coexist with electric charges. This is because the field associated with a magnetic-monopole source describes a state that is not in the domain of the Hamiltonian. This is shown in Sec. VIII.

Theorem III. The field equations for massless particles of spin 0, $\frac{1}{2}$, and 1 are "conformally invariant," but the group of conformal transformations cannot (except in two special cases) be implemented by unitary operators acting on the physical states; this is simply because the domain of the Hamiltonian is not conformally invariant. The exceptions are the realistic case of chiral neutrinos and the unrealistic case of self-dual photons. In all other cases considered the conformal invariance is spontaneously broken (Secs. III, V, and VIII).

The problems of massless particles and conformal symmetry in curved spaces have been studied extensively in the literature,[1] but always from the point of view of the formal invariance of the field equations. The nature of our results makes it clear what is the new element in this paper, and also why it was necessary to restrict ourselves to a space of constant curvature.[2]

de Sitter space can be visualized as (the covering space of) a hyperboloid $y_\alpha{}^2 \equiv y_0{}^2 - y_1{}^2 - y_2{}^2 - y_3{}^2 + y_5{}^2 = \rho^{-1}$ (where ρ is the curvature constant) in a five-dimensional pseudo-Euclidean space with signature $+---+$.[3] The most useful set of intrinsic coordinates is $\vec{y}, t$, where the time t is defined by Eq. (2.6). The group of motions is the universal covering group of SO(3, 2) and the generators are denoted $L_{\alpha\beta}$, $\alpha, \beta = 0, 1, 2, 3, 5$. The irreducible representations that are relevant for elementary particles are denoted $D(E_0, s)$.[4] Here E_0 is the lowest among the eigenvalues E of L_{05} and s is the angular momentum of the lowest eigenspace. Whenever we mention "the Casimir operator" we mean the bilinear invariant Q. In $D(E_0, s)$ we have

$$Q \equiv \tfrac{1}{2} L_{\alpha\beta} L_{\alpha\beta} = E_0(E_0 - 3) + s(s+1)\,. \tag{1.1}$$

In the flat-space limit $\rho \to 0$ the limits of $L_{\mu\nu}$ and $P_\mu \equiv \rho^{1/2} L_{\mu 5}$ ($\mu, \nu = 0, 1, 2, 3$) become the generators of the Poincaré group. Thus, $\rho^{1/2}E$ is the energy and s is the spin. The mass, well defined only in flat space, is given by

$$m = \lim \rho^{1/2} E_0\,. \tag{1.2}$$

A representation of the Poincaré group with $m \neq 0$ is thus obtained as a limit that lets $E_0 \to \infty$.[5] Conversely, any fixed choice of E_0 describes a particle that is massless in the flat-space limit. This fact leaves a great deal of latitude for the definition of "massless" in de Sitter space. We expected that to study and perhaps resolve this ambiguity would give us a deeper understanding of the nature of masslessness. We therefore studied low values of E_0, looking for special phenomena associated with particular representations. The results are as follows: (i) The very remarkable Dirac singleton representations[6] have too few states to allow a field-theoretical application. (Details are not reported here.) (ii) For $s=0$, the only "special" values of E_0 are those associated with conformal invariance, $E_0 = 1$ or 2. (iii) For $s = \frac{1}{2}$ the value $E_0 = \frac{3}{2}$ is distinguished by the existence of a chirality operator and by conformal invariance. (iv) For spin 1, only $E_0 = 2$ is consistent with gauge invariance; this is also the value associated with conformal invariance. (v) For spin 2, only $E_0 = 3$ allows gauge invariance. (Details are not reported here.)

The weight diagrams for $s = 0$, $\frac{1}{2}$, and 1 are shown in Figs. 1, 2, and 3.

To complete this study of massless particles in de Sitter space we should have considered $s = \frac{3}{2}$ and $s = 2$ as well. In particular, the following question suggests itself: Can Einstein's theory of gravitation be reinterpreted as a field theory of massless spin-2 gravitons in a space of constant curvature? In the (unlikely) case that this should be denied we would regard as fortuitous the fact that it can be done in flat space.[7]

[Some unusual conventions: de Sitter space coordinates y_α, $\alpha = 0, 1, 2, 3, 5$; Dirac matrices $\gamma_\alpha = (\gamma_0, \gamma_1, \gamma_2, \gamma_3, i)$; $'\phi$, $'\psi$, $'A_\alpha$, $'F_{\alpha\beta}$ are the "duals" of ϕ, ψ etc.]

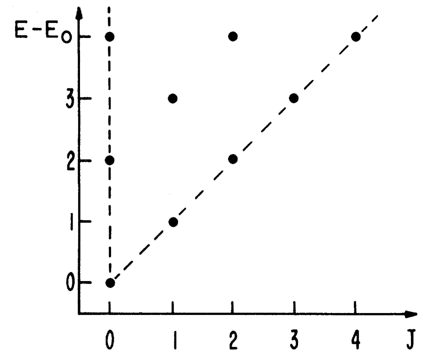

FIG. 1. Weight diagram for $D(E_0, 0)$. Each dot represents an SO(3) irreducible $(2J+1)$-dimensional multiplet. The patterns extend upwards without bound, in the angle bounded by the two broken lines.

II. SPIN 0, GENERAL

Let $\phi(y)$ be a scalar field, on which the action of the infinitesimal generators $L_{\alpha\beta}$ of SO(3, 2) is given by

$$L_{\alpha\beta} = i(y_\alpha \partial_\beta - y_\beta \partial_\alpha)\,. \tag{2.1}$$

The second-order Casimir operator is

$$Q = \tfrac{1}{2} L_{\alpha\beta} L_{\alpha\beta} = \hat{N}(\hat{N} + 3) - y^2 \partial^2\,, \tag{2.2}$$

$$\hat{N} \equiv y_\alpha \partial_\alpha\,, \tag{2.3}$$

and the wave equation is

$$[\,Q - E_0(E_0 - 3)]\,\phi(y) = 0\,. \tag{2.4}$$

As was explained in the Introduction, E_0 is the lowest value on the spectrum of L_{05}.

When E_0 is sufficiently positive one finds the following complete set[8] of solutions of Eq. (2.4)[9]:

$$\phi_{ELM} = C_{EL}\, Y_{LM}(\hat{y}) e^{-iEt\rho^{1/2}} |\vec{y}\,|^{E-E_0}\, Y^{-E} \times {}_2F_1(-K, -K-L-\tfrac{1}{2}; E_0 - \tfrac{1}{2}; -1/\rho\vec{y}^{\,2})\,. \tag{2.5}$$

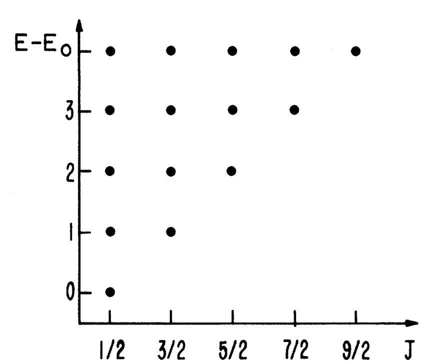

FIG. 2. Weight diagram for $D(E_0, \frac{1}{2})$.

Here C_{EL} is a normalization constant and

$$Y \equiv (y_0{}^2 + y_5{}^2)^{1/2} = (\vec{y}^2 + 1/\rho)^{1/2} ,$$
$$e^{it\rho^{1/2}} \equiv (y_5 - iy_0)/Y , \qquad (2.6)$$
$$K \equiv \tfrac{1}{2}(E - E_0 - L) .$$

The function ϕ_{ELM} is a simultaneous eigenfunction of L_{05}, $\vec{L}^2$, and L_{12}, with eigenvalues $E, L(L+1)$, and M in the range

$$K = 0, 1, \ldots ; \quad L = 0, 1, \ldots ,$$

and, of course, $M = -L, \ldots, L$. With a proper choice of C_{EL} one has[9]

$$\int \phi^*_{ELM}(y)\phi_{E'L'M'}(y)(dy) = \delta_{EE'}\,\delta_{LL'}\,\delta_{MM'} , \qquad (2.7)$$

$$(dy) \equiv dt\, d^3\vec{y} = 2\rho^{-1/2}\delta(y^2 - \rho^{-1})d^5y . \qquad (2.8)$$

These integrals converge when $E_0 > \frac{3}{2}$.

The action of the differential operators (2.1) induces an algebraically irreducible matrix representation of the SO(3, 2) algebra on the basis. This representation will be denoted $D(E_0, 0)$. The matrices are Hermitian if C_{EL} is determined by (2.7). Using the same expressions for the C_{EL} when $E_0 \le \frac{3}{2}$ (analytic continuation in E_0), one finds that the algebraic representation remains irreducible and Hermitian when $\frac{1}{2} < E_0 \le \frac{3}{2}$.[9] The representation can be integrated to an irreducible representation of the SO(3, 2) group, by unitary operators acting in a Hilbert space, if $E_0 > \frac{1}{2}$. When $E_0 > \frac{3}{2}$ we can define the Hilbert space by

$$(\phi_1, \phi_2) = \int \phi_1^*(y)\phi_2(y)(dy) . \qquad (2.9)$$

When $\frac{1}{2} < E_0 \le \frac{3}{2}$ this integral does not converge and we have to define the Hilbert space as the set of functions of the form

$$\phi(y) = \sum C_{ELM}\phi_{ELM}(y) \qquad (2.10)$$

with square summable coefficients (the l^2 norm):

$$\sum |C_{ELM}|^2 < \infty . \qquad (2.11)$$

We wish to explore low values of E_0. We want to know what values of E_0 are "exceptional," and whether unitary representations in the range $\frac{1}{2} < E_0 \le \frac{3}{2}$ are relevant to field theory. The only value of E_0 for which special phenomena have been found is $E_0 = 2$; this case is already known to be of interest in connection with conformal invariance.

III. SPIN 0, E_0=2

In flat space one is concerned with special irreducible representations of the conformal group

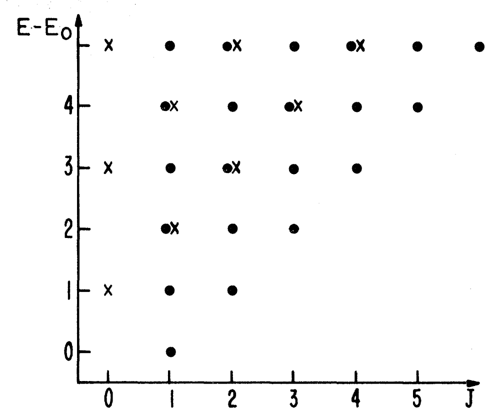

FIG. 3. Weight diagram for $D(E_0, 1)$. Each dot or cross represents an SO(3) irreducible multiplet. When $E_0 = 2$ the weights indicated by crosses belong to an invariant subspace that carries $D(3, 0)$—compare Fig. 1—and the dots give the weight diagram for the representation $D'(2, 1)$ used by electrodynamics.

that remain irreducible when restricted to the Poincaré group.[10] Although the conformal group in de Sitter space has the same structure as the conformal group in flat space, there is no irreducible representation whose SO(3, 2) restriction is equivalent to $D(E_0, 0)$. However there is a very well-known representation that reduces to the sum of only two irreducible SO(3, 2) components, namely, $D(2, 0) \oplus D(1, 0)$. These have the same value of the Casimir operator, $Q = -2$, and may therefore be expected to have the same wave equation. Please see the Appendix.

In fact, Eq. (2.4) has a second set of solutions, $'\phi_{ELM}$ say, given by the same formula (2.5), but with E_0 replaced by $E_0' \equiv 3 - E_0$. Thus

$$\phi_{ELM} \Longleftrightarrow D(E_0, 0) \quad (E_0 = 2)$$
$$'\phi_{ELM} \Longleftrightarrow D(E_0', 0) \quad (E_0' = 3 - E_0 = 1) .$$

The prescription given in the preceding section, for normalizing the wave functions when $E_0 > \frac{1}{2}$, determines all normalization constants except for a single multiplicative factor common to all ϕ'_{ELM}.

The operators $L_{6\alpha}$, that with (2.1) complete the algebra of infinitesimal conformal transformations, can be represented by (see the Appendix)

$$L_{6\alpha} = \rho^{1/2}(y_\beta L_{\beta\alpha} - iy_\alpha)$$
$$= i\rho^{-1/2}\partial_\alpha - i\rho^{1/2}\hat{N}y_\alpha . \qquad (3.1)$$

They do not, in general, commute with Q, but the wave equation (2.4) is nevertheless invariant if the eigenvalue $E_0(E_0 - 3)$ is equal to -2. They are not symmetric with respect to (2.9), but this is irrelevant since the basis functions $'\phi_{ELM}$ are not normalizable in that sense. The operators

(2.1) and (3.1) induce on the basis functions ϕ_{ELM}, $'\phi_{ELM}$ an irreducible matrix representation of the conformal algebra. In particular, $L_{6\alpha}$ transforms the ϕ_{ELM} into the $'\phi_{ELM}$ and vice versa. The matrices are Hermitian provided only that the previously undetermined common normalizer of all the $'\phi_{ELM}$ is chosen appropriately. The representation can be integrated in the same sense that $D(E_0, 0)$ can be integrated when $\frac{1}{2}< E_0 \leq \frac{3}{2}$, on the l^2 norm.

The wave equation (2.4), in terms of the variables $\vec{y}$ and t, reads

$$\left(H - \frac{1}{\rho Y^2}\,\partial_t{}^2\right)\phi(\vec{y}, t) = 0\,, \tag{3.2}$$

$$H \equiv \partial_i(\delta_{ij} + \rho y_i\, y_j)\partial_j + 2\rho\,. \tag{3.3}$$

The canonical quantum-mechanical metric is given by

$$(\phi_1, \phi_2) = \int \phi_1^* i\overleftrightarrow{\partial}_t \phi_2 \frac{d^3y}{\rho Y^2}\;; \tag{3.4}$$

it has some interesting properties. As far as the ϕ_{ELM} are concerned one finds that orthonormality as defined by (2.7) means exactly the same as

$$(\phi_{ELM}, \phi_{E'L'M'}) = \delta_{EE'}\,\delta_{LL'}\,\delta_{MM'}\,. \tag{3.5}$$

The $'\phi_{ELM}$ could not be normalized by (2.7), but they are normalizable in the sense of (3.4). In fact, the normalization condition

$$('\phi_{ELM}, '\phi_{E'L'M'}) = \delta_{EE'}\,\delta_{LL'}\,\delta_{MM'} \tag{3.5$'$}$$

is precisely the one that makes the matrices $L_{6\alpha}$ Hermitian.

Thus, the $'\phi_{ELM}$ seem to have gained respectability—but there is a catch: the lowest-energy eigenfunctions

$$\phi_{200} \propto (y_0 + iy_5)^{-2} \text{ and } '\phi_{100} \propto (y_0 + iy_5)^{-1} \tag{3.6}$$

are not orthogonal, although their energies are different. Orthogonality between two states with unequal energies depends on the fact that ($P_0 \equiv \rho^{1/2} L_{05} = id/dt$)

$$\begin{aligned}\rho^{1/2}(E_1 - E_2)(\phi_1, \phi_2) &= -i\,\frac{d}{dt}\,(\phi_1, \phi_2)\\ &= \int [\,(P_0\phi_1)^* i\overleftrightarrow{\partial}_t \phi_2\\ &\qquad - \phi_1^* i\overleftrightarrow{\partial}_t (P_0\phi_2)]\, d^3y/\rho Y^2\,.\end{aligned} \tag{3.7}$$

This vanishes if P_0 is self-adjoint on a domain $D(P_0)$ with respect to the metric (3.4) and if ϕ_1, ϕ_2 both belong to $D(P_0)$. Since the functions (3.6) are not orthogonal they cannot both be in $D(P_0)$.

Using the wave equation (3.2) we can rewrite (3.7) as follows

$$= \int [\,(H\phi_1)^*\phi_2 - \phi_1^*(H\phi_2)]\, d^3y\,. \tag{3.8}$$

In spherical coordinates (3.3) takes the form

$$H = r^{-2}\partial_r(r^2 + \rho r^4)\partial_r - \frac{\vec{L}^2}{r^2} + 2\rho\,. \tag{3.9}$$

This is formally symmetric with respect to the ordinary L^2 metric, hence, the failure of (3.8) to vanish has to do with poor convergence in the limit $r\to\infty$. If ϕ_1 and ϕ_2 have the same angular quantum numbers, then (3.8) reduces to

$$\int_0^\infty dr[\,(\partial_r r^4 \partial_r R_1)^* R_2 - R_1^*(\partial_r r^4 \partial_r R_2)]\,, \tag{3.10}$$

where $R_i(t,r)$ is the radial part of $\phi_i(t, r, \Omega)$. All the other terms are strongly convergent and therefore cancel out. Rearranging we get

$$\int_0^\infty dr\partial_r[\,(r^4\partial_r R_1)^* R_2 - R_1^*(r^4\partial_r R_2)]\,. \tag{3.11}$$

This vanishes if and only if

$$\lim_{r\to\infty}\,[\,(r^4\partial_r R_1)^* R_2 - R_1^*(r^4\partial_r R_2)] = 0\,. \tag{3.12}$$

Equation (2.5) shows that, as $r\to\infty$, ϕ_{ELM} behaves like $r^{-2} + cr^{-4} + \cdots$, while $'\phi_{ELM}$ behaves like $r^{-1} + c'r^{-3} + \cdots$. Therefore, let us take

$$R_i = a_i(t)r^{-1} + b_i(t)r^{-2} + \cdots,\quad i = 1, 2\,. \tag{3.13}$$

Then (3.12) reduces to

$$a_1^* b_2 - b_1^* a_2 = 0\,. \tag{3.14}$$

This means that, if $\phi = R(t,r)Y_{LM}(\Omega)$ and $R(t,r)$ has the expansion $R = a(t)r^{-1} + b(t)r^{-2} + \cdots$ for large r, then $a(t)/b(t)$ must be a fixed real number (or infinite)—the same for all ϕ in $D(P_0)$. Because the energy spectrum associated with ϕ_{ELM} [and hence with $b(t)$] is different from that of $'\phi_{ELM}$ [i.e., $a(t)$], we conclude that the only possibilities are (i) $a(t) = 0$, (ii) $b(t) = 0$. Thus, $D(P_0)$ may be taken to be spanned by the ϕ_{ELM} *or* the $'\phi_{ELM}$, but not by both.

It follows that the physical states do not carry a representation of the conformal group (or even the algebra), this in spite of the invariance of the wave equation, and in spite of the fact that the algebraic representation induced on the solutions is equivalent to a unitary one. We are in the presence of a perfect example of a spontaneously broken symmetry.

It may be objected that the argument was based on the interpretation of the theory as a quantum-mechanical system, that may be irrelevant since a complete quantum-mechanical interpretation is impossible in view of the appearance of negative-energy states. However, the requirement of completeness with respect to (3.4) can be based on the unitarity of the field-theoretical S matrix in perturbation theory. The solution of the inhomogeneous wave equation

$$(Q+2)\phi(y)=f(y)$$

is

$$\phi(y)=\int D_R(y,y')f(y')(dy'),$$

where[11]

$$D_R(y,y')=\sum \phi_I(y)\phi_I^*(y')\theta(t-t'). \qquad (3.15)$$

Here the sum runs over the physical asymptotic states. Thus

$$(Q+2)\int D_R(y,y')f(y')(dy')=f(y),$$

which is equivalent to

$$\int\sum \phi_I(y)\phi_I^*(y')i\overleftrightarrow{\partial}_{t'}f(y')\frac{d^3y'}{\rho Y'^2}\bigg|_{t'=t}=f(y).$$

Therefore, the wave functions that enter into the sum that defines $D_R(y,y')$ must be complete with respect to (3.4). On the other hand, unitarity of the S matrix requires that these wave functions be precisely those of the physical asymptotic states.

[The sum in (3.15), extended over the normalized $\phi_{ELM}(y)$, was calculated in Ref. 3, for arbitrary E_0. When $E_0=2$ it reduces to

$$\sum \phi_{ELM}(y)\phi_{ELM}^*(y')=(\rho/4\pi^2)(z^2-1)^{-1}, \qquad (3.16)$$

with $z\equiv\rho y_\alpha y'_\alpha$. In the limit $\rho\to 0$ this becomes the familiar $(-1/2\pi^2)/(x-x')^2$. Similarly,

$$\sum{}' \phi_{ELM}(y)'\phi_{ELM}^*(y')=(\rho/4\pi^2)z(z^2-1)^{-1}$$

which has the same flat-space limit.]

IV. SPIN $\frac{1}{2}$, GENERAL

Let $\psi_a(y)$, $a=1,2,3,4$, be a spinor field. The infinitesimal generators of SO(3, 2) are

$$L_{\alpha\beta}=i(y_\alpha\partial_\beta-y_\beta\partial_\alpha)+\tfrac{1}{4}i(\tilde\gamma_\alpha\gamma_\beta-\tilde\gamma_\beta\gamma_\alpha)$$
$$\equiv M_{\alpha\beta}+\Sigma_{\alpha\beta}, \qquad (4.1)$$

where $\gamma_\alpha=(\gamma_0,\vec\gamma,i)$, $\tilde\gamma_\alpha=(\gamma_0,\vec\gamma,-i)$, and $\gamma_0,\vec\gamma$ are the usual Dirac matrices. We note the following expressions for the Casimir operator:

$$Q\equiv\tfrac{1}{2}L_{\alpha\beta}L_{\alpha\beta}=\hat N(\hat N+3)-y^2\partial^2+\kappa+\tfrac{5}{2}$$
$$=\kappa(\kappa+4)+\tfrac{5}{2}$$
$$=E_0(E_0-3)+\tfrac{3}{4}. \qquad (4.2)$$

Here $\hat N\equiv y_\alpha\partial_\alpha$ as before, and

$$\kappa\equiv\Sigma_{\alpha\beta}M_{\alpha\beta}=2i\Sigma_{\alpha\beta}y_\alpha\partial_\beta$$
$$=\hat N-\tilde\gamma_\alpha y_\alpha\gamma_\beta\partial_\beta. \qquad (4.3)$$

The last expression for Q in (4.2) is an application of Eq. (1.1) to the case $s=\frac{1}{2}$, the others are simple to verify directly.

Equation (4.2) shows that Q is essentially factorized by κ, so that an appropriate choice of wave equation is

$$(\kappa-N)\psi=0$$

or

$$(2i\Sigma_{\alpha\beta}y_\alpha\partial_\beta-N)\psi=0. \qquad (4.4)$$

According to (4.2) the eigenvalue N of κ is related to E_0 by either $E_0=-N-\frac{1}{2}$ or $E_0=N+\frac{7}{2}$. We may take ψ to be homogeneous of degree N, then as (4.3) shows, Eq. (4.4) reduces to the simpler

$$\gamma_\alpha\partial_\alpha\psi=0. \qquad (4.5)$$

However, this equation[12] does not by itself fix E_0; therefore, it does not define an irreducible representation, and it is not a suitable wave equation. Equation (4.4) was proposed by Dirac.

The Lagrangian for (4.4) is [with $\bar\psi\equiv\psi^*\gamma_0$ and the volume element (dy) defined by (2.8)]

$$\mathcal{L}=\int\bar\psi(2i\Sigma_{\alpha\beta}y_\alpha\partial_\beta-N)\psi(dy). \qquad (4.6)$$

The current

$$J_\beta\equiv 2\bar\psi\Sigma_{\alpha\beta}y_\alpha\psi \qquad (4.7)$$

satisfies $y_\beta J_\beta=0$ and, if ψ satisfies (4.4),

$$\partial_\beta J_\beta=0 \qquad (4.8)$$

or

$$\frac{d}{dt}J^t+\partial_iJ^i=0. \qquad (4.9)$$

Here we have introduced the variables $t,\vec y$ [see Eq. (2.6)] and

$$J^t\equiv(y_0J_5-y_5J_0)Y^{-2}=\psi^\dagger\Gamma^t\psi, \qquad (4.10)$$

$$\Gamma^t\equiv 1+i(\vec\gamma\cdot\vec y/Y)\exp(i\gamma_0\rho^{1/2}t). \qquad (4.11)$$

Integrating (4.9) we find

$$\frac{d}{dt}\int J^td^3y=0, \qquad (4.12)$$

which expresses the fact that $\int J^td^3y$ is invariant with respect to the transformations generated by P_0.

Since

$$P_0\equiv\rho^{1/2}L_{05}=i\frac{d}{dt}-\tfrac{1}{2}\rho^{1/2}\gamma_0, \qquad (4.13)$$

the time dependence of a stationary state is contained in a factor of the form

$$\exp[-i\rho^{1/2}(E+\tfrac{1}{2}\gamma_0)t] .$$

It is convenient to reduce this to a pure phase factor by introducing

$$\tilde{\psi}(t,\vec{y}) \equiv \exp(\tfrac{1}{2}i\rho^{1/2}\gamma_0 t)\psi(t,\vec{y}) . \qquad (4.14)$$

Substituting this into the wave equation (4.4) we obtain the Schrödinger equation

$$\left(\Gamma^0 i\frac{d}{dt} - H\right)\tilde{\psi} = 0 \qquad (4.15)$$

with

$$\Gamma^0 = \Gamma^t|_{t=0} = 1 + i\vec{\gamma}\cdot\vec{y}/Y, \qquad (4.16)$$

$$H = \tfrac{1}{2}\{Y\Gamma^0, i\gamma_0\vec{\gamma}\cdot\vec{\partial}\} - (N+2)\rho^{1/2}\gamma_0 . \qquad (4.17)$$

As either (4.12) or (4.15) shows, the canonical quantum-mechanical metric is

$$(\psi_1, \psi_2) = \int \tilde{\psi}_1^\dagger \Gamma^0 \tilde{\psi}_2 d^3y . \qquad (4.18)$$

The matrix Γ^0 is Hermitian and positive definite.

When N is sufficiently negative one finds[13] for (4.4) a complete, discrete system of basis vectors $\psi^N_{EJM}(y)$, simultaneous eigenvectors of L_{05}, $\vec{L}^2$, and L_{12}. The spectrum is

$$E - E_0 = 0, 1, 2, \ldots; \quad J = \tfrac{1}{2}, \tfrac{3}{2}, \ldots, E - E_0 + \tfrac{1}{2}, \qquad (4.19)$$

and $E_0 = -N - \frac{1}{2}$. The wave function for the ground state is

$$\tilde{\psi}^N_{E_0, 1/2, M} \propto Y^N u , \qquad (4.20)$$

where u is a constant spinor such that $\gamma_0 u = u$. The algebraic representation induced on ψ^N_{EJM} by the differential operators (4.1) is Hermitian and irreducible if $N < -\frac{3}{2}$; that is, $E_0 > 1$. The wave functions are normalizable in the metric (4.18) under the same condition.

For other values of N one finds for (4.4) a different set of solutions, with essentially the same properties, but with ground-state energy $E_0' = N + \frac{7}{2}$. They are given by

$$'\psi^N_{EJM} = \beta\psi^{-N-4}_{EJM} , \qquad (4.21)$$

where β is the matrix introduced by Gürsey and Lee[12]:

$$\beta \equiv -i\rho^{1/2}\gamma\gamma_\alpha y_\alpha , \quad \gamma \equiv i\gamma_0\gamma_1\gamma_2\gamma_3 . \qquad (4.22)$$

Indeed, one easily verifies that

$$\{\beta, \kappa+2\} = 0 , \qquad (4.23)$$

so that

$$(\kappa - N)'\psi^N = -\beta(\kappa + N + 4)\psi^{-N-4} = 0 . \qquad (4.24)$$

The ground-state wave function is

$$'\tilde{\psi}^N_{E_0', 1/2, M} \propto \gamma\Gamma^0 Y^{-N-3} u , \qquad (4.25)$$

where u is a constant spinor satisfying $\gamma_0 u = u$. The algebraic representation induced on $'\psi^N_{EJM}$ by (4.1) is Hermitian and irreducible if $N > -\frac{5}{2}$; that is, $E_0' > 1$. Noting that $\Gamma^0\gamma\Gamma^0 = \gamma/\rho Y^2$, we see that the wave functions are normalizable in the metric (4.18) under the same condition.

To summarize: When $N \le -\frac{5}{2}$ the first set of solutions form a complete orthonormal basis for the representation $D(-N-\frac{1}{2}, \frac{1}{2})$. When $N \ge -\frac{3}{2}$ the second set of solutions carry the representation $D(N+\frac{7}{2}, \frac{1}{2})$. When $-\frac{5}{2} < N < -\frac{3}{2}$ both sets are normalizable and this interval deserves some further investigation. The midpoint, $N = -2$ (or $-N - \frac{1}{2} = N + \frac{7}{2}$) is also the only case in which the chiral projections $\frac{1}{2}(1 \pm \beta)$ are defined on the domain of the Hamiltonian.

V. SPIN $\frac{1}{2}$, $E_0 = \frac{3}{2}$

The operators $L_{\alpha 6}$, that with (4.1) complete the algebra of infinitesimal conformal transformations, are (see the Appendix)

$$L_{6\alpha} = \rho^{1/2}(y_\beta L_{\beta\alpha} - \tfrac{3}{2}i y_\alpha)$$
$$= i\rho^{-1/2}\partial_\alpha - i\rho^{1/2}(\hat{N}+1)y_\alpha + \tfrac{1}{2}i\rho^{1/2}\tilde{\gamma}_\beta y_\beta \gamma_\alpha . \qquad (5.1)$$

The commutator $[L_{6\alpha}, \kappa]$ is equal to $iy_\alpha(\kappa+2)$; hence, our wave equation (4.4) is invariant only if $N = -2$. In this case $E_0 = E_0' = \frac{3}{2}$ and the functions ψ^N, $'\psi^N$ carry two identical copies of the representation $D(\frac{3}{2}, \frac{1}{2})$ of SO(3, 2). Note that (4.21) reduces to

$$'\psi^N_{EJM} = \beta\psi^N_{EJM} \quad (\text{when } N = -2) \qquad (5.2)$$

or

$$'\tilde{\psi}^N_{EJM} = \tilde{\beta}\tilde{\psi}^N_{EJM} , \qquad (5.3)$$

$$\tilde{\beta} \equiv \rho^{1/2} Y\gamma\Gamma^0 . \qquad (5.4)$$

The matrices β and $\tilde{\beta}$ satisfy

$$\beta^2 = \tilde{\beta}^2 = 1 . \qquad (5.5)$$

The operators $L_{\alpha 6}$ transform the ψ^N into the $'\psi^N$ and vice versa. Together, ψ^N and $'\psi^N$ carry a representation of the conformal algebra. Unlike the spin-0 case, however, this representation is reducible. The operators (5.1) commute with β and a pair of conjugate[14] irreducible representations are induced in the chirality subspaces spanned by

$$\tfrac{1}{2}('\psi^N_{EJM} \pm \psi^N_{EJM}) = \frac{1 \pm \beta}{2}\psi^N_{EJM} . \qquad (5.6)$$

As in the spin-0 case, we must determine which states span $D(P_0)$. The analog of (3.8) is

$$\int [(H\tilde{\psi}_1)^\dagger \tilde{\psi}_2 - \tilde{\psi}_1^\dagger (H\tilde{\psi}_2)]\, d^3y\,. \tag{5.7}$$

Using (4.17) we reduce this to ($i/2$ times)

$$\int d^3y\, \vec{\partial}\cdot (Y\tilde{\psi}_1^\dagger \{\vec{\gamma}\gamma_0, \Gamma^0\}\tilde{\psi}_2)\,. \tag{5.8}$$

The condition for this to vanish is, simply ($r \equiv |\vec{y}|$),

$$\lim_{r\to\infty} r^3 \int \tilde{\psi}_1^\dagger \gamma_0 \tilde{\psi}_2 d\Omega = 0\,. \tag{5.9}$$

Asymptotically, up to a common factor,

$$\tilde{\psi}^N_{EJM} = r^{-2}u + r^{-2}(1 - i\vec{\gamma}\cdot\hat{y})v\,, \tag{5.10}$$

$${}'\tilde{\psi}^N_{EJM} = r^{-1}\gamma(1 + i\vec{\gamma}\cdot\hat{y})u\,. \tag{5.11}$$

The spinors u and v depend on t and on the angles, and both are in the positive eigenspace of γ_0; $\hat{y}$ is the direction of $\vec{y}$ and $r = |\vec{y}|$. Take, for $i = 1, 2$,

$$\tilde{\psi}_i = r^{-2}u_i + r^{-1}\gamma(1 + i\vec{\gamma}\cdot\hat{y})u_i' + r^{-2}(1 - i\vec{\gamma}\cdot\hat{y})v_i\,; \tag{5.12}$$

then (5.9) reduces to

$$\int (u_1^\dagger \gamma\vec{\gamma}\cdot\hat{y}u_2' - u_1'^\dagger \gamma\vec{\gamma}\cdot\hat{y}u_2)d\Omega = 0\,. \tag{5.13}$$

This is identically zero in special cases—for example, if $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are in the same energy eigenspace—because of the cancellations brought about by the integration over angles. For P_0 to be self-adjoint, however, $D(P_0)$ must be defined by the condition that, if $\tilde{\psi} \in D(P_0)$ then u' must be a fixed real multiple of u—the same for all $\tilde{\psi}$.

A complete set of physical basis vectors that span $D(P_0)$ is given by

$$\psi^{(\omega)}_{EJM}(y) = e^{i\omega\beta}\psi^N_{EJM}(y) \tag{5.14}$$

where ω is a fixed real number. If $\omega = \pm\,\pi/2$, then these states are states of fixed chirality; in that case only do the physical states carry a (unitary, irreducible) representation of the conformal group.

One implication of this result is that, if neutrinos of both chiralities exist, then the states must be distinguished by a superselection rule; that is, by an exactly conserved quantum number (the muonic number).

VI. SPIN-1 FIELDS

Let $A_\alpha(y)$, $\alpha = 0, 1, 2, 3, 5$, be a vector field. The action of the infinitesimal generators $L_{\alpha\beta}$ of SO(3, 2) on this field is given by

$$(L_{\alpha\beta}A)_\sigma = i(y_\alpha\partial_\beta - y_\beta\partial_\alpha)A_\sigma + i\delta_{\alpha\sigma}A_\beta - i\delta_{\beta\sigma}A_\alpha\,. \tag{6.1}$$

Let $M_{\alpha\beta}$ and $\Sigma_{\alpha\beta}$ be the differential and constant parts of $L_{\alpha\beta}$ and note the formulas

$$M_{\alpha\beta}M_{\beta\gamma} = y_\alpha\partial^2 y_\beta + \partial_\alpha y^2\partial_\beta - y_\alpha(\hat{N}+7)\partial_\beta - \partial_\alpha(\hat{N}-2)y_\beta - \delta_{\alpha\beta}\,, \tag{6.2}$$

$$\tfrac{1}{2}M_{\alpha\beta}M_{\alpha\beta} = \hat{N}(\hat{N}+3) - y^2\partial^2, \quad \hat{N} \equiv y_\alpha\partial_\alpha \tag{6.3}$$

$$\tfrac{1}{2}\Sigma_{\alpha\beta}\Sigma_{\alpha\beta} = 4\,, \tag{6.4}$$

$$(M_{\alpha\beta}\Sigma_{\alpha\beta})_{\sigma\tau} \equiv \kappa_{\sigma\tau} = 2\partial_\sigma y_\tau - 2y_\sigma\partial_\tau - 2\delta_{\sigma\tau}\,. \tag{6.5}$$

The Casimir operator is thus

$$Q \equiv \tfrac{1}{2}L_{\alpha\beta}L_{\alpha\beta} = \hat{N}(\hat{N}+3) - y^2\partial^2 + \kappa + 4\,. \tag{6.6}$$

If E_0 is the lowest eigenvalue of L_{05}, then by Eq. (1.1) the value of Q is $(E_0 - 1)(E_0 - 2)$ and the wave equation is

$$[\hat{N}(\hat{N}+3) - y^2\partial^2 + \kappa + 4 - (E_0-1)(E_0-2)]A_\alpha = 0\,. \tag{6.7}$$

Because the equations

$$\partial_\alpha A_\alpha = 0, \quad y_\alpha A_\alpha = 0 \tag{6.8}$$

are invariant, they must be satisfied in an irreducible representation.

Differential subsidiary conditions give rise to great difficulties when one attempts to introduce interactions. The only known remedy is the Fierz-Pauli scheme, which consists of finding a Lagrangian such that both the wave equation and the subsidiary condition are consequences of the variational equations. We write down the most general wave equation

$$\mathcal{L}_{\alpha\beta}A_\beta = 0 \tag{6.9}$$

with the following properties. (1) Equation (6.9) is equivalent to the set (6.7), (6.8). (2) It is a differential equation of second order. This means that $\mathcal{L}_{\alpha\beta}$ is a sum of $Q\delta_{\alpha\beta}$ and the type of terms that occur on the right-hand side of Eq. (6.2), plus a term of the form $y_\alpha y_\beta$, with seven arbitrary, complex coefficients. (3) The Lagrangian

$$\mathcal{L} = \int A_\alpha^* \mathcal{L}_{\alpha\beta}A_\beta(dy) \tag{6.10}$$

must be real.

We find that $\mathcal{L}_{\alpha\beta}$ must be

$$\mathcal{L}_{\alpha\beta} = [(\hat{N}+E_0)(\hat{N}+3+E_0) - y^2\partial^2]\delta_{\alpha\beta} + y_\alpha a y_\beta + y_\alpha\partial^2 y_\beta + \partial_\alpha y^2\partial_\beta - y_\alpha(\hat{N}+4)\partial_\beta - \partial_\alpha(\hat{N}+1)y_\beta\,, \tag{6.11}$$

where a is an arbitrary real constant. Now

$$y_\alpha\mathcal{L}_{\alpha\beta}A_\beta = [a - (E_0-1)(E_0-2)]\,y_\alpha A_\alpha\,,$$

$$\partial_\alpha\mathcal{L}_{\alpha\beta}A_\beta = -(E_0-1)(E_0-2)\partial_\alpha A_\alpha + (\hat{N}+5)a y_\alpha A_\alpha\,.$$

Equation (6.9) thus implies (6.8), *unless*

$(E_0-1)(E_0-2)=0$. The only interesting exceptional case is $E_0=2$, since there is no unitary representation with $E_0=1$. The term $y_\alpha a y_\beta$ turns out to be uninteresting and will henceforth be dropped.

This result is closely analogous to what is well known in flat space: The Fierz-Pauli scheme is applicable to spin-1 fields unless the mass vanishes. The failure of this scheme for electrodynamics is of course fundamental; it is the *sine qua non* of local gauge invariance. The choice $E_0=2$ is the only one that can lead to gauge-invariant electrodynamics in de Sitter space.

Solutions of (6.9) of the form $A_\alpha=\partial_\alpha\Lambda+y_\alpha\Lambda'$ do not exist unless $(E_0-1)(E_0-2)=0$, and in that case they solve (6.9) identically. Hence, Eq. (6.9), if $E_0=2$, is gauge invariant; that is, invariant under the transformation

$$A_\alpha \rightarrow A_\alpha+\partial_\alpha\Lambda+y_\alpha\Lambda' , \tag{6.12}$$

with arbitrary scalar fields Λ and Λ'. The wave equation (6.7) is invariant under (6.12) only if Λ and Λ' are restricted by $Q\Lambda=Q\Lambda'=0$. The choice between the fully gauge invariant (6.9) and the less invariant (6.7) has a well-known exact analog in flat space.

Locally gauge-invariant electrodynamics in de Sitter space is constructed by minimal coupling substitution

$$\partial_\alpha \rightarrow \partial_\alpha - ieA_\alpha \tag{6.13}$$

into the field equations for charged particles. These field equations involve ∂_α only in the combination $y_\alpha\partial_\beta-y_\beta\partial_\alpha$; therefore, it would be more accurate to give the minimal substitution in the form

$$y_\alpha\partial_\beta-y_\beta\partial_\alpha \rightarrow y_\alpha\partial_\beta-y_\beta\partial_\alpha-ie(y_\alpha A_\beta-y_\beta A_\alpha), \tag{6.14}$$

which shows that the component $y_\alpha\Lambda'$ is uncoupled. In the Appendix we show that this theory takes the familiar generally covariant form when it is expressed in terms of intrinsic spacetime variables.

VII. SPIN-1 BASIS

It is not very difficult to write down the most general $A_\alpha(y)$ that is a simultaneous eigenvector of L_{05}, $\vec{L}^2$, and L_{12}, besides satisfying the wave equation (6.7) and the subsidiary conditions (6.8). The weight diagram, including the correct multiplicities, can be obtained by inspection of the character of the representation, or very simply by reduction of the product representation:

$$D_5\otimes D(E_0,0)=D(E_0,1)\oplus D(E_0+1,0)\oplus D(E_0-1,0). \tag{7.1}$$

Here D_5 is the irreducible five-dimensional representation. Using the known weight diagram for $D(E_0,0)$, we easily calculate that of $D(E_0,1)$—see Fig. 1 and Fig. 3.

We take basis functions that are homogeneous of degree $-E_0$ in y_α; thus

$$\partial_\alpha A_\alpha=y_\alpha A_\alpha=(\hat{N}+E_0)A_\alpha=0. \tag{7.2}$$

Then κ reduces to -2 and the wave equation to $\partial^2 A_\alpha=0$.

The ground state is the unique triplet that satisfies (7.2) and

$$(L_{05}-E_0)A^{E_0,1}=0=(iL_{i0}-L_{i5})A^{E_0,1}, \tag{7.3}$$

namely ($\alpha=0,1,2,3,5$; $\vec{d}$ is polarization vector)

$$A_\alpha^{E_0,1}(\vec{d})=(y_0+iy_5)^{-E_0}\left(\frac{\vec{d}\cdot\vec{y}}{y_0+iy_5},\vec{d},\frac{i\vec{d}\cdot\vec{y}}{y_0+iy_5}\right). \tag{7.4}$$

Acting with the generators (6.1) on (7.4) we obtain the wave functions for all the states of the weight diagram in Fig. 3. The matrix representation of SO(3, 2) thus induced on the basis vectors is irreducible, and can be made Hermitian by suitable normalization of the basis, provided $E_0>2$. We show explicitly what happens in the limiting case $E_0=2$.

Applying the raising operators $iL_{i0}+L_{i5}$ to (7.4) we obtain the nine states with $E=E_0+1$, $J=0$, 1, and 2:

$$a_i(iL_{i0}+L_{i5})A^{E_0,1}(\vec{d}) = \tfrac{2}{3}\vec{d}\cdot\vec{a}A^{E_0+1,0}+A^{E_0+1,1}+A^{E_0+1,2}, \tag{7.5}$$

where the singlet wave function is

$$A_\alpha^{E_0+1,0}=(y_0+iy_5)^{-E_0-1}\left(\frac{(E_0+1)\vec{y}^2}{y_0+iy_5}+3iy_5,\ (E_0+1)\vec{y},\ \frac{i(E_0+1)\vec{y}^2}{y_0+iy_5}-3iy_0\right). \tag{7.6}$$

Next we calculate

$$d_i(iL_{i0}-L_{i5})A^{E_0+1,0}=-(E_0-2)A^{E_0,1}(\vec{d}) \tag{7.7}$$

which shows, when compared with (7.5), that the representation cannot be made Hermitian if $E_0\le 2$.[15]

When $E_0=2$ the right-hand side of (7.7) vanishes, and this suggests that, when $E_0=2$, the state

$A^{E_0+1,0}$ is the ground state of an invariant subspace. That this is indeed the case is shown by the fact that, when $E_0 \Rightarrow 2$, the wave function (7.6) reduces to

$$A_\alpha^{E_0+1,0} \Rightarrow (\partial_\alpha y^2 + y_\alpha)(y_0 + iy_5)^{-3} \tag{7.8}$$

which belongs to an invariant subspace of "gauge fields" of the form

$$A_\alpha = (\partial_\alpha y^2 + y_\alpha)\Lambda \quad . \tag{7.9}$$

It is clear that this subspace carries the representation $D(3, 0)$, so that the basis states that span it have the weights of that representation. These basis states are indicated by crosses in Fig. 3. Let W^0 denote this invariant subspace.

Let W be the space spanned by all the basis states, then a unitary irreducible representation is induced in the quotient space W/W^0. This is the representation that must be associated with photons. We denote it $D'(2, 1)$ to distinguish it from the limit of the family $D(E_0, 1)$; thus

$$\lim_{E_0 \to 2} D(E_0, 1) = D'(2, 1) \oplus D(3, 0) . \tag{7.10}$$

The weight diagram for $D'(2, 1)$ is given by Fig. 3 when the crosses are ignored. All the weights are simple and there are no states with $J = 0$.

Note that the electromagnetic field carries an irreducible representation of the de Sitter group in a somewhat abstract manner. The generators of $D'(2, 1)$ are not precisely the differential operators (6.1). Instead, they are given by (6.1) with the additional instruction that states belonging to W^0 are to be ignored whenever they appear. Expressing this fact somewhat elliptically we may say that the electromagnetic field is not strictly a vector field. Once again, this conclusion is in close accord with the situation in flat space.

As regards questions of completeness and the identification of the physical states, there is a further complication that we have neglected to mention. When $E_0 = 2$ there is a second set of solutions of the wave equation (6.9), and on them, a second copy of $D'(2.1)$. This is the subject of the next section.

VIII. SPIN 1. CONFORMAL INVARIANCE

The generators $L_{\alpha 6}$ that with (6.1) complete the conformal algebra are (see the Appendix)

$$\begin{aligned} L_{6\alpha} &= \rho^{1/2}(y_\beta L_{\beta\alpha} - iy_\alpha) \\ &= i\rho^{-1/2}\,\partial_\alpha - i\rho^{1/2}\hat{N}y_\alpha + \Sigma_\alpha , \\ \Sigma_\alpha A_\sigma &= i\rho^{1/2}(y_\sigma A_\alpha - \delta_{\alpha\sigma} y_\beta A_\beta) . \end{aligned} \tag{8.1}$$

These operators take a solution of the fully gauge-invariant wave equation (6.9) into another solution; hence, (6.9) is invariant under infinitesimal conformal transformations. Furthermore, the SO(3, 2)-invariant subspace of fields of the form $y_\alpha\Lambda + \partial_\alpha\Lambda'$ remains invariant under conformal transformations as well; therefore, terms of this form may be ignored in the following calculations.

Consider the ground state (7.4); the compact generators L_{i6}, $i = 1, 2, 3$, have the following effect when $E_0 = 2$:

$$a_i L_{6i} A^{E_0,1}(\vec{d}) = i\rho^{-1/2}(y_0 + iy_5)^{-2}\left(\frac{\rho a\cdot y d\cdot y - a\cdot d}{y_0 + iy_5},\ \rho a\cdot y\vec{d},\ i\,\frac{\rho a\cdot y d\cdot y - a\cdot d}{y_0 + iy_5}\right) .$$

Adding a gauge term, namely,

$$\partial_\alpha\left(i\rho^{-1/2}\,\frac{\rho a\cdot y d\cdot y - a\cdot d}{2(y_0 + iy_5)^2}\right),$$

we reduce this to

$$a_i L_{6i} A^{E_0,1}(\vec{d}) = \frac{i\rho^{1/2}}{2(y_0 + iy_5)^{-2}}\,(0, (\vec{a}\times\vec{d})\times\vec{y}, 0) . \tag{8.2}$$

Also, the lowering operator gives

$$(L_{60} + iL_{65})A^{E_0,1}(\vec{d}) = \partial_\alpha\left(\frac{-i\rho^{1/2}\,y\cdot d}{y_0 + iy_5}\right) \approx 0, \tag{8.3}$$

while $L_{60} + iL_{65}$ applied to the state (8.2) gives zero. The meaning of these results is as follows.

The six states given by (7.4) and (8.2) carry a representation of the compact subalgebra SO(4) = SU(2)⊗SU(2) of the conformal algebra; namely, in conventional notation, the representation $D(1, 0) \oplus D(0, 1)$ of SO(4). These states are the ground states of a fully reducible (into two irreducible components) representation of the conformal algebra. Suitable normalization of the basis makes this representation Hermitian and integrable in the l^2 sense. An irreducible representation of the conformal group is obtained by taking for ground states three suitable linear combinations of (7.4) and (8.2); these are close analogs of the chiral projections associated with spin-$\frac{1}{2}$ fields.

Let A_α^{EJM}, $'A_\alpha^{EJM}$ denote the basis vectors of the pair of SO(3, 2) representations having as ground states (7.4), (8.2), respectively. We seek an analog of the chiral operator β of the spin-$\frac{1}{2}$ theory, defined by

$$'A_\alpha^{EJM}=\beta A_\alpha^{EJM}, \quad \beta^2=1\,. \tag{8.4}$$

Since no local operator with these properties exist we try to relate the corresponding field strengths. Define

$$F_{\alpha\beta}\equiv\partial_\alpha A_\beta-\partial_\beta A_\alpha\,, \tag{8.5}$$

$$F^\perp_{\alpha\beta}=F_{\alpha\beta}-\rho y_\alpha y_\gamma F_{\gamma\beta}-\rho y_\beta y_\gamma F_{\alpha\gamma}\,. \tag{8.6}$$

Then the field equation (6.9) reduces to

$$\partial_\alpha F^\perp_{\alpha\beta}=0\,. \tag{8.7}$$

Evidently, another solution is given by

$$'F_{\alpha\beta}=\tfrac{1}{2}\rho^{1/2}\epsilon_{\alpha\beta\gamma\sigma\tau}y_\gamma F_{\sigma\tau}\,. \tag{8.8}$$

Hence, if A_α is a solution of the wave equation, then another solution is given up to a gradient by

$$'A_\alpha=\tfrac{1}{2}\rho^{1/2}\epsilon_{\alpha\beta\gamma\sigma\tau}\int^y y'_\beta\partial_\gamma A_\sigma(y')dy'_\tau\,. \tag{8.9}$$

When we apply this transformation to (7.4) we obtain (8.2); hence, (8.9) defines the operator β:

$$'A_\alpha=\beta A_\alpha\,. \tag{8.10}$$

The conformal transformations commute with β, and the chiral projections $(1\pm\beta)/2$ give two conjugate[14] irreducible representations of the conformal group.

In order to decide whether the conformal group is implemented on the physical states, we repeat the analysis already applied to spin-0 and to spin-$\frac{1}{2}$ fields. It should be noted that our procedure cannot be fully justified in this case because of the problems of indefinite metric associated with gauge invariance.

Since we are concerned with matrix elements between states that are represented by wave functions that satisfy the subsidiary conditions (6.8), the wave operator (6.11) can be simplified to

$$\mathfrak{L}_{\alpha\beta}\to[(\hat N+1)(\hat N+2)-\rho^{-1}\partial^2]\,\delta_{\alpha\beta}\,. \tag{8.11}$$

In order to reduce the time dependence of A_α^{EJM} to a simple exponential factor we define

$$\tilde A_\alpha\equiv(A_0\cos\rho^{1/2}t+A_5\sin\rho^{1/2}t,\vec{A},$$
$$A_5\cos\rho^{1/2}t-A_0\sin\rho^{1/2}t)\,. \tag{8.12}$$

This transformation is analogous to (4.14). The analog of (3.8) and (5.7) is

$$\int[(H\tilde A_1)^\dagger_\alpha\tilde A_{2\alpha}-\tilde A^\dagger_{1\alpha}(H\tilde A_2)_\alpha]\,d^3y\,, \tag{8.13}$$

where $H_{\alpha\beta}$ is the ∂_t-independent part of the wave operator for $\tilde A_\alpha$. It is evident that the only relevant part of H is, as in the case of spin 0,

$$r^{-2}\partial_r r^4\partial_r\,. \tag{8.14}$$

For $\tilde A_{1\alpha}$ and $\tilde A_{2\alpha}$ we take linear combinations of $\tilde A_\alpha^{EJM}$ and $'\tilde A_\alpha^{EJM}$. Asymptotically

$$\tilde A_\alpha^{EJM}\propto r^{-2}e_\alpha, \quad '\tilde A_\alpha^{EJM}\propto r^{-1}f_\alpha\,, \tag{8.15}$$

where e_α and f_α depend on t and on the angles. Take, for $i=1,2$,

$$\tilde A_{i\alpha}=r^{-1}f_{i\alpha}+r^{-2}e_{i\alpha}+\cdots\,. \tag{8.16}$$

Inserting (8.6) and (8.14) into (8.13), we get, following precisely the procedure of the two previous cases, that the vanishing of (8.13) leads to

$$\int(f^*_{1\alpha}e_{2\alpha}-e^*_{1\alpha}f_{2\alpha})d\Omega=0\,. \tag{8.17}$$

Therefore, to make P_0 self-adjoint the domain $D(P_0)$ must be characterized by $e_\alpha=cf_\alpha$, where c is a fixed real number.

The conclusions regarding implementability of the conformal group on the physical states are thus precisely the same in electrodynamics as in neutrino theory. Nature, however, has elected quite different options in each case.

As far as I know, a complete Lagrangian field theory of photons interacting with both electric and magnetic charges does not exist. Let us therefore define a monopole as follows: Suppose that $F^0_{\alpha\beta}$ is the electromagnetic field for a state in which the only source is an electric charge, and let $'F^0_{\alpha\beta}$ be the dual of $F^0_{\alpha\beta}$. Then, if a state exists for which $'F^0_{\alpha\beta}$ is the electromagnetic field, its source will be called a magnetic charge or monopole. Now our results show that, if $F^0_{\alpha\beta}$ describes a state in $D(P_0)$, then $'F^0_{\alpha\beta}$ does not, hence magnetic charges cannot coexist with electric charges.

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APPENDIX

The conformal group is the invariance group of the light cone. It has the same structure in de Sitter space as in flat space; namely, it is the twofold covering group of SO(4, 2). Let z_A be the six real coordinates of pseudo-Euclidean space with the metric given by

$$g_{AB}z_Az_B\equiv z_A{}^2\equiv z_0{}^2-z_1{}^2-z_2{}^2-z_3{}^2+z_5{}^2-z_6{}^2\,; \tag{A1}$$

then SO(4, 2) is the pseudo-orthogonal linear group of transformations $z\to\Lambda z$ that leave $z_A{}^2$ invariant. Consider the cone $z_A{}^2=0$ and define

$$x_\mu=\frac{z_\mu}{z_5+z_6}\,,\quad \mu=0,1,2,3\,. \tag{A2}$$

The transformations $z\to\Lambda z$ induce the conformal group of transformations on the four-dimensional Minkowski space with coordinates x_μ and the usual metric. Similarly, if we put

$$y_\alpha=\rho^{1/2}\frac{z_\alpha}{z_6}\,,\quad \alpha=0,1,2,3,5 \tag{A3}$$

then the transformations $z\to\Lambda z$ induce the conformal group of transformations on the four-dimensional de Sitter space with coordinates y_α, $y_\alpha{}^2=1/\rho$.

Let $f(z)$ be a function defined on the cone $z_A{}^2=0$; then the generators of $f(z)\to f(\Lambda^{-1}z)$ are the set of fifteen differential operators

$$L_{AB}=i(z_A\partial_B-z_B\partial_A)\,. \tag{A4}$$

Suppose $f(z)$ is homogeneous of degree l (the degree of homogeneity is a conformal invariant), and define

$$\phi(y)=z_6{}^{-l}f(z)\,. \tag{A5}$$

Then the action of $L_{6\alpha}$ on $\phi(y)$ is given by

$$L_{6\alpha}=\rho^{1/2}(y_\beta L_{\beta\alpha}+ily_\alpha)\,. \tag{A6}$$

The most convenient way of including spin is to take (A6) as the general definition of $L_{6\alpha}$; the commutation relations for SO(4, 2) are satisfied by L_{AB} provided only that $L_{\alpha\beta}$ satisfy those of SO(3, 2). The number l is called the conformal degree of the representation.

The special representations encountered in our work have very peculiar algebraic properties. From (A6) there follows by direct computation

$$i\rho^{1/2}[L_{6\alpha},Q]=y_\beta\{L_{\beta A},L_{\alpha A}\}-2l(l+2)y_\alpha\,, \tag{A7}$$

where $Q=\frac{1}{2}L_{\alpha\beta}L_{\alpha\beta}$ is the SO(3, 2) Casimir operator. The relationship between Q and the wave operator suggests that, in a conformally invariant theory, the left-hand side vanishes, which would imply that

$$\{L_{BA},L_{CA}\}=2l(l+2)\delta_{BC}\,. \tag{A8}$$

Putting $B=C=6$ we get

$$L_{6\alpha}L_{6\alpha}=-2l(l+2) \tag{A9}$$

and then, putting $B=C=\beta$ and summing:

$$Q=2l(l+2)\,. \tag{A10}$$

On the other hand, one may easily verify that the lowering operator $L_{60}+iL_{65}$ annihilates the ground state only if $l=-E_0$. This additional constraint, together with Eq. (1.1), finally gives

$$l=-E_0=-s-1\,. \tag{A11}$$

These conjectures are all verified in the cases $s=0,\frac{1}{2},1$.[16] In the case of electrodynamics, however, the formulas hold for the operators that act on $F_{\alpha\beta}$ rather than A_α. The conformal degree of A_α is -1, as indicated by Eq. (8.1). This can be shown directly, either from the invariance of (6.9) or from the invariance of the subspace of gauge fields.

In Minkowski space, (A5) is replaced by

$$\hat\phi(x)=(z_5+z_6)^{-l}f(z)\,. \tag{A12}$$

Thus, for scalar fields:

$$\hat\phi(x)=[2/(1+x^2)]\,\phi(y)\,. \tag{A13}$$

Also

$$(dy)=[2/(1+x^2)]^4d^4x\,, \tag{A14}$$

$$Q+2=-[2/(1+x^2)]^{-1}(\partial/\partial x^\mu)^2[2/(1+x^2)]^{-1}\,, \tag{A15}$$

so that the Lagrangian, in the case of conformal invariance ($E_0=1$), reduces to that of the massless Klein-Gordon equation in Minkowski space.

However, the mapping between de Sitter space and Minkowski space is singular, and the causal structure is not preserved. In fact,

$$\rho(y_\alpha-y'_\alpha)^2=[2/(1+x^2)][2/(1+x'^2)]\,(x_\mu-x'_\mu)^2\,. \tag{A16}$$

The separation between y and y' is spacelike, lightlike, or timelike in the de Sitter group invariant sense if the left-hand side is negative, zero, or positive. We see that this notion is not Poincaré invariant. In particular, a three-dimensional surface that is de Sitter spacelike is not Minkowski spacelike. This means that the definition of the quantum-mechanical metric is different for the two interpretations, so that the results obtained in de Sitter space do not automatically apply to flat space.

Intrinsic coordinates for de Sitter space will be denoted x^μ, $\mu=0,1,2,3$. It is convenient to take $x^\mu(y)$ to be homogeneous of degree zero in y_α. When dealing with intrinsic coordinates for the curved manifold it is necessary to distinguish between covariant and contravariant indices. Define

$$y_{\alpha\mu} \equiv \partial y_\alpha / \partial x^\mu, \quad x_\alpha^\mu \equiv \partial x^\mu / \partial y_\alpha \, ,$$

then

$$y_\alpha y_{\alpha\mu} = y_\alpha x_\alpha^\mu = 0 \, ,$$

$$x_\alpha^\mu y_{\alpha\nu} = \delta_\nu^\mu, \quad x_\alpha^\mu x_\alpha^\nu = g^{\mu\nu}, \quad y_{\alpha\mu} y_{\alpha\nu} = g_{\mu\nu} \, ,$$

$$(-g)^{-1/2} \partial_\mu (-g)^{1/2} = x_\alpha^\nu \partial_\mu y_{\alpha\nu} = - y_{\alpha\nu} \partial_\mu x_\alpha^\nu \, .$$

Define $F_{\alpha\beta}$ as in Eq. (8.5), and

$$F^{\mu\nu} \equiv x_\alpha^\mu x_\beta^\nu F_{\alpha\beta}, \quad A^\mu \equiv x_\alpha^\mu A_\alpha \, .$$

Then $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the field equation (6.9) reads

$$\partial_\mu (-g)^{1/2} F^{\mu\nu} + \rho(E_0 - 1)(E_0 - 2)(-g)^{1/2} A^\nu = 0 \, .$$

The substitution (6.14) takes the form $\partial_\mu \to \partial_\mu - ieA_\mu$.

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[1]See, for example, R. Penrose, Proc. R. Soc. (London) A284, 159 (1965).

[2]See also the first paper in this series, C. Fronsdal, Rev. Mod. Phys. 37, 221 (1965).

[3]A detailed description of global de Sitter space is given in the second paper in the series, C. Fronsdal, Phys. Rev. D 10, 589 (1974).

[4]See Ref. 2 for a general discussion.

[5]The limit $\rho \to 0$, P_μ finite, is a Wigner-Inonu contraction. It is a very straightforward matter for SO(3, 2), though rather complicated for SO(4, 1), see C. Martin, Ann. Inst. H. Poincaré 20, 373 (1974).

[6]P. A. M. Dirac, Ann. Math. 36, 657 (1935).

[7]S. N. Gupta, Phys. Rev. 96, 1683 (1954).

[8]In all cases "negative-energy" solutions corresponding to charge-conjugate states exist, though they will not be mentioned again.

[9]Details in Ref. 3.

[10]See, e.g., L. Castell, Nucl. Phys. B4, 343 (1967); G. Mack and Abdus Salam, Ann. Phys. (N.Y.) 53, 174 (1969); M. Flato, J. Simon, and D. Sternheimer, Ann. Phys. (N. Y.) 61, 78 (1970). Also see *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Colorado Associated University Press, Boulder, Colorado, 1971), Vol XIII.

[11]The invariant causal structure was examined in detail in Ref. 3.

[12]Equation (4.5) was proposed by F. Gürsey and T. D. Lee, Proc. Natl. Acad. Sci. USA 49, 179 (1963). In addition, they required that ψ be "independent of y^2." Taking this to mean that $\hat{N} = 0$ we get $E_0 = -\frac{1}{2}$ and $E_0' = \frac{7}{2}$. In this case the operator β (see below) does not commute with the Hamiltonian and the chiral projections $\frac{1}{2}(1 \pm \beta)$ are not defined.

[13]C. Fronsdal and R. B. Haugen, preceding paper, Phys. Rev. D 12, 3810 (1975) (the third paper in this series).

[14]The conjugation meant here is the permutation of the factors of the compact subgroup SU(2)⊗ SU(2).

[15]This construction has not been carried out except for the lowest-lying states. [For spin 0 and spin $\frac{1}{2}$ it was completed in full detail in Refs. 3 and 13.] It is easy to show (Ref. 2) that $E_0 > s$ is a necessary condition for the Hermiticity of $D(E_0, s)$. It is known (Harish-Chandra) that a sufficient condition is $E_0 > s + 2$. When $s = 0$ and $\frac{1}{2}$ our explicit calculations (Refs. 3, 13) show that the necessary and sufficient conditions are $E_0 \geq \frac{1}{2}$ and $E_0 > 1$, respectively. The limit is reached when certain matrix elements change sign, and extensive experience with this type of phenomena (including the two cases just cited) shows that the sign change always occurs along the border of the weight diagram. That is why our investigation justifies our assertion that the necessary and sufficient condition for Hermiticity when $s = 1$ is $E_0 \geq 2$, although a rigorous proof of sufficiency has not been given.

[16]The representations associated with $s = 0$ and $\frac{1}{2}$ are familiar from the group-theoretical treatment of the Coulomb and Kepler problems.