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Elementary particles in a curved space. III

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Irreducible representations of the 3 + 2 de Sitter group are associated with free Dirac fields in a space of constant, positive curvature.

I. INTRODUCTION

Our program is to transfer conventional physical theory to the arena of a space of constant positive curvature. The role ordinarily played by the Poincaré group is here assumed by the de Sitter group SO(3, 2). Irreducible representations of this group, suitable for describing massive elementary particles, were found in I.[1] Details for the case of spinless fields were given in II[2]; here we present analogous results for spin $\frac{1}{2}$. Previous work in this field[3] has mostly concentrated on the case of negative curvature—in this case the group is SO(4, 1)—in spite of the well-known difficulty of an energy spectrum with no lower bound.

Our approach to quantized free Dirac fields (in Sec. IV) is quite different from that of previous work. It is based on a Fock space construction over the discrete one-particle eigenstates of energy and angular momentum that span an irreducible representation.

II. WAVE FUNCTIONS IN MOMENTUM SPACE

A. General considerations

The de Sitter group is the connected part of the group of pseudoorthogonal, real, five-dimensional matrices. The metric is $+---+$. If $L_\alpha{}^\beta$ is the matrix

$$(L_\alpha{}^\beta)_\gamma{}^\delta = \delta_\gamma{}^\beta \delta_\alpha{}^\delta ,$$

then a parametrization of Λ is given by the six independent components of the real, antisymmetric tensor $\theta_{\alpha\beta}$ through the formula

$$\Lambda(\theta) = \exp(\theta_{\alpha\beta} L^{\alpha\beta}) .$$

The simplest realization of the group action is in terms of functions defined over the real cone C:

$$b_\alpha b^\alpha \equiv b_0{}^2 - \vec{b}^2 + b_5{}^2 = 0 , \qquad \vec{b}^2 \equiv b_1{}^2 + b_2{}^2 + b_3{}^3 \tag{2.1}$$

in a five-dimensional pseudo-Euclidean space of signature $+---+$. While this is an adequate setting for some true representations of SO(3, 2) it is not suitable for the more general case of representations up to a factor (representations of the universal covering group) that we wish to include. Let C^* be the cone (2.1) with the origin $b_\alpha = 0$ removed, then the appropriate carrier space is the universal covering $\overline{C}^*$ of C^*; this is isomorphic to $C^* \otimes I$, where I is the set of integers. Nevertheless, we indicate functions on $\overline{C}^*$ as, e.g., $g(b)$, as if they were functions on C. Their true nature will be evident.

A spinor is a set of four functions $g_a(b)$, $a = 1, 2, 3, 4$. The de Sitter group acts on the set of spinors as follows:

$$T(\Lambda):\ g_a(b) \to [D(\Lambda)g]_a(b) = S(\Lambda)_{aa'} g_{a'}(\Lambda^{-1}b) . \tag{2.2}$$

The matrices $S(\Lambda)$ are the four dimensional realization of SO(3, 2) given by

$$S(\Lambda) = \exp[\tfrac{1}{2} i \theta_{\alpha\beta} \Sigma^{\alpha\beta}] , \tag{2.3}$$

where

$$\Sigma^{\alpha\beta} = -\Sigma^{\beta\alpha} = \tfrac{1}{2} i \gamma^\alpha \gamma^\beta , \quad \alpha < \beta \tag{2.4}$$

and

$$\gamma^\alpha = (\gamma^\mu, i) . \tag{2.5}$$

The γ^μ, $\mu = 0, 1, 2, 3$, are the usual Dirac matrices. From now on the spinor index a will be dropped whenever possible. For the infinitesimal transformations

$$(\Lambda^{-1}b)_\alpha = b_\alpha + \theta_{\alpha\beta} b_\beta , \tag{2.6}$$

the generators are

$$L_{\alpha\beta} = i(b_\alpha \partial_\beta - b_\beta \partial_\alpha) + \Sigma_{\alpha\beta} . \tag{2.7}$$

The basis functions will be taken to be homogeneous in b_α. Since the operator $b^\alpha \partial_\alpha$ commutes with all the generators, the degree of homogenity must be fixed in an irreducible representation:

$$b^\alpha \partial_\alpha g(b) = N g(b) . \tag{2.8}$$

The Casimir operator also reduces to a multiple of the identity

$$\tfrac{1}{2}L_{\alpha\beta}L^{\alpha\beta}g(b)=Qg(b)\,. \tag{2.9}$$

Now

$$\tfrac{1}{2}L_{\alpha\beta}L^{\alpha\beta}=N(N+3)+\tfrac{5}{2}+2i\Sigma^{\alpha\beta}b_\alpha\partial_\beta\,; \tag{2.10}$$

so we have

$$2i\Sigma^{\alpha\beta}b_\alpha\partial_\beta g(b)=\kappa g(b)\,, \tag{2.11}$$

where κ is likewise fixed. The value of κ is related to the value of N; by direct calculations we find that

$$\kappa(\kappa+3)=N(N+3)\,. \tag{2.12}$$

Equation (2.9) has the character of a wave equation: It helps to project out an irreducible representation. Simpler, though essentially equivalent, wave equations can be found, however. The operator $\gamma^\alpha b_\alpha$ is not invariant, but its null space is invariant. It is easy to verify that the equation

$$\gamma^\alpha b_\alpha g(b)=0 \tag{2.13}$$

is equivalent to (2.11) with $\kappa=-N-3$. To discuss the invariant differential equation

$$\gamma^\alpha \partial_\alpha g(b)=0 \tag{2.14}$$

we must first define an extension of $g(b)$ off the cone. The extension is obviously not unique. [In particular, if $g(b)$ is an extension, then so is $g(b)+cb^\alpha b_\alpha$, where c is a constant spinor.] Roughly, the existence of an extension that satisfies (2.14) is equivalent to (2.11) with $\kappa=N$.

Finally, κ and N are related to the lowest value $E_0\equiv\min(E)$ of the eigenvalue E of the operator L_{05}. The Casimir operator can be written

$$Q=\vec{J}^2+L_{05}(L_{05}-3)+(iL_{i0}+L_{i5})(iL_{i0}-L_{i5})$$
$$\left(\vec{J}^2\equiv\sum_{i<j}L_{ij}{}^2=\text{total angular momentum}\right). \tag{2.15}$$

The lowest eigenspace is annihilated by the lowering operator $iL_{i0}-L_{i5}$, while $E=E_0$ and $\vec{J}^2=\frac{3}{4}$; hence Q reduces to

$$Q=E_0(E_0-3)+\tfrac{3}{4}\,. \tag{2.16}$$

Comparison with (2.10)–(2.12) yields four distinct possibilities:

$$\kappa=N,\quad E_0=-N-\tfrac{1}{2}\ \ (\text{or } E_0=N+\tfrac{7}{2}), \tag{2.17}$$

$$\kappa=-N-3,\quad E_0=N+\tfrac{5}{2}\ \ (\text{or } E_0=-N+\tfrac{1}{2})\,. \tag{2.18}$$

The values enclosed in parentheses will be less important. Recall that (2.18) is associated with the wave equation $\gamma^\alpha b_\alpha g(b)=0$, while (2.17) can be related to $\gamma^\alpha\partial_\alpha g(b)=0$.

B. Basis vectors for the case $\gamma^\alpha b_\alpha=0$

Any solution of Eq. (2.13) that is also an eigenfunction of L_{05}, L_{12}, and $\vec{J}^2$, with eigenvalues E, M, and $J(J+1)$, is a linear combination of two spinors of the form

$$g^\pm_{EJM}(b)\equiv b^{N-1}\left(\frac{b_0-ib_5}{b_0+ib_5}\right)^{(E-1/2)/2}\tilde{\gamma}^\alpha b_\alpha \mathcal{Y}^\pm_{JM}(\hat{b})\,, \tag{2.19}$$

where $\hat{b}$ is the direction of $\vec{b}$ and

$$\tilde{\gamma}^\alpha=(\gamma^\mu,-i)\,. \tag{2.20}$$

The factor $\tilde{\gamma}^\alpha b_\alpha$ takes care of (2.13). Since the rank of this matrix is 2, there is some ambiguity in the choice of the $\mathcal{Y}$'s. We have taken

$$\mathcal{Y}^+_{JM}=\begin{bmatrix}\left(\dfrac{J+M}{2J}\right)^{1/2}Y_{J-1/2,M-1/2}\\ \left(\dfrac{J-M}{2J}\right)^{1/2}Y_{J-1/2,M+1/2}\\ 0\\ 0\end{bmatrix},$$
$$\mathcal{Y}^-_{JM}=\begin{bmatrix}-\left(\dfrac{J+1-M}{2J+2}\right)^{1/2}Y_{J+1/2,M-1/2}\\ \left(\dfrac{J+1+M}{2J+2}\right)^{1/2}Y_{J+1/2,M+1/2}\\ 0\\ 0\end{bmatrix}. \tag{2.21}$$

Next, we want to determine the range of the indices, and normalization coefficients $(E,J)^\pm$, that make

$$E^\pm_{EJM}(b)\equiv(E,J)^\pm g^\pm_{EJM}(b) \tag{2.22}$$

the basis for a unitary matrix representation of the group. The coefficients may be found by the requirement that $L_{i0}\pm iL_{i5}$ be mutually Hermitian adjoint. Because the $\mathcal{Y}^\pm_{JM}$ already form a normalized basis for the rotation subgroup, the coefficients are independent of M and it is sufficient to consider the case $M=\frac{1}{2}$, $i=3$. After simple, but lengthy calculations one finds

$$(iL_{30}\pm L_{35})g^{+}_{EJ1/2}=\frac{(2J+1)^{1/2}(2J+3)^{1/2}}{4J+4}\left\{\begin{matrix}N-E-J\\ E+N-J-1\end{matrix}\right\}g^{+}_{E\pm1,J+1,1/2}$$

$$-\frac{1}{4J(J+1)}\left\{\begin{matrix}N-E-J\\ E+N+J+1\end{matrix}\right\}g^{-}_{E\pm1,J,1/2}+\frac{(2J-1)^{1/2}(2J+1)^{1/2}}{4J}\left\{\begin{matrix}N-E+J\\ E+N+J+1\end{matrix}\right\}g^{+}_{E\pm1,J-1,1/2}\,, \tag{2.23a}$$

$$(iL_{30}\pm L_{35})g^{-}_{EJ1/2}=\frac{(2J+1)^{1/2}(2J+3)^{1/2}}{4J+4}\left\{\begin{matrix}N-E-J-1\\ E+N-J\end{matrix}\right\}g^{-}_{E\pm1,J+1,1/2}$$

$$-\frac{1}{4J(J+1)}\left\{\begin{matrix}N-E+J+1\\ E+N-J\end{matrix}\right\}g^{+}_{E\pm1,J,1/2}+\frac{(2J+1)^{1/2}(2J-1)^{1/2}}{4J}\left\{\begin{matrix}N-E+J+1\\ E+N+J\end{matrix}\right\}g^{-}_{E\pm1,J-1,1/2}\,. \tag{2.23b}$$

Inspection of (2.23) reveals the existence of several invariant subspaces. Of course, J and M are restricted to half-odd-integer values, and the fractional part of E is an invariant. Hermiticity of L_{05} requires E to be real, but its fractional part is otherwise arbitrary for a representation of the universal covering group of SO(3, 2). In practice we fix this number by fixing $E_0=\min(E)$.

Equations (2.23) show that g^{+}_{EJM} and $g^{+}_{E'J'M'}$ belong to the same irreducible representation only if $E-E'$ is even, while g^{+}_{EJM} and $g^{-}_{E'J'M'}$ are connected only if $E-E'$ is odd. This state of affairs is illustrated in Figs. 1 and 2.

Equations (2.23) show that a finer decomposition, based on the sign of $E+J$ or $E-J$, is possible for special values of N. Since we are interested in representations in which E is bounded below we discuss only two cases.

Let k be the integer

$$k=(E-E_0)-(J-\tfrac{1}{2})\,. \tag{2.24}$$

(i) If $N+E-\frac{1}{2}$ is integer, then the subspace

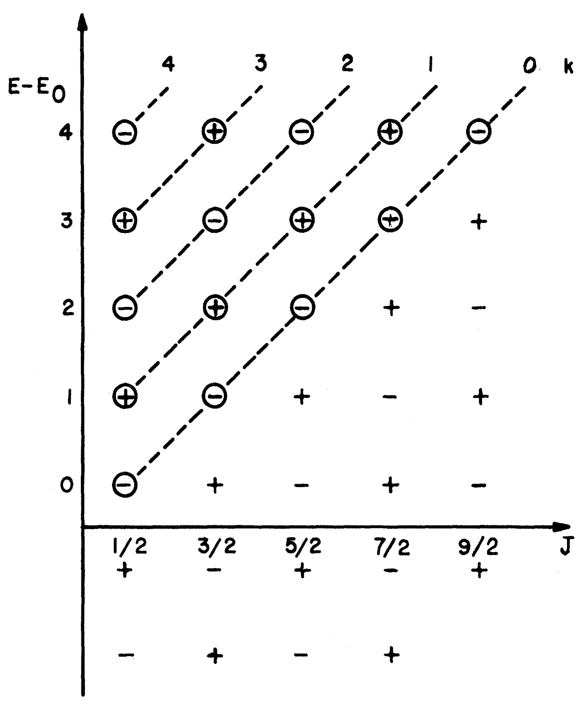

FIG. 1. The weights associated with (2.24) are shown circled. The signs refer to the superscript on the basis functions.

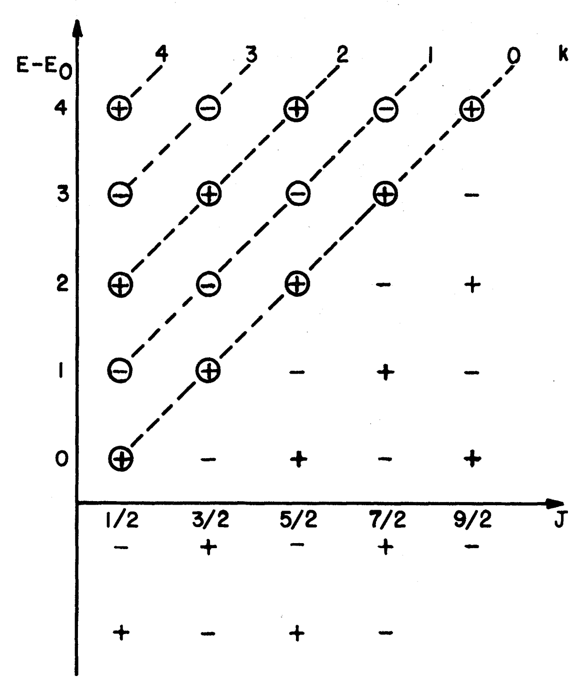

FIG. 2. The weights associated with the subspace W^{+} are shown circled, they are the weights of the representation induced in $(W^{+}\otimes W^{-})/W^{-}$. The remaining weights belong to W^{-}.

spanned by

$$\left.\begin{array}{ll} g^-_{EJM}, & k=0,2,\ldots \\ g^+_{EJM}, & k=1,3,\ldots \end{array}\right\} E_0=-N+\tfrac{1}{2} \tag{2.25}$$

is invariant. It carries an irreducible representation if $N<-\frac{1}{2}$. The weight diagram is shown in Fig. 1. Each circled point represents the $(2J+1)$-dimensional subspace spanned by $g^{\pm}_{EJM}$ for fixed E and J and the sign refers to the superscript.

(ii) If $E-N-\frac{5}{2}$ is integer, consider the subspace W^+ spanned by

$$\left.\begin{array}{ll} g^+_{EJM}, & k=0,2,\ldots \\ g^-_{EJM}, & k=1,3,\ldots \end{array}\right\} E_0=N+\tfrac{5}{2}, \tag{2.26}$$

and the complimentary subspace W^- defined similarly, but with negative values of k. The weight diagrams for both are shown in Fig. 2. The subspace W^- is invariant, and the quotient space $(W^+\oplus W^-)/W^-$ carries an irreducible representation if $N>-\frac{3}{2}$.

As far as algebraic structure is concerned, the above two irreducible representations (for the same value of E_0) are identical. Only the first is realized by differential operators; nevertheless, the second is the one we prefer. The reason is the same as in the case of spin-0 representations and has to do with the possibility of defining the invariant norm by means of an integral, as we shall see. There is a natural bijection between W^+ and $(W^+\oplus W^-)/W^-$ that allows us to define a representation in W^+. This differs from (2.23) in that any basis function that belongs to W^- appearing on the right-hand side is to be ignored. We obtain a representation in W^+, but the action is not given by the differential operators.

Returning now to the question of the coefficients in (2.22), we use (2.23), impose Hermiticity on the matrices that act on $F^{\pm}_{EJM}$, and obtain for $E_0=N+\frac{5}{2}$

$$(E,J)^+=\left[\frac{(E_0-\frac{1}{2})!\,(E_0-2)!(K+J)!K!}{(\frac{1}{2})!\,(K+E_0+J-1)!(K+E_0-2)!}\right]^{1/2}, \tag{2.27a}$$

$$(E,J)^-=\left[\frac{(E_0-\frac{1}{2})!\,(E_0-2)!(K+J+1)!K!}{(\frac{1}{2})!\,(K+E_0+J-1)!(K+E_0-1)!}\right]^{1/2}, \tag{2.27b}$$

where

$$K\equiv\text{integral part of } \tfrac{1}{2}k=0,1,2,\ldots\,. \tag{2.28}$$

According to (2.22) and (2.27), the normalized basis functions $F^{\pm}_{EJM}$ are infinite if $K=-1,-2,\ldots$.

This infinite renormalization of the basis vectors in W_- reduces matrix elements of operators that connect W_+ to W_- to zero, thus automatically accomplishing the required modification of (2.23), as discussed in the preceding paragraph. The functions

$$\left.\begin{array}{ll} F^+_{EJM}(b), & k=0,2,\ldots \\ F^-_{EJM}(b), & k=1,3,\ldots \end{array}\right\} k=(E-E_0)-(J-\tfrac{1}{2}) \tag{2.29}$$

form the basis for an irreducible, and if $E_0>1$, Hermitian, integrable representation of the algebra. The generators are given by differential operators except for the rule that all basis functions belonging to W_-, that appear as a differential operator acts on a function (2.29), is to be erased, or equivalently, by (2.22), (2.23), and (2.27).

As (2.29) shows, the sign-index is redundant and may henceforth be suppressed. The indices EJM will sometimes be indicated collectively by the single letter I.

The Hilbert space in which our representation is integrable consists of the functions

$$F(b)=\sum_I C_I\,F_I(b) \tag{2.30}$$

with norm

$$\sum_I |C_I|^2<\infty\,. \tag{2.31}$$

Next, we shall find a means to express the norm in the form of an integral.

C. Function space for the case $\gamma^\alpha b_\alpha=0$

The following development is possible only for the case $E_0=N+\frac{5}{2}$. The difficulty that is encountered when one tries to treat the case $E_0=\frac{1}{2}-N$ is exactly the same as for spin-0 representations.[4]

The basis functions are given by (2.22) and (2.19) in terms of the isotropic five vector b_α. In order to obtain a compact domain for integration, we define angles θ and ϕ by the direction $\hat{b}$ of $\vec{b}$, and a third angle u by

$$e^{iu}=(b_0+ib_5)/|\vec{b}|\,. \tag{2.32}$$

The homogeneity of $F(b)$ allows these functions to be replaced by functions depending only on the angles

$$F(b)=(2\pi)^{1/2}|\vec{b}|^{-3}(b_0-ib_5)^{E_0+1/2}H(\hat{b},u)\,. \tag{2.33}$$

In particular,

$$H_{EJM}(\hat{b},u)=(2\pi)^{-1/2}(E,J)e^{-iu(E-E_0)}\times(1-e^{iu}\vec{\gamma}\cdot\hat{b})\mathcal{Y}_{JM}(\hat{b}) \tag{2.34}$$

from which it is seen that (2.31) implies that $H(\hat{b},u)$ has an analytic extension into the domain $|e^{iu}|>1$.

It is now straightforward to find an integral kernel K such that

$$(F,F')\equiv\sum_I C_I^* C_I' = \int H^*(\hat{b},u)\gamma_0 K(\hat{b},u;\hat{b}',u') \times H'(\hat{b},u')\,du\,du'\,d\Omega d\Omega' , \qquad (2.35)$$

namely,

$$K(\hat{b},u;\hat{b}',u') = (8\pi^2)^{-1}[e^{2i(u'-u)}+1-2\hat{b}\cdot\hat{b}'e^{i(u'-u)}]^{-E_0-1/2} . \qquad (2.36)$$

This must be understood, for real $u-u'$, as the boundary value of a function analytic in $|e^{i(u'-u)}| <1$.

The special case of (2.35),

$$\delta_{II'} = \int H_I^*(\hat{b},u)\gamma_0 K(\hat{b},u;\hat{b}',u') \times H_{I'}(\hat{b}',u')\,du\,du'\,d\Omega d\Omega' \qquad (2.37)$$

can easily be verified by direct integration.

[The result (2.35)–(2.36) can also be derived as follows. The invariant integral

$$\int F^*(b)(b^\alpha b_\alpha')^{-N-3}F(b')\delta(b'^2)\,d^5b\,d^5b' \qquad (2.38)$$

is only logarithmically divergent when the exponent has the value shown. Changing variables and ignoring the divergent factor $\int db_5 db_5'/b_5 b_5'$ one obtains (2.35) with (2.36). This formal procedure can be made rigorous by smearing over N.]

D. Function space for the case $\gamma^\alpha \partial_\alpha=0$

So far we have investigated functions satisfying (2.13). The alternative, (2.14), may be studied in a similar way, but the results are obtained much more easily as follows. Let a function $h(\hat{b},u)$, "dual" to $H(\hat{b},u)$, be defined by

$$h(\hat{b},u) = \int K(\hat{b},u;\hat{b}',u')H(\hat{b}',u')du'd\Omega' . \qquad (2.39)$$

With this definition the inner product (2.35) takes the simple form

$$(F,F')\equiv(H,H') = \int h^*(\hat{b},u)\gamma_0 H'(\hat{b},u)\,du\,d\Omega . \qquad (2.40)$$

[We may also define a function $f(b)$,

$$f(b) = (2\pi)^{1/2}(b_0+ib_5)^{-E_0-1/2}h(\hat{b},u) . \qquad (2.41)$$

This is dual to $F(b)$ is the sense that it can be expressed formally as

$$f(b)\sim\int(b^\alpha b_\alpha')^{-N-3}F(b')\,\delta(b'^2)d^5b' , \qquad (2.42)$$

so that (2.38) reduces to

$$\int F^*(b)f(b)\delta(b^2)d^5b .$$

Of course, these integrals must be regularized before they make any sense; nevertheless, (2.42) strongly suggests that the functions $f(b)$ satisfy (2.14): $\gamma^\alpha\partial_\alpha F(b)=0$. In fact, (2.42) gives the extension of (2.41) off the cone that one needs to satisfy this wave equation.]

The normalized basis functions $h_{EJM}(b,u)$ are defined by (2.38) and (2.34). Explicit integration gives

$$h_{EJM}(b,u) = (2\pi)^{-1/2}\alpha_{EJ}e^{-iu(E-E_0)} \times(1-\beta_{EJ}\vec{\gamma}\cdot\hat{b}e^{iu})\mathcal{Y}_{JM}(\hat{b}) , \qquad (2.43)$$

with

$$\alpha_{EJ}^+ = \frac{1}{(E,J)^+}\frac{K+E_0-1}{E_0-1} , \quad \alpha_{EJ}^- = \frac{1}{(E,J)^-}\frac{K+J+E_0}{E_0-1} , \qquad (2.44a)$$

$$\beta_{EJ}^+ = \frac{K}{K+E_0-1} , \quad \beta_{EJ}^- = \frac{J+K+1}{J+K+E_0} . \qquad (2.44b)$$

It is easy to verify that

$$\delta_{II'} = \int h_{EJM}^*(\hat{b},u)\gamma_0 H_{E'J'M'}(\hat{b},u)\,du\,d\Omega . \qquad (2.45)$$

For later reference we note that (2.31) guarantees that $h=\sum C_I h_I$ has an analytic extension in the domain $|e^{iu}|>1$.

E. Momentum space

The projection of the cone $b^\alpha b_\alpha=0$ on the "momentum" hyperboloid $p^\mu p_\mu=m^2$ is given by

$$p_\mu = imb_\mu/b_5 , \quad m\equiv\rho^{1/2}E_0 . \qquad (2.46)$$

Since the functions $F(b)$ and $f(b)$ are homogeneous, functions of p_μ are obtainable by setting

$$\Psi(p)\equiv b_5{}^{-E_0+5/2}F(b) , \qquad (2.47a)$$

$$\psi(p)\equiv b_5{}^{E_0+1/2}f(b) , \qquad (2.47b)$$

Because $\gamma^\alpha b_\alpha F=0$,

$$(\gamma^\mu p_\mu - m)\Psi(p) = 0 . \qquad (2.48)$$

We have discussed the equation $\gamma^\alpha\partial_\alpha f(b)$—following (2.14) and following (2.42)—and it is obvious that it is simpler to replace this equation by the invariant equation (2.11) with $\kappa = N = -E_0 - \frac{1}{2}$:

$$2i\Sigma^{\alpha\beta} b_\alpha \partial_\beta f(b) = Nf(b) .$$

This yields a wave equation for $\psi(p)$ that can easily be verified directly:

$$(m\gamma^\mu \partial_\mu + p^\mu \partial_\mu + E_0 + \tfrac{1}{2})\psi(p) = 0 . \qquad (2.49)$$

The action of the generators is given by (2.7) and reduces to

$$L_{\mu\nu} = M_{\mu\nu} + \Sigma_{\mu\nu} , \quad M_{\mu\nu} \equiv i(p_\mu \partial_\nu - p_\nu \partial_\mu) \qquad (2.50)$$

on either $\Psi(p)$ or $\psi(p)$. Acting on $\Psi(p)$,

$$P_\mu \equiv \rho^{1/2} L_{\mu 5} = \left(1 + \frac{1}{2E_0}\right) p_\mu - \tfrac{1}{2}\rho^{1/2}\gamma_\mu + \frac{i}{E_0} M_{\mu\nu} p^\nu$$
$$= p_\mu + \frac{i}{E_0} L_{\mu\nu} p^\nu , \qquad (2.51)$$

while on $\psi(p)$

$$P_\mu = \left(1 + \frac{1}{2E_0}\right) p_\mu - \tfrac{1}{2}\rho^{1/2}\gamma_\mu - \frac{i}{E_0} p^\nu M_{\mu\nu}$$
$$= p_\mu - \frac{i}{E_0} p^\nu L_{\mu\nu} + \frac{1}{2E_0} p_\mu (\gamma^\nu p_\nu - m) . \qquad (2.52)$$

Equation (2.46) signals the need for an analytic continuation from real b_α to p_μ. Nor have we forgotten the complication explained just after Eq. (2.29). We now show that this unpleasantness evaporates on carrying out the continuation to real p_μ.

The problem is to rewrite (2.39) in terms of Ψ and ψ, integrating over real p_μ. The integral (2.39) may be interpreted as a contour integral in the plane of $z = e^{iu}$, the contour being the unit circle. According to the remark just after (2.45), $h^*(\hat{b}, u)$ is the boundary value of a function that is analytic for $|z| < 1$. The functions $H(\hat{b}, u)$ are not analytic there, in fact, (2.34) shows that the basis functions have poles at $z = 0$. We can easily find a set of functions $H(\hat{b}, u)$ that is dense in Hilbert space and that is analytic in $|z| < 1$ except for poles at $z = 0$, but it is sufficient to consider the basis vectors. We are thus led to consider the integral

$$\oint A(z) z^{-n-1} dz ,$$

where $A(z)$ is analytic for $|z| < 1$ and the contour is a simple curve in $|z| < 1$ enclosing the origin. This integral is equal to the real integral

$$\int A(x) \frac{1}{n!} \delta^{(n)}(x) dx .$$

In this way the original integrals, over $|z| = 1$, can be rewritten as integrals over the real axis; that is, over real p_μ, provided the factors z^{-n-1} are replaced by suitable distributions. The result is that the inner product takes the form

$$(\Psi, \Psi') = \int \psi^*(p)\gamma_0 \Psi(p)(dp)_+ ,$$
$$(dp)_+ \equiv d^4p\,\delta(p^2 - m^2)\,\theta(p_0) , \qquad (2.53)$$

while the new basis functions are

$$\Psi_{EJM}(p) = (E, J)\left(\frac{p_0 + m}{2m}\right)^{E_0 + 1/2} \mathfrak{D}^{E - E_0}(p)\mathcal{Y}_{JM}(\hat{p}) , \qquad (2.54)$$

$$\psi_{EJM}(p) = \alpha_{EJ}\left(\frac{2m}{p_0 + m}\right)^{E_0 + 1/2}\left(\frac{p}{p_0 + m}\right)^{E - E_0}$$
$$\times \left[1 - \beta_{EJ}\frac{\vec{\gamma}\cdot\vec{\mathrm{p}}}{p_0 - m}\right]\mathcal{Y}_{JM}(\hat{p}) . \qquad (2.55)$$

The distributions $\mathfrak{D}^{E - E_0}(p)$ are defined by

$$\int A(p)\mathfrak{D}^n(p)\frac{p^2 dp}{2p_0}$$
$$= \frac{1}{n!}\left(\frac{p_0(p_0 + m)}{m}\frac{\partial}{\partial p}\right)^n A(p)\frac{\gamma^\mu p_\mu + m}{p_0 + m}\bigg|_{p=0} . \qquad (2.56)$$

The inner product (2.53) can also be expressed entirely in terms of $\Psi(p)$—but not in terms of $\psi(p)$. Here is a collection of formulas that follow immediately from our earlier results[2] concerning the functions H and h:

$$(\Psi, \Psi') = \int \Psi^*(p)\gamma_0 K(p, p')\Psi(p')(dp)_+(dp')_+ , \qquad (2.57)$$

$$\psi(p) = \int K(p, p')\Psi(p')(dp')_+ , \qquad (2.58)$$

$$K(p, p') = \frac{1}{4\pi}\left(\frac{2m^2}{p^\mu p'_\mu + m^2}\right)^{E_0 + 1/2} . \qquad (2.59)$$

The sum formula

$$\sum_I \psi_I(p)\psi_I^*(p')\gamma_0$$
$$= K(p, p')\left[1 - \frac{1}{4}\frac{2E_0 + 1}{E_0 - 1}\frac{(\gamma p - m)(\gamma p' - m)}{pp' + m^2}\right] \qquad (2.60)$$

is easily obtained from invariance and the fact that only four terms contribute when $\vec{\mathrm{p}}' = 0$.

If $|I\rangle = |EJM\rangle$ are orthonormal basis states, then

$$\sum C_I |I\rangle = \int \Psi(p)|p\rangle (dp)_+$$
$$= \int \psi(p)|p)(dp)_+ , \qquad (2.61)$$

where

$$|p\rangle \equiv \sum \psi_I^*(p)|I\rangle , \quad |p) \equiv \sum_I \Psi_I^*(p)|I\rangle , \qquad (2.62)$$

$$|I\rangle = \int \Psi_I(p)|p\rangle$$
$$= \int \psi_I(p)|p\rangle . \qquad (2.63)$$

All this is very similar to corresponding results for spin 0.

The appearance of distributions marvelously removes the unpleasantness referred to after (2.52). The partial integration that is implied by (2.56) has the effect that the differential operators always end up acting on $\psi(p)$ rather than on $\Psi(p)$, and finally allows the direct identification between generators and differential operators expressed by (2.50)–(2.52).

III. WAVE FUNCTIONS IN CONFIGURATION SPACE

A. General considerations

Configuration space is the covering space of the hyperboloid

$$y^\alpha y_\alpha \equiv y_0^2 - \vec{y}^2 + y_5^2 = \rho^{-1} . \qquad (3.1)$$

We consider spinor functions $\psi_a(y)$ on which the action of the de Sitter group is exactly the same as on the spinor functions $g_a(b)$. The spinor index will be suppressed.

In an irreducible representation the Casimir operator

$$Q = \tfrac{1}{2}L_{\alpha\beta}L^{\alpha\beta}$$
$$= \hat{N}(\hat{N}+3) - \rho^{-1}\partial_\alpha\partial^\alpha + 2i\Sigma^{\alpha\beta}y_\alpha\partial_\beta + \tfrac{5}{2} \qquad (3.2)$$

must be fixed. We have simplified the writing by defining the operator

$$\hat{N} = y^\alpha \partial_\alpha . \qquad (3.3)$$

The idea of reducing $\hat{N}$ to a constant by imposing a fixed degree of homogeneity on $\psi(y)$ is seen by (3.1) to be both possible and artificial. The value of Q that is of interest is, of course, that given by (2.16):

$$Q\psi = [E_0(E_0-3) + \tfrac{3}{4}]\psi . \qquad (3.4)$$

Similarly, we must have

$$2i\Sigma^{\alpha\beta}y_\alpha\partial_\beta\psi = \kappa\psi \qquad (3.5)$$

with the same value of κ as adopted in Sec. II, namely,

$$\kappa = -E_0 - \tfrac{1}{2} . \qquad (3.6)$$

Actually (3.5), (3.6) implies (3.4); for "squaring" (3.5) gives

$$\kappa(\kappa+3) = \hat{N}(\hat{N}+3) - \rho^{-1}\partial_\alpha^2$$

or

$$Q = \kappa(\kappa+4) + \tfrac{5}{2} = E_0(E_0-3) + \tfrac{3}{4} .$$

Equation (3.5) is the wave equation proposed by Dirac.[5] As shown by Gürsey and Lee it can be replaced by[3]

$$\gamma^\alpha\partial_\alpha\psi(y) = 0 . \qquad (3.7)$$

The simplest way to achieve this is to impose the homogeneity condition

$$\hat{N}\psi(y) = y^\alpha\partial_\alpha\psi(y) = \kappa\psi(y) . \qquad (3.8)$$

In this case Eq. (3.7) is equivalent to (3.5), (3.6). Of course we also have

$$\partial_\alpha\partial^\alpha\psi(y) = 0 .$$

B. Basis functions

The simplest derivation of basis functions $\psi_{EJM}(y)$ satisfying (3.7) and (3.8) is through the Fourier transform

$$\psi(y) \sim \int e^{-ib^\alpha y_\alpha} F(b) d^5 b\, \delta(b^\alpha b_\alpha) . \qquad (3.9)$$

Then (3.7) and (3.8) follow from $\gamma^\alpha b_\alpha F(b) = 0$ and the fact that $F(b)$ is homogeneous of degree $-\kappa-3$. The factor $\tilde{\gamma}^\alpha b_\alpha$ in $F_{EJM}(b)$—Eq. (2.19)—can be replaced by $i\tilde{\gamma}^\alpha(\partial/\partial y^\alpha)$ and taken outside the integral, which then reduces to the corresponding integral in the earlier treatment of the spin-zero case. Unfortunately, (3.9) diverges for all cases of interest, but this difficulty may be circumvented by analytic continuation in E_0. The result, which may be verified *a posteriori* (this is essential to obtain the correct normalization) is

$$\psi^\pm_{EJM}(y) = \tfrac{1}{2}\rho^{1/2(3/2-E_0)}\left[\frac{(E_0-\frac{1}{2})!}{\pi\frac{1}{2}!\,(E_0-2)!}\right]^{1/2} C^\pm_{EJM}$$
$$\times \tilde{\gamma}^\alpha\partial_\alpha\, y^{E-E_0}(y_5 - iy_0)^{1/2-E}\, \mathcal{Y}^\pm_{JM}(\hat{y})\, {}_2F_1(-K, -K-L-\tfrac{1}{2}; E_0-1; 1-R^2/y^2) , \qquad (3.10)$$

where $\hat{y}$ and y are the direction and magnitude of $\vec{y}$, $R \equiv (y_0^2+y_5^2)^{1/2}$,

$$C^+_{EJM} = [(E,J)^+(K+J+E_0-1)]^{-1},$$

$$C^-_{EJM} = [(E,J)^-(K+E_0-1)]^{-1},$$

and $L = J-\frac{1}{2},\ J+\frac{1}{2}$ for ψ^+, ψ^-, respectively. To guarantee (3.7), (3.8) no substitution based on the constraint (3.1) should be effected. The normalization is such that, for $E_0 > 1$

$$\delta_{I,I} = \int \psi_I^*(y)\gamma_0\psi_{I'}(y)(dy)\,, \tag{3.11}$$

$$(dy) \equiv 2\rho^{-1/2}d^5y\,\delta(y^\alpha y_\alpha - \rho^{-1})\,. \tag{3.12}$$

The geometry of the space-time manifold was already discussed in detail. Notation: x will denote a point in the manifold, whatever coordinates (e.g., y_μ) are used; the invariant distance z is defined by $z = \rho y^\alpha y'_\alpha$. We need to know the propagator function

$$\langle x|x'\rangle_{1/2} \equiv \sum \psi_I(y)\psi_I(y')^*\gamma_0 \tag{3.13}$$

for which we find the expression

$$\langle x|x'\rangle_{1/2} = -\frac{N+3+\kappa}{2(E_0-1)}\langle x|x'\rangle_0, \tag{3.14}$$

where $N = -E_0 - \frac{1}{2}$, κ is the operator defined in Eq. (3.5), and $\langle x|x'\rangle_0$ is the propagator function for the spin-zero case,[2] with E_0 replaced by $E_0 + \frac{1}{2}$. Thus,

$$\langle x|x'\rangle_{1/2} = -\frac{\rho^2}{8\pi^2\frac{1}{2}!}\,\frac{(E_0-\frac{1}{2})!}{(E_0-1)!}\,(N+3+\kappa)[z+(z^2-1)^{1/2}]^{-1/2-E_0}\,{}_2F_1\left(E_0+\tfrac{1}{2},\tfrac{3}{2},E_0,\ \frac{z-(z^2-1)^{1/2}}{z+(z^2-1)^{1/2}}\right)$$

$$= \frac{\rho^2}{(2\pi)^3}(N+3+\kappa)(z^2-1)^{-1/2}Q^1_{E_0-3/2}(z)e^{2\pi i n(E_0+1/2)}\,, \tag{3.15}$$

where n is the relative leaf number. The analytic structure of this function is the same as that of $\langle x|x'\rangle_0$.

In the zero-curvature limit, $\rho\to 0$, $E_0\to\infty$, $m = \sqrt{\rho}\ E_0$ fixed, Eq. (3.14) reduces to

$$\langle x|x'\rangle_{1/2} = \frac{m - i\gamma^\mu\partial_\mu}{2m}\langle x|x'\rangle_0\,. \tag{3.16}$$

IV. FREE QUANTIZED FIELDS

Fock space may be constructed over the discrete set of one-particle normalized basis states. Let $|EJM\rangle$, or more simply $|I\rangle$, denote the basis states associated with the wave functions $\psi_I(x)$. Define the charge-conjugate wave functions

$$\psi_I^c(x) = i\gamma_2\psi_I^*(x)\,, \tag{4.1}$$

and let $|I^c\rangle$ be the associated basis. We note that $\psi_I^c(x)$ satisfies the wave equation (3.8):

$$(\kappa + E_0 + \tfrac{1}{2})\psi_I^c(x) = (\kappa + E_0 + \tfrac{1}{2})\psi_I(x) = 0\,, \tag{4.2}$$

and that $\psi_I^c(x)$ is orthogonal to $\psi_{I'}(x)$ in the inner product defined by (3.11). The states $|I^c\rangle$ have negative energy before quantization.

Now let $|\Omega\rangle$ denote the vacuum state and define creation and destruction operators by

$$a_I^*|\Omega\rangle = |I\rangle,\quad b_I^*|\Omega\rangle = |I^c\rangle\,,$$

$$a_I|\Omega\rangle = b_I|\Omega\rangle = 0$$

$$\{a_I, a_{I'}^*\} = \delta_{II'} = \{b_I, b_I^*\}\,,$$

all other anticommutators being zero. The free quantized Dirac field is defined by

$$\Phi_{1/2}(x) = \sum_I [a_I\psi_I(x) + b_I^*\psi_I^c(x)]\,.$$

The anticommutator is

$$\{\Phi_{1/2}(x), \overline{\Phi}_{1/2}(x')\} = \sum_I [\psi_I(x)\overline{\psi}_I(x') + \psi_I^c(x)\overline{\psi}_I^c(x')]\,.$$

The first term is given by (3.14) and we have

$$\{\Phi_{1/2}(x), \overline{\Phi}_{1/2}(x')\} = \langle x|x'\rangle_{1/2} + (i\gamma_2)\langle x|x'\rangle^*_{1/2}(-i\gamma_2)$$

$$= \langle x|x'\rangle_{1/2} - \langle x'|x\rangle_{1/2}$$

$$= -\frac{N+3+\kappa}{2(E_0-1)}(\langle x|x'\rangle_0 - \langle x'|x\rangle_0)$$

$$= -\frac{N+3+\kappa}{2(E_0-1)}\,\mathrm{disc}\,\langle x|x'\rangle_0,$$

since $\langle x|x'\rangle_0$ is a real analytic function. The causal structure is thus the same as for spin zero; this case was discussed in detail in Ref. 2. In the flat-space limit disc $\langle x|x'\rangle_0$ reduces to the usual scalar invariant function $\Delta_m(x-x')$, while the factor $-(N+3+\kappa)/2(E_0-1)$ goes to $(m - i\gamma^\mu\partial_\mu)/2m$ as we already pointed out.

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[4]See Ref. 2. For spin 0, the tractable case is $E_0=N+3$ and the nontractable case is $E_0=-N$.