PHYSICAL REVIEW D VOLUME 12, NUMBER 12 15 DECEMBER 1975

Gauge fields arising from spacetime symmetries and gravitational theories. II

A. G. Agnese and P. Calvini

Istituto di Scienze Fisiche dell'Università, Genova, Italy and Istituto Nazionale di Fisica Nucleare, Sezione di Genova, Italy

(Received 24 June 1975)

The consequences of covariance of the matter Lagrangian under gauge transformations of the second kind of the 15-parameter conformal group are investigated. In order to accomplish this, 44 fields (16 vierbein fields $h_k{}^\mu$, 24 rotational gauge fields $A^{ij}{}_\mu$, and 4 scale gauge fields ϕ_μ) are introduced. Furthermore, a Brans-Dicke-type scalar field is used to construct a simple "vacuum" Lagrangian, which is easily reducible to the Einstein Lagrangian; the derived dynamical theory is investigated in the most general case and in various particular cases. Connections are established to many works on this subject. Finally a link is established between the mass of the scalar field and the cosmological constant of Einstein's theory. A possible value is calculated.

I. INTRODUCTION

From the epoch of Einstein up to our days many gravitational theories have been proposed that are more or less deep modifications of general relativity. Theoretical and even philosophical considerations have played a very important role in their history since experiments are never extremely accurate and can rarely say a decisive word.

So in a field where physicists are compelled to use axiomatics, we find it very interesting to investigate the consequences of some symmetries of spacetime from the gauge field point of view. A very attractive feature of this theoretical method is the fact that general relativity can be obtained as a gauge field theory from translational invariance[1] (spacetime homogeneity), but its most fascinating characteristic is that we can go further considering larger spacetime symmetries, as was already remarked in the preceding paper I. There, we developed a mathematical formalism which applies to any Lie group representing some symmetry of spacetime; in this paper we consider the 15-parameter Lie group of conformal transformations of coordinates and derive a gravitational theory which reflects this larger symmetry. Following the prescription of constructing a theory easily reducible to Einstein's theory we introduce, in addition to 44 gauge fields (16 translational gauge fields $h_k{}^\mu$, 24 rotational gauge fields $A^{ij}{}_\mu$, 4 dilatation gauge fields ϕ_μ), a scalar field σ playing nearly the same role of the scalar field of Brans-Dicke cosmologies.[2,3] Next, we consider some limiting cases and show to what works of literature we are led by certain restrictions on spacetime symmetries. Finally, a link is established between the cosmological constant of the universe and the mass of the scalar σ.

II. INVARIANCE UNDER 15-PARAMETER CONFORMAL GROUP C_0

The 15-parameter Lie group of conformal coordinate transformations has aroused much interest in physicists for it is the most general group under which electrodynamics is covariant.[4] The infinitesimal expressions of these transformations can be written as

$$\delta x^k = \alpha^k \quad \text{(4 parameters—translations)}, \tag{1}$$

$$\delta x^k = \epsilon^k{}_j x^j = \tfrac{1}{2}\epsilon^{ij} S^k_{L(ij)l}\, x^l \quad \text{(6 parameters—rotations)}, \tag{2}$$

$$\delta x^k = \rho x^k \quad \text{(1 parameter—dilatations)}, \tag{3}$$

$$\delta x^k = (\beta^k x_j - \beta_j x^k)x^j - \beta^j x_j x^k \quad \text{(4 parameters—transformations of acceleration)}, \tag{4}$$

where relations (4) can be regarded as point-dependent rotations and dilatations. In effect, we can also write them as

$$\delta x^k = \tfrac{1}{2}(\beta^i x^j - \beta^j x^i) S^k_{L(ij)l}\, x^l + (-\beta^j x_j) x^k \tag{5}$$

so that the role of the parameters $\epsilon^{ij} = -\epsilon^{ji}$ and ρ of (2) and (3) is played by the terms $\beta^i x^j - \beta^j x^i$ and $-\beta^j x_j$, respectively. Transformations (3) are interpreted as a change in units ($c = \hbar = 1$) so that all physical quantities are to transform according to their dimensional number (see Refs. 3, 5–8). With the choice of conformal group, transformations (2) and (3) of paper I become

$$\delta x^k = \alpha^k + \rho x^k + \tfrac{1}{2}\epsilon^{ij} S^k_{L(ij)l}\, x^l + \tfrac{1}{2}(\beta^i x^j - \beta^j x^i) S^k_{L(ij)l}\, x^l - \beta^j x_j x^k \tag{6}$$

and

$$\delta\chi=\tfrac{1}{2}\epsilon^{ij}M_{L(ij)}\chi+\rho d\chi + \tfrac{1}{2}[2(\beta^i x^j-\beta^j x^i)]M_{L(ij)}\chi+(-2\beta^j x_j)d\chi\,, \tag{7}$$

where d is the dimensional number of the matter field under consideration. The requirement of invariance under the first 11 parameters gives the conservation laws [see (8′) and (9′) of paper I]

$$\left(\frac{\partial L}{\partial\chi_{,k}}\chi_{,l}-\delta^k{}_l L\right)_{,k}\equiv T^k{}_{l,k}=0\,, \tag{8}$$

$$\left(-\frac{\partial L}{\partial\chi_{,k}}M_{L(ij)}\chi+T^k{}_i x_j-T^k{}_j x_i\right)_{,k}\equiv M^{(T)k}_{(ij),k}=0\,, \tag{9}$$

$$M^{(T)k}_{(ij)}=S^k_{(ij)}+M^{(0)k}_{(ij)}\,,\quad S^k_{(ij)}=-\frac{\partial L}{\partial\chi_{,k}}M_{L(ij)}\chi\,,$$

and

$$\left(-\frac{\partial L}{\partial\chi_{,k}}d\chi+T^k{}_t x^t\right)_{,k}\equiv D^k{}_{,k}=0\,,$$

$$D^k=D^{(0)k}+D^{(1)k},\quad D^{(1)k}=-\frac{\partial L}{\partial\chi_{,k}}d\chi \tag{10}$$

which express conservation laws of momentum, angular momentum plus spin, and dilatation current. Invariance under the remaining 4 parameters of acceleration is granted if the matter Lagrangian satisfies the following relations:

$$\frac{\partial L_M}{\partial\chi_{,k}}(M_{L(ik)}-\eta_{ik}d)\chi=0\,. \tag{11}$$

Following the procedure outlined in paper I (Sec. III), we introduce gauge transformations of the second kind derived from (6) and (7): If the 15 constant infinitesimal parameters are made point-dependent, only 11 infinitesimal functions are needed because acceleration gauge transformations of the first kind are already spacetime-dependent rotations and dilatations. In other words, the term $\rho(x)-\beta^j(x)x_j$ is equivalent to $\rho'(x)$ if $\rho(x)$, $\beta^j(x)$, and $\rho'(x)$ are arbitrary functions; the same can be said for rotations. Then gauge transformations of the second kind derived from (6) and (7) can be written as

$$\delta x^\mu=\xi^\mu{}_{(x)}\,, \tag{12}$$

$$\delta\chi=\tfrac{1}{2}\epsilon^{ij}(x)M_{L(ij)}\chi+\rho(x)d\chi\,. \tag{13}$$

Then only 11×4 gauge fields are needed, interacting with matter through covariant derivatives of the form

$$\chi_{;k}=h_k{}^\mu\chi_{;\mu} =h_k{}^\mu(\chi_{,\mu}+\tfrac{1}{2}A^{ij}{}_\mu M_{L(ij)}\chi+\phi_\mu d\chi)\,. \tag{14}$$

The translational gauge fields $h_k{}^\mu$ are referred to in the literature as vierbein fields (see Refs. 1, 6, and 7), the rotational gauge fields $A^{ij}{}_\mu$ are generalizations of Ricci coefficients (Refs. 6 and 7), and lastly the 4 ϕ_μ functions are the dilatation gauge fields.[6] All these fields are coupled minimally with matter, following the usual procedures outlined in Refs. 1, 6, 7 and 9 (see also Sec. I of paper I).

So we have introduced 44 gauge fields to preserve invariance of action under gauge transformations of the second kind of the conformal group. In the next section we shall investigate the derived geometry.

III. SPACETIME GEOMETRY

The aim of this section is to show what the formulas of Sec. IV of paper I become if we specify the Lie generators; for instance, the four-dimensional representations are to be set as follows:

$$A=1,\ldots,6:\ S^k_{(A)l}\to S^k_{L(ij)l}=\delta^k_i\eta_{jl}-\delta^k_j\eta_{il}\,; \tag{15}$$

$$A=7:\ S^k_{(7)l}=\delta^k_l\,. \tag{16}$$

Then the metric tensor $g^{\mu\nu}$ transforms as [cf. (19) of paper I]

$$\delta g^{\mu\nu}=\xi^\mu{}_{,\alpha}g^{\alpha\nu}+\xi^\nu{}_{,\alpha}g^{\mu\alpha}-2\rho g^{\mu\nu}\,. \tag{17}$$

This is a Weyl transformation (see Refs. 4–6, and 10–12). So the element of length ds^2 has dimension 2, $\sqrt{-g}$ has dimension 4 as follows from (20) of paper I, and the Lagrangian density must have dimension -4 in order to give dimensionless action (generalized coordinates are dimensionless too).

Following relations (22) of paper I, covariant derivatives of covariant quantities with dimensional number d must be written as

$$H_{;k}=h_k{}^\mu(H_{,\mu}+\tfrac{1}{2}A^{ij}{}_\mu M_{L(ij)}H+\phi_\mu dH+\Gamma^\epsilon_{\rho\mu}\Sigma^\rho_\epsilon H)\,. \tag{18}$$

Formulas that give us expressions of spacetime connections as functions of gauge fields are derived from (24) of paper I as

$$\Gamma^\mu_{\beta\nu}=\phi_\nu\delta^\mu_\beta+\tfrac{1}{2}A^{ij}{}_\nu S^\mu_{L(ij)\beta}+h^l{}_{\beta,\nu}h_l{}^\mu\,. \tag{19}$$

We can see that the connections depend upon all the gauge fields previously introduced and upon the derivatives of the metric gauge fields (vierbein fields). Then the connections can be written[6] as the sum of three terms, one taking account only of the metric field (Christoffel symbols), another containing the scale gauge fields, and a third term containing torsion. This sum is

$$\Gamma^{\alpha}_{\beta\gamma} = {}^0\Gamma^{\alpha}_{\beta\gamma} + (\delta^{\alpha}_{\beta}\phi_{\gamma} + \delta^{\alpha}_{\gamma}\phi_{\beta} - g_{\beta\gamma}\phi^{\alpha}) + \tfrac{1}{2}[C^{\alpha}{}_{\beta\gamma} + g^{\mu\alpha}(g_{\epsilon\beta}C^{\epsilon}{}_{\gamma\mu} + g_{\epsilon\gamma}C^{\epsilon}{}_{\beta\mu})], \qquad (20)$$

where ${}^0\Gamma^{\alpha}_{\beta\gamma}$ are the usual Christoffel symbols.

So we are in a Weyl space with torsion (unsymmetrical connections). The gauge field strengths of relations (26) of paper I are [let $A \to (ij) = -(ji)$]

$$F^{ij}{}_{\mu\nu} = A^{ij}{}_{\mu,\nu} - A^{ij}{}_{\nu,\mu} - A^{i}{}_{k\mu}A^{kj}{}_{\nu} + A^{i}{}_{k\nu}A^{kj}{}_{\mu} = -F^{ji}{}_{\mu\nu} \text{ (for } A \text{ running from 1 to 6)}, \qquad (21)$$

and

$$F^{(7)}{}_{\mu\nu} = \phi_{\mu\nu} = \phi_{\mu,\nu} - \phi_{\nu,\mu} \quad \text{(for } A = 7). \qquad (22)$$

Then the Riemann tensor is no longer skewsymmetric with respect to the first two indexes, as we have from (28) of paper I:

$$R_{ab\mu\nu} = F_{ab\mu\nu} + \eta_{ab}\phi_{\mu\nu}. \qquad (23)$$

The matter equation of motion in the case of interaction is (29) of paper I. The generalized conservation laws derived from (31) and (32) of paper I are

$$\mathcal{T}^{\epsilon}{}_{\mu;\epsilon} - C_{\epsilon}\mathcal{T}^{\epsilon}{}_{\mu} + C^{\alpha}{}_{\mu\beta}\mathcal{T}^{\beta}{}_{\alpha} = \tfrac{1}{2}R^{ij}{}_{\mu\alpha}\mathcal{S}^{\alpha}_{(ij)} + \phi_{\mu\alpha}\mathcal{D}^{(1)\alpha}, \qquad (24)$$

$$\mathcal{S}^{\epsilon}_{(ij);\epsilon} - C_{\epsilon}\mathcal{S}^{\epsilon}_{(ij)} = \mathcal{T}_{ji} - \mathcal{T}_{ij}, \qquad (25)$$

$$\mathcal{D}^{(1)\epsilon}{}_{;\epsilon} - C_{\epsilon}\mathcal{D}^{(1)\epsilon} = \mathcal{T}^{\epsilon}{}_{\epsilon}, \qquad (26)$$

where $\mathcal{T}^{\alpha}{}_{\beta}$ and $\mathcal{D}^{(1)\alpha}$ are the obvious generalizations of the quantities $T^{i}{}_{j}$ and $D^{(1)i}$ [see (33) and (34) of paper I; (8), (9), and (10) of this paper].

So we have specified the spacetime geometry induced by the requirement of invariance under gauge transformations of the second kind of the group C_0.

IV. ACTION FOR THE GAUGE FIELDS AND EQUATIONS OF MOTION

We now introduce a Lagrangian density function for the gauge fields; it must be invariant under transformations (12) and (13) for the action is a scalar with dimensional number $d=0$.[6,7] So we have to construct with gauge field strengths $F^{ij}{}_{\mu\nu}$ and $\phi_{\mu\nu}$ scalar quantities with $d=-4$ and then multiply them by $\sqrt{-g}$ which has 4 as dimensional number.

Weyl,[13] in order to satisfy all these requirements, chose a Lagrangian of the type R^2; equations of motion were obtained rather with difficulty and were not at all similar to Einstein's. Our purpose being to generalize general relativity, we prefer to construct a Lagrangian rather similar to that of Einstein and such that with a few hypotheses it gives back the usual metric theory.

So we introduce a new scalar massive field σ (with $d=-1$) interacting with the gauge fields previously introduced and with all massive matter fields. An interaction is proposed of the type $m \to \mu\sigma$ so that in the matter Lagrangian wherever mass terms appear we must replace masses m_i by $\mu_i\sigma$ (μ_i are dimensionless constants). So masses are to transform in the following way under the action of (12) and (13):

$$\delta m = \delta(\mu\sigma) = \mu\,\delta\sigma = -\rho\mu\sigma = -\rho m, \qquad (27)$$

and this is consistent with conditions $c=\hbar=1$ (masses are to transform as inverse lengths).

Then instead of Weyl's Lagrangian, we prefer to use a Lagrangian density of the type proposed by Brans and Dicke[2,3,14] where the field σ plays the role of the scalar field. The form proposed for the Lagrangian is

$$\mathcal{L} = \sqrt{-g}\,(\sigma^2 R + a\phi_{\mu\nu}\phi^{\mu\nu} + b\sigma^4 + c\sigma^{;\mu}\sigma_{;\mu} + L_M) \qquad (28)$$

(a, b, c are arbitrary parameters)

where

$$\sigma_{;\mu} = \sigma_{,\mu} - \phi_{\mu}\sigma \qquad (29)$$

and

$$R = -g^{\epsilon\nu}R^{\rho}{}_{\epsilon\rho\nu} = -h_{i}{}^{\mu}h_{j}{}^{\nu}F^{ij}{}_{\mu\nu} = F. \qquad (30)$$

In (28) the term L_M represents the matter Lagrangian: in it we have substituted ordinary derivatives of matter fields with "covariant" ones as usual and, moreover, masses m_i with $\mu_i\sigma$ terms.

Then, if we apply the usual variational principles, we obtain the following equations for the gauge fields $h_k{}^{\mu}, A^{ij}{}_{\mu}, \phi_{\mu}$ and for the scalar field σ [equations of interacting matter are (29) of paper I]:

$$\sigma(R+2b\sigma^2) - \frac{c}{\sqrt{-g}}(\sqrt{-g}\,\sigma^{;\mu})_{;\mu} + cC_{\mu}\sigma^{;\mu} = -\frac{1}{2\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta\sigma}, \qquad (31)$$

$$-c\sigma^{;\mu}\sigma - 2a\left[\frac{1}{\sqrt{-g}}(\sqrt{-g}\,\phi^{\mu\nu})_{;\nu} - C_{\nu}\phi^{\mu\nu} - \tfrac{1}{2}C^{\mu}_{\alpha\beta}\phi^{\alpha\beta}\right] = -\frac{1}{2\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta\phi_{\mu}}, \qquad (32)$$

$$\sigma^2(C_{\epsilon}B^{\mu\epsilon}_{ij} + \tfrac{1}{2}C^{\mu}_{\epsilon\nu}B^{\epsilon\nu}_{ij}) - \frac{1}{\sqrt{-g}}(\sqrt{-g}\,\sigma^2 B^{\mu\nu}_{ij})_{;\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta A^{ij}{}_{\mu}}, \qquad (33)$$

$$\sigma^2(F_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)=-\frac{1}{2\sqrt{-g}}\left(a\mathcal{T}^{(\phi)}_{\mu\nu}+c\mathcal{T}^{(\sigma)}_{\mu\nu}+h_{k\mu}\frac{\delta\mathfrak{L}_M}{\delta h_k{}^\nu}\right), \quad (34)$$

where

$$B^{\mu\nu}_{ij}=h_i{}^\mu h_j{}^\nu-h_i{}^\nu h_j{}^\mu, \quad (35)$$

$$F_{\mu\nu}=-\frac{R^\alpha{}_{\mu\alpha\nu}+R_\mu{}^\alpha{}_{\nu\alpha}}{2}=\frac{R_{1\mu\nu}+R_{2\mu\nu}}{2}, \quad (36)$$

and $\mathcal{T}^{(\phi)}$ and $\mathcal{T}^{(\sigma)}$ are the canonical energy-momentum tensor densities for the fields ϕ_μ and σ, respectively. They have the form

$$\mathcal{T}^{(\phi)}_{\mu\nu}=\sqrt{-g}\,(4\phi^\alpha{}_\mu\phi_{\alpha\nu}-g_{\mu\nu}\phi_{\alpha\beta}\phi^{\alpha\beta}) \quad (37)$$

and

$$\mathcal{T}^{(\sigma)}_{\mu\nu}=\sqrt{-g}\left[2\sigma_{;\mu}\sigma_{;\nu}-g_{\mu\nu}\left(\frac{b}{c}\sigma^4+\sigma^{;\alpha}\sigma_{;\alpha}\right)\right]. \quad (38)$$

So right-hand side terms in (31), (32), (33), and (34) can be considered as "sources" for the gauge fields and for σ, but, as is shown by (34), ϕ_μ and σ fields also occur in determining the spacetime metric structure through their canonical momenta. The above-mentioned sources obey certain identities deriving from second-kind gauge covariance of the matter Lagrangian density. In effect, we have the relations

$$-\frac{\delta\mathfrak{L}_M}{\delta\sigma}=-\mu_i\frac{\partial\mathfrak{L}_M}{\partial m_i}, \quad (39)$$

$$-\frac{\delta\mathfrak{L}_M}{\delta\phi_\mu}=-\frac{\partial\mathfrak{L}_M}{\partial\chi_{;\mu}}d\chi=\mathfrak{D}^{(1)\mu}, \quad (40)$$

$$-\frac{\delta\mathfrak{L}_M}{\delta A^{ij}{}_\mu}=-\frac{\partial\mathfrak{L}_M}{\partial\chi_{;\mu}}M_{L(ij)}\chi=\mathcal{S}^\mu_{(ij)}, \quad (41)$$

$$-\frac{\delta\mathfrak{L}_M}{\delta h_k{}^\mu}h_{k\nu}=-\left(\frac{\partial\mathfrak{L}_M}{\partial\chi_{;\nu}}\chi_{;\mu}-g_{\mu\nu}\mathfrak{L}_M\right)=-\mathcal{T}^{(M)}_{\mu\nu}. \quad (42)$$

So the identities that are obeyed by the sources are the conservation laws (24), (25), and (26) of Sec. III.

Formulas (31), (32), (33), and (34) represent the equations of gravitation in the framework of a very general dynamical theory which is covariant under gauge transformations of the second kind of the conformal group. Furthermore, it has the advantage, with respect to any other one derived from a different Lagrangian, of being easily reducible to gravitational theories well known in the literature.

V. GRAVITATIONAL THEORIES DERIVED FROM BRANS-DICKE–TYPE LAGRANGIAN

In this section we shall illustrate in detail what has already been noted in Sec. IV: the Lagrangian density (28) can be considered a common generalization of different Lagrangian functions, each one corresponding to a particular gravitational theory with fixed covariance properties. Now we want to show what restrictions are needed and consequently what gravitational theories are obtained.

(a) We put scale gauge fields $\phi_\mu=0$, thereby destroying second-kind gauge covariance of the theory with respect to transformations (3). Then the Lagrangian density (28) must be written as

$$\mathfrak{L}=\sqrt{-g}(\sigma^2R+b\sigma^4+c\sigma_{,\mu}\sigma^{,\mu}+L_M). \quad (43)$$

The derived gravitational equations are obtained by putting $a=0$ in Eqs. (31), (32), (33), and (34), ignoring ϕ_μ terms in covariant derivatives, and eliminating the scalar σ from the matter Lagrangian (as scale invariance is no longer required in this case); furthermore, we have $F_{\mu\nu}=R_{1\mu\nu}=R_{2\mu\nu}$, which we call $R_{\mu\nu}$ [see relations (23)]. Then the equations of motion (31), (32), (33), and (34) become

$$\sigma(R+2b\sigma^2)-\frac{c}{\sqrt{-g}}(\sqrt{-g}\,\sigma^{,\mu})_{;\mu}+c\sigma^{,\mu}C_\mu=0, \quad (44)$$

$$\sigma^2(C_\epsilon B^{\mu\epsilon}_{ij}+\tfrac{1}{2}C^\mu_{\epsilon\nu}B^{\epsilon\nu}_{ij})-\frac{1}{\sqrt{-g}}(\sqrt{-g}\,\sigma^2B^{\mu\nu}_{ij})_{;\nu}=-\frac{1}{\sqrt{-g}}\frac{\delta\mathfrak{L}_M}{\delta A^{ij}{}_\mu}, \quad (45)$$

$$\sigma^2(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)=-\frac{1}{2\sqrt{-g}}\left(c\mathcal{T}^{(\sigma)}_{\mu\nu}+h_{k\mu}\frac{\delta\mathfrak{L}_M}{\delta h_k{}^\nu}\right). \quad (46)$$

This gravitational theory is more general than that of Brans and Dicke[2,3] for it takes account of rotational degrees of freedom (spin gauge fields are independent variables) and admits a massive scalar field. As an important particular case we may put $b=0$ in the Lagrangian density (43) and give up the rotational degrees of freedom; then (43) becomes

$$\mathfrak{L}=\sqrt{-g}\,(\sigma^2R^0+c\sigma_{,\mu}\sigma^{,\mu}+L_M), \quad (47)$$

where R^0 is the Riemannian curvature in a torsionless spacetime. In fact if we lose the rotational degrees of freedom, we must replace the gauge fields $A^{ij}{}_\mu$ with Ricci rotational coefficients $A^{0ij}{}_\mu$, which can be expressed in terms of the metric fields and their derivatives. So, only the scalar field σ and the vierbein field $h_k{}^\mu$ survive as independent variables.

Finally, if we put $\sigma^2=\phi$, the Lagrangian (47) becomes just that of Brans and Dicke, i.e.,

$$\mathfrak{L}=\sqrt{-g}\left(\phi R^0-\omega\frac{\phi^{,\mu}\phi_{,\mu}}{\phi}+L_M\right),\quad \omega=-\frac{c}{4}. \tag{48}$$

For the equations of motion see Refs. 2 and 3.

(b) Another restriction on the Lagrangian density (28) may be that the scalar field is constant ($\sigma=1$): so covariance only with respect to the Poincaré group is preserved. Owing to the fact that we give up dilatation covariance, ϕ_μ fields are also no longer useful and the Lagrangian (28) can be written as

$$\mathfrak{L}=\sqrt{-g}\,(R+b+L_M)\,. \tag{49}$$

The derived equations of motion are

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}(R+b)=-\frac{1}{2\sqrt{-g}}\,\mathcal{T}^{(M)}_{\mu\nu}\,, \tag{50}$$

$$\sqrt{-g}\,(h_k{}^\mu C^k{}_{ij}-h_j{}^\mu C_i+h_i{}^\mu C_j)=\mathcal{S}^\mu_{(ij)}\,, \tag{51}$$

which are exactly those of Refs. 7(b) and 15. Again, if we give up the rotational degrees of freedom, we are led to Einstein's Lagrangian

$$\mathfrak{L}=\sqrt{-g}\,(R^0+\Lambda)+\sqrt{-g}\,L_M\,, \tag{52}$$

where $\Lambda=b$ is the cosmological constant.

(c) When deriving equations of motion (31), (32), (33), and (34), we considered all field variables as independent ones. In order to decrease the number of degrees of freedom and gain in simplicity of the geometrical context, we look for a relation between scale gauge fields and the scalar σ. As ϕ_μ and σ must obey the transformation rules

$$\delta\phi_\mu=-\xi^\nu{}_{,\mu}\phi_\nu-\rho_{,\mu}\,, \tag{53}$$

$$\delta\sigma=-\rho\sigma\,, \tag{54}$$

we immediately see that a relation of the form

$$\phi_\mu=\phi_\mu(\sigma;\sigma_{,\alpha}) \tag{55}$$

can have a single solution, viz.,

$$\phi_\mu=\frac{\sigma_{,\mu}}{\sigma}=(\ln\sigma)_{,\mu}\,, \tag{56}$$

The imposing of this restraint implies

(1) the vanishing of $\sigma_{;\mu}=\sigma_{,\mu}-\sigma_{,\mu}(\sigma/\sigma)$,

(2) the vanishing of the gauge field strengths $\phi_{\mu\nu}=\phi_{\mu,\nu}-\phi_{\nu,\mu}$,

(3) the transition from a Weyl spacetime with torsion to an ordinary Riemann-Cartan spacetime (still with torsion, but simpler). This is an immediate consequence of the integrability conditions expressed by (2). (See Refs. 6–17.)

Then our old Lagrangian (28) becomes

$$\mathfrak{L}=\sqrt{-g}(\sigma^2R+b\sigma^4+L_M) \tag{57}$$

where the matter Lagrangian L_M depends on the scalar σ through both the substitution (56) and the interaction with mass terms. Form now on we exclude this last form of interaction, thus admitting the possibility of the existence of scale-breaking mass terms. In this line of thought the term $b\sigma^4$ in (57) may be changed into $b\sigma^2$. The gravitational equations derived from Lagrangian (57) are

$$\sigma(R+b)=-\frac{1}{2\sqrt{-g}}\,\frac{\delta\mathfrak{L}_M}{\delta\sigma}\,, \tag{58}$$

$$\sigma^2[R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}(R+b)]=-\frac{1}{2\sqrt{-g}}\,\mathcal{T}^{(M)}_{\mu\nu}\,, \tag{59}$$

$$\sigma^2(C_\epsilon B^{\mu\epsilon}_{ij}+\tfrac{1}{2}C^\mu_{\epsilon\nu}B^{\epsilon\nu}_{ij})-\frac{1}{\sqrt{-g}}(\sqrt{-g}\,\sigma^2B^{\mu\nu}_{ij})_{;\nu}=-\frac{1}{\sqrt{-g}}\,\frac{\delta\mathfrak{L}_M}{\delta A^{ij}{}_\mu} \tag{60}$$

From Lagrangian (57) in the limit of $\sigma=1$ we get again the results of case b: Kibble's gravitational theory[7(b)] and Einstein's can be straightforwardly obtained and again the arbitrary parameter b is identified with the cosmological constant Λ.

VI. A COSMOLOGICAL IMPLICATION

In this section we want to show that a link can be established between the cosmological constant of the Einstein Lagrangian (52) and the mass of the scalar field σ. We start by writing the Brans-Dicke Lagrangian (48) expressed in terms of the scalar σ instead of the field ϕ of Refs. 2–3,

$$\mathfrak{L}=\sqrt{-g}\,[\sigma^2R^0-8\omega(\tfrac{1}{2}\sigma_{,\mu}\sigma^{,\mu})+L_M]\,; \tag{61}$$

next we impose that the scalar is massive; then (61) becomes

$$\mathfrak{L}=\sqrt{-g}\,[\sigma^2R^0-8\omega(\tfrac{1}{2}\sigma^{,\mu}\sigma_{,\mu}-\tfrac{1}{2}\mu_0{}^2\sigma^4)+L_M]\,, \tag{62}$$

where the presence of the mass $m_0=\mu_0\sigma$ generates the term $4\omega m_0{}^2\sigma^2$ which is to be identified with $b\sigma^4$ of Lagrangian (28). So we have for the mass m_0 the following expression

$$m_0=\left(\frac{b\sigma^2}{4\omega}\right)^{1/2}. \tag{63}$$

In formula (52) we have identified the arbitrary parameter b to be numerically equal to the cosmological constant when σ is unity. Then we can substitute the term $b\sigma^2$ in (63) with Λ. A value of Λ in agreement with the age of the universe and with Hubble's constant is about 10^{-35} sec^{-2} for a model of a universe which is isotropic, homogeneous, and closed.[18] So, if we suppose that the dimensionless parameter ω of Refs. 2–3 is positive and of the order of magnitude of unity, we get for m_0 the value

$$m_0 \cong 10^{-12}\ \mathrm{sec}^{-1}, \tag{64}$$

i.e., in ordinary units (we had put $c = \hbar = 1$)

$$m_0 \cong 10^{-68}\ \mathrm{kg} \cong 10^{-42} M_p, \tag{65}$$

where M_p is the proton mass.

VII. DISCUSSION

After this survey of gravitational theories, we point out the usefulness of the gauge field method. With such a theoretical tool we have collected a number of theories as originated from a single principle.

Bregman in Ref. 6 obtained the case c of this work. Omote in Ref. 12 analyzed the scale gauge fields ϕ_μ and the scalar without including the rotational gauge fields. So our equations (31)–(34) are more general. In special restrictive gauges these equations give the results of the above-mentioned authors.

We also call attention to the interesting paper by Dirac.[14] Essentially our mass term b^4 in Lagrangian (28) corresponds to the term c^4 in his action density [Eq. (5.3)]. Dirac states that this term is connected with cosmology and does not carry the analysis further. The discussion of this term in our work leads to the connection between the scalar massive field and the cosmological constant (Sec. VI).

ACKNOWLEDGMENTS

The authors thank Professor C. Becchi and Professor A. Wataghin for helpful discussions.

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