Exact solution of the Lifshitz equations governing the growth of fluctuations in cosmology

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We present the exact solution of the Lifshitz equations governing the cosmological evolution of an initial fluctuation. Lifshitz results valid for $c_s^2 = 0$ and $c_s^2 = 1/3$ are extended in closed form to any equation of state of the form $p = c_s^2\epsilon$. The solutions embody all the results found previously for special cases of c_s^2. It is found that the growth of any initial fluctuation is only like t^n with $n \leq 4/3$ and hence insufficient to produce galaxies unless the initial fluctuation is very large. A possible way to produce very large initial fluctuations by modifying the equation of state by including gravitational interactions, as originally suggested by Sakharov, is also examined. It is found that a phase transition can occur at baryonic density of 1 nucleon per cubic Planck length, or equivalently at a time $t \approx 10^{-43}$ sec. At those early times the masses allowed by causality requirements are, however, too small to be of interest in galaxy formation.

I. INTRODUCTION

The origin of galaxies is among the most fundamental cosmological problems that are still lacking a satisfactory solution. The original suggestion by Jeans[1] that an initial random density fluctuation could grow fast enough (exponentially) in time to become a galaxy within the known age of the universe endured until 1946. Then Lifshitz[2] showed that, once the expansion of the universe was properly incorporated by using general relativity, for at least two special types of equations of state $p = c_s^2\epsilon$, with the speed of sound $c_s^2 = 0$ and $c_s^2 = \frac{1}{3}$, the exponential growth of an initial fluctuation was degraded to a power law with an exponent of the order of unity. Since 1946 Lifshitz's results have been quoted[3] as a proof of the inability of any thermal density fluctuation to grow to galactic size.

However, a different equation of state (different c_s^2) in the ultrahigh-density regime at the beginning of the cosmic expansion might modify Lifshitz's results to the extent that galaxy formation may be possible. This suggestion gains added strength from recent studies[4] of the behavior of matter above nuclear densities which indicate that c_s^2 is probably unity in the ultrahigh-density regime.

Further, it is possible that the initial density fluctuation may be much larger than the expected value owing to random thermal fluctuations. This could occur, for example, if the ultrahigh-density "cosmic soup" passed through a critical point during the course of the cosmic expansion. If δ is a measure of the relative density change in a region ($\delta \sim \Delta\rho/\rho$) then $\delta \sim N^{-1/2}$ for an average thermal fluctuation in an ideal gas (N is the number of particles contained in the fluctuation) while $\delta \sim N^{-1/6}$ at a critical point.[5] Consequently, for a galaxy-sized fluctuation ($N \sim 10^{68}$ baryons) δ could be enhanced by a factor of $N^{-1/6}/N^{-1/2} \sim 10^{23}$ if the cosmic soup were to pass through a critical point. Since initially

$$\delta(t) \sim \delta_0 t^n \tag{1.1}$$

this enhancement in δ_0 could make a major difference in the possibility of galaxy formation even if the exponent n were to stay of order unity.

The outline of this paper is as follows. We first present in closed form the exact solution of the cosmological equations of Lifshitz for any equation of state of the general form

$$p = c_s^2\epsilon , \tag{1.2}$$

where p is the pressure, ϵ is the total energy density, and c_s is the (constant) sound velocity divided by c. Equation (1.2) is an excellent approximation to the high-density equation of state since in general the speed of sound is a very insensitive function of ϵ at high density. After obtaining the condition for growth of the initial density perturbation in a form similar to the Jeans criterion, we determine the dependence of the exponent n of (1.1) on c_s^2 and show that $n \leq \frac{4}{3}$ for all possible c_s^2. The condition for growth of any small density perturbation by self-gravitation is impossible until after decoupling. Consequently, the maximum possible enhancement of the initial small density perturbation is by a factor $\lesssim 10^5$.

Since relatively large initial density perturbations are required for galaxy formation, possible methods of obtaining such large perturbations are examined. An example given by Terletsky[6] is discussed and it is noted that, since the universe is evolving, one must take into account the minimum formation time of any initial perturbation. We find that the universe must be of order 10^4 sec old before density perturbations of order $\Delta\epsilon/\epsilon \sim 1$ have time to form by any means whatsoever. This is used to eliminate passage of the very early universe through a critical point as a means for enhancing initial density fluctuations.

Having clarified the two remaining doubts concerning Lifshitz's treatment, we conclude that the growth of density fluctuations generated by random processes near the origin of time cannot be considered a viable mechanism for galaxy formation.

II. EQUATIONS GOVERNING THE GROWTH OF SMALL INITIAL DENSITY PERTURBATIONS

The first analysis of the growth in time due to gravitation of small density perturbations was performed by Jeans.[1] With Newtonian gravitation in a static universe he used the Navier-Stokes and continuity equations

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho_{tot}}\vec{\nabla}\rho_{tot} + \vec{\varphi}\,, \qquad \vec{\nabla}\cdot\vec{\varphi} = -4\pi G\rho_{tot}\,, \tag{2.1a}$$

$$\frac{\partial \rho_{tot}}{\partial t} + \vec{\nabla}\cdot(\rho_{tot}\vec{v}) = 0 \tag{2.1b}$$

and showed that small mass density perturbations of the form

$$\rho_{tot} = \rho + \rho_1\,, \quad \rho_1 \ll \rho \tag{2.2}$$

satisfy

$$\left(\frac{\partial^2}{\partial t^2} - v_s^{\,2}\vec{\nabla}^2\right)\delta(t) = 4\pi G\rho\,\delta(t)\,, \quad \delta(t) \equiv \rho_1(t)/\rho \tag{2.3}$$

where ρ is the average density, ρ_{tot} is the total density, and v_s is the sound velocity

$$v_s^{\,2} \equiv \left(\frac{\partial p}{\partial \rho}\right)_S = \frac{p_1}{\rho_1}\,. \tag{2.4}$$

Equation (2.3) has solutions like

$$\delta(t) \propto e^{i(\vec{k}\cdot\vec{r} - \omega t)} \tag{2.5}$$

with ω and $\vec{k}$ related by

$$\omega^2 = k^2 v_s^{\,2} - 4\pi G\rho\,. \tag{2.6}$$

Hence for any wave number k satisfying

$$k < k_j \equiv \left(\frac{4\pi G\rho}{v_s^{\,2}}\right)^{1/2} \tag{2.7}$$

the solution (2.5) for $\delta(t)$ will grow exponentially in time (the Jeans criterion).

Since the universe is actually not static but is expanding with a rate $\dot{R}/R$ the above analysis is not applicable. The best known analysis using Newtonian gravitation in an expanding universe is that by Bonner.[7] By considering that

$$\rho = \rho_0\left[\frac{R_0}{R(t)}\right]^3, \quad \vec{v} = \vec{r}\left[\frac{\dot{R}(t)}{R(t)}\right], \quad \dot{R} \equiv \frac{\partial R}{\partial t} \tag{2.8a}$$

where $R(t)$ is a scale factor satisfying the Einstein equation

$$\dot{R}^2 + \kappa = \frac{8\pi G\rho}{3}R^2 \tag{2.8b}$$

(κ is the curvature parameter), Bonnor showed that $\delta(t)$ satisfies

$$\ddot{\delta} + \frac{2\dot{R}}{R}\dot{\delta} + \left(\frac{v_s^{\,2}q^2}{R^2} - 4\pi G\rho\right)\delta = 0\,, \quad \delta \propto e^{i\vec{r}\cdot\vec{q}/R}\,. \tag{2.9}$$

The presence of $\dot{\delta}$, due entirely to the cosmic expansion, changes the exponential growth of δ in the static universe into a power-law growth in the expanding universe. The growing solution of (2.9) in flat spacetime ($\kappa = 0$) is well known to behave like

$$\delta(t) \propto t^{2/3} \tag{2.10}$$

for an equation of state $p \propto \rho^\gamma$, $\gamma > \frac{4}{3}$. The condition for growth is that the wave number $\vec{q}$ satisfy

$$\frac{q}{R} \lesssim \left(\frac{6\pi G\rho}{v_s^{\,2}}\right)^{1/2}, \tag{2.11}$$

which is similar to the Jeans criterion (2.7) if $k \equiv q/R$ is identified as the physical wave number.

Even though the Newtonian analysis in an expanding universe gives the main result (2.10) of a polynomial increase in time of δ, it is still not sufficiently general to be applied to the case when matter is very dense, $p \sim \rho c^2$, near the origin of the universe. In this high-density regime one must include the perturbations of spacetime in the analysis as well as the matter perturbations. Lifshitz[2,8] originated this perturbational technique and obtained approximate solutions for the cases $v_s^{\,2} \cong 0$ and $v_s^{\,2} = \frac{1}{3}c^2$. For simplicity we adopt the equivalent set of equations given by Weinberg.[9]

Define the perturbations $h_{\mu\nu}$, U_1^ν, p_1, and ϵ_1 by

$$g_{\mu\nu}^{tot} = g_{\mu\nu} + h_{\mu\nu}\,, \quad U_{tot}^\nu = U^\nu + U_1^\nu\,, \qquad p_{tot} = p + p_1\,, \quad \epsilon_{tot} = \epsilon + \epsilon_1\,, \tag{2.12}$$

where $g_{\mu\nu}$, U^ν, p, and ϵ are, respectively, the metric tensor, four-velocity, pressure, and total energy density for the Robertson-Walker metric ($\kappa = 0$)

$$g_{tt} = -c^2\,, \quad g_{ti} = 0\,, \quad g_{ij} = R^2(t)\delta_{ij}\,, \quad U^t = 1\,, \quad U^i = 0\,, \tag{2.13a}$$

$$\dot{R}^2 = \frac{8\pi G\epsilon}{3c^4}R^2\,, \quad \dot{R} \equiv \frac{1}{c}\frac{\partial R}{\partial t} \tag{2.13b}$$

$$\dot{\epsilon}R^3 + 3(\epsilon + p)R^2\dot{R} = 0\,. \tag{2.13c}$$

An appropriate choice of gauge (coordinate condition) and the normalization of the four-velocity give the conditions

$$h_{tt} = h_{ti} = U_1^t = 0\,. \tag{2.14}$$

The equations

$$R^{\rm tot}_{\mu\nu} = -\frac{8\pi G}{c^2}\left(T^{\rm tot}_{\mu\nu} - \tfrac{1}{2} g^{\rm tot}_{\mu\nu} T^{\rm tot\lambda}_{\lambda}\right), \quad T^{\mu\nu}_{;\nu} = 0\,, \tag{2.15}$$

when linearized in the perturbations, give the three pertinent equations as

$$\ddot{h} - \frac{2\dot{R}}{R}\dot{h} + 2\left(\frac{\dot{R}^2}{R^2} - \frac{\ddot{R}}{R}\right)h = -\frac{8\pi G}{c^4}R^2(\epsilon_1 + 3p_1)\,, \quad h \equiv \sum_{k=1}^{3} h_{kk} \tag{2.16a}$$

$$\dot{\epsilon}_1 + \frac{3\dot{R}}{R}(\epsilon_1 + p_1) = -(\epsilon + p)\frac{1}{c}\left[\frac{\partial}{\partial t}\left(\frac{h}{2R^2}\right) + i\vec{q}\cdot\vec{U}_1\right]\,, \tag{2.16b}$$

$$\frac{1}{c^2}\frac{\partial}{\partial t}[i\vec{q}\cdot\vec{U}_1 R^5(\epsilon + p)] = q^2 R^3 p_1\,, \tag{2.16c}$$

where $h_{ij}, \epsilon_1, p_1, \vec{U}_1 \propto e^{i\vec{q}\cdot\vec{x}}$ have been assumed.

III. THE SOLUTION

The time dependences of R and ϵ are determined by (2.13b) and (2.13c). Equation (2.13c) gives

$$\frac{d\epsilon}{dR}R + 3(1 + c_s^2)\epsilon = 0\,, \tag{3.1a}$$

$$\frac{\epsilon}{\epsilon^*} = \left(\frac{R}{R^*}\right)^{-3(1+c_s^2)}, \tag{3.1b}$$

which in conjunction with (2.13b) gives

$$\frac{R}{R^*} = \tau^{2/3(1+c_s^2)}\,, \quad \frac{\epsilon}{\epsilon^*} = \tau^{-2}\,, \quad \tau \equiv t/t_* \tag{3.2a}$$

$$t_*^{-1} \equiv \frac{3c}{2}(1 + c_s^2)\left(\frac{8\pi G\epsilon_*}{3c^4}\right)^{1/2}, \tag{3.2b}$$

where ϵ_* and R_* are integration constants. Note that $\epsilon \propto \tau^{-2}$ regardless of the value of c_s^2.

Define new dimensionless variables by

$$\delta \equiv \frac{\epsilon_1}{\epsilon + p} = \frac{1}{1 + c_s^2}\frac{\epsilon_1}{\epsilon}\,, \tag{3.3a}$$

$$\sigma \equiv i\vec{q}\cdot\vec{U}_1 \frac{R^2}{q^2 t_* c^2}\,, \tag{3.3b}$$

$$\lambda \equiv R^2\left(\frac{h}{2R^2}\right)' \equiv R^2\frac{\partial}{\partial\tau}\left(\frac{h}{2R^2}\right), \tag{3.3c}$$

and use the equation of state

$$p = c_s^2\epsilon\,, \quad p_1 = c_s^2\epsilon_1\,, \quad c_s = \frac{v_s}{c} \tag{3.4}$$

in Eqs. (2.1b) to get

$$\lambda' = -2\left(\frac{1 + 3c_s^2}{3 + 3c_s^2}\right)\left(\frac{\epsilon}{\epsilon_*}\right)R^2\delta\,, \tag{3.5a}$$

$$R^2\delta' = -q^2c^2t_*^2\sigma - \lambda\,, \tag{3.5b}$$

$$\sigma' = 3c_s^2\frac{R'}{R}\sigma + c_s^2\delta\,. \tag{3.5c}$$

Elimination of λ' and σ' and use of (3.2) gives

$$\tau^2\delta'' + \frac{4}{3(1 + c_s^2)}\tau\delta' + \left(\frac{1 + 3c_s^2}{3 + 3c_s^2}\right)^2\left(b^2\tau^{2(1+3c_s^2)/(3+3c_s^2)} - \frac{6 + 6c_s^2}{1 + 3c_s^2}\right)\delta + \frac{2b^2}{1 + c_s^2}\left(\frac{1 + 3c_s^2}{3 + 3c_s^2}\right)^2\tau^{(3c_s^2-1)/(3+3c_s^2)}\sigma = 0\,, \tag{3.6a}$$

and (3.5c) can be written as

$$(\sigma\tau^{-2c_s^2/(1+c_s^2)})' = c_s^2\tau^{-2c_s^2/(1+c_s^2)}\delta\,, \tag{3.6b}$$

$$b \equiv \frac{qt_*v_s}{R_*}\frac{3 + 3c_s^2}{1 + 3c_s^2}\,. \tag{3.6c}$$

With the coordinate transformation

$$z \equiv b\tau^{(1+3c_s^2)/(3+3c_s^2)} \tag{3.7}$$

Eqs. (3.6) become

$$z^2\frac{d^2\delta}{dz^2} + \frac{2}{1 + 3c_s^2}z\frac{d\delta}{dz} + \left(z^2 - \frac{6 + 6c_s^2}{1 + 3c_s^2}\right)\delta + \frac{2}{1 + c_s^2}b^{(3+3c_s^2)/(1+3c_s^2)}z^{(3c_s^2-1)/(1+3c_s^2)}\sigma = 0\,, \tag{3.8a}$$

$$\frac{d}{dz}(b^{(3+3c_s^2)/(1+3c_s^2)}z^{-6c_s^2/(1+3c_s^2)}\sigma) = c_s^2\frac{3 + 3c_s^2}{1 + 3c_s^2}z^{(2-6c_s^2)/(1+3c_s^2)}\delta\,. \tag{3.8b}$$

Equations (3.8) are solved as follows: Multiply (3.8a) by $z^{(1-9c_s^2)/(1+3c_s^2)}$, differentiate, and use (3.8b) to eliminate σ. In the resulting third-order differential equation make the substitution

$$X \equiv z^{-2/(1+3c_s^2)}\frac{d}{dz}(z^{(3+3c_s^2)/(1+3c_s^2)}\delta)\,, \tag{3.9}$$

which yields finally

$$z^2\frac{d^2X}{dz^2}-\frac{12c_s^2}{1+3c_s^2}z\frac{dX}{dz}+\left(z^2+\frac{18c_s^2-2}{3c_s^2+1}\right)X=0\,. \tag{3.10}$$

This has the general solution

$$X(z)=z^{(1+15c_s^2)/(2+6c_s^2)}[AJ_\nu(z)+BJ_{-\nu}(z)]\,,\quad \nu\neq\text{integer} \tag{3.11a}$$

$$X(z)=z^{(1+15c_s^2)/(2+6c_s^2)}[AJ_\nu(z)+BN_{-\nu}(z)]\,,\quad \nu=\text{integer} \tag{3.11b}$$

$$\nu\equiv\frac{3}{2}\frac{1-c_s^2}{1+3c_s^2}\,, \tag{3.11c}$$

where J_ν is the Bessel function of order ν, N_ν is the Neumann function of order ν, and A and B are integration constants. The final result for $\delta(z)$ is

$$z^{(3+3c_s^2)/(1+3c_s^2)}\delta(z)=A\int z^{5/2}J_\nu(z)dz+B\int z^{5/2}J_{-\nu}(z)dz+C\,, \tag{3.12a}$$

$$z\equiv b\tau^{(1+3c_s^2)/(3+3c_s^2)}\,, \tag{3.12b}$$

where $J_{-\nu}(t)$ is replaced by $N_\nu(z)$ for $c_s^2=\frac{1}{9}$ and $c_s^2=1$, and from (3.8b)

$$z^{-6c_s^2/(1+3c_s^2)}\sigma(z)=c_s^2\left(\frac{2-6c_s^2}{1+3c_s^2}\right)b^{-(3+3c_s^2)/(1+3c_s^2)}\int z^{(2-6c_s^2)/(1+3c_s^2)}\delta(z)dz+\sigma_0\,. \tag{3.12c}$$

σ_0 is determined by substitution of Eqs. (3.12) back into (3.8a) while C is determined to make δ regular at the origin if necessary. Knowing δ and σ, λ is determined from (3.5b). Hence, Eqs. (2.16) are completely solved in general for all values of c_s^2.

IV. CONDITIONS FOR GROWTH (JEANS CRITERION)

The general solution for $\delta(z)$, Eq. (3.12a), relates $\Delta\epsilon/\epsilon$ to time. In order for the initial density perturbation to form a galaxy, $\Delta\epsilon/\epsilon$ must grow in time. However, $J_\nu(z)$ and $N_\nu(z)$ have the property that they oscillate for $z\gg1$ (sound waves) so no growth is possible in this regime. Consequently $\delta(z)$ can grow appreciably only for $z<1$.

The condition for growth, $z<1$, can be translated into a criterion analogous to the Jeans criterion by use of Eqs. (3.7), (3.6b), and (3.2) and one finds

$$k\equiv\frac{q}{R}<k_J\equiv\left[\frac{2(1+3c_s^2)^2}{3}\frac{\pi G\epsilon}{v_s^2c^2}\right]^{1/2} \tag{4.1}$$

as the condition for growth of the density perturbation in time. Note that $c_s^2\cong0$ in (4.1) does not agree with (2.11) because the equation of state $p\propto\epsilon^\gamma$ is not of the form $p=c_s^2\epsilon$; $p\propto\epsilon^\gamma$ is valid at low densities but not at high densities, whereas $p=c_s^2\epsilon$ is valid at high densities but not at low densities.

To find the time dependence of the initial growth of a density perturbation at very early times, one must first require the condition for growth ($z<1$) and then use this condition to simplify the general solution (3.12a). If $\nu\neq0$ the Bessel functions for small z take the form

$$J_\nu(z)\sim z^\nu\,,\quad N_\nu(z)\sim z^{-\nu}\,,\quad \nu\neq0 \tag{4.2a}$$

while for $\nu=0$

$$J_0(z)\sim1\,,\quad N_0(z)\sim\ln z-\alpha\,,\quad \alpha\equiv\ln2-E \tag{4.2b}$$

where E is Euler's constant. From (3.12a) and (3.7) one finds

$$\delta(t)\sim\delta_0At^{(2+6c_s^2)/(3+3c_s^2)}+\delta_0Bt^{(9c_s^2-1)/(3c_s^2+3)}\,,\quad c_s^2\neq1 \tag{4.3a}$$

$$\delta(t)\sim\delta_0\{At^{4/3}+Bt^{4/3}[\tfrac{2}{3}\ln(t/t_*)+\ln b-(\alpha+\tfrac{2}{7})]\}\,,\quad c_s^2=1 \tag{4.3b}$$

where the A and B serve only to identify the origin in Eq. (3.12b) of each of the terms in (4.3). Since the A and B label independent modes of time dependence in the general equation for δ, they do the same in (4.3). In order to evolve $\delta(t)$ as rapidly as possible one chooses that mode which gives the most rapid growth. Consequently one finds

$$\delta(t)\sim\delta_0t^n\,, \tag{4.4a}$$

$$n=\frac{2+6c_s^2}{3+3c_s^2} \tag{4.4b}$$

as the most rapid possible mode of growth for the density ratio $\delta(t)$, which requires that $n\leq\frac{4}{3}$ for any c_s^2.

For the special cases $c_s^2=0$ and $c_s^2=\frac{1}{3}$, Eqs. (4.3) give

$$\delta(t)\sim\delta_0(At^{2/3}+Bt^{-1/3})\,,\quad c_s^2=0 \tag{4.5a}$$

$$\delta(t)\sim\delta_0(At+Bt^{1/2})\,,\quad c_s^2=\tfrac{1}{3} \tag{4.5b}$$

which agree with Lifshitz's results. The case $c_s^2=\frac{1}{3}$ can be solved exactly for arbitrary z to give

$$\delta_{1/3}(z)=\left[A\left(2\frac{\sin z}{z}+2\frac{\cos z-1}{z^2}-\cos z\right)+B\left(2\frac{\cos z}{z}-2\frac{\sin z}{z^2}+\sin z\right)\right]\delta_0\,, \tag{4.6a}$$

$$\sigma_{1/3}(z)=\frac{3R_*^2}{2q^2t_*^2c^2}[A(2-z\sin z-2\cos z)+B(2\sin z-z\cos z)]\delta_0\,, \quad (4.6b)$$

$$z=\frac{2qt_*c}{3R_*}\left(\frac{t}{t_*}\right)^{1/2}. \quad (4.6c)$$

A growth law of the form (4.4a) with $n\leq\frac{4}{3}$ and $\delta_0\sim N^{-1/2}$ is insufficient to lead to the formation of galaxies. The reason is simple. Consider the ratio of the gravitational energy of a sphere of radius k^{-1} to its thermal energy

$$\frac{G(\epsilon c^{-2}k^{-3})^2/k^{-1}}{\epsilon c^{-2}v_s^2k^{-3}}=\frac{G\epsilon}{v_s^2c^2}\frac{1}{k^2}\sim\left(\frac{k_J}{k}\right)^2. \quad (4.7)$$

Consequently, the criterion (4.1) for growth of a density perturbation asserts that the gravitational energy of the perturbation should be greater than its thermal energy. But this is valid only if the perturbation is causal, i.e., the time required for light (or gravitational effects) to travel across the perturbation must be larger than the time during which the perturbation existed. The maximum time a perturbation can exist is t, the age of the universe. Hence

$$\pi k_{\max}{}^{-1}=ct\,, \quad (4.8)$$

and since (4.1) is the criterion for growth the density perturbation *cannot* grow until $k_{\max}<k_J$ or

$$\frac{3\pi^2}{2(1+3c_s^2)^2}\frac{v_s^2c^2}{\pi G\epsilon}<c^2t^2\,. \quad (4.9)$$

Using (3.2) to eliminate ϵ and t, (4.9) becomes

$$3\pi c_s\frac{1+c_s^2}{1+3c_s^2}<1\,,\quad c_s^2\lesssim\tfrac{1}{90} \quad (4.10)$$

which is a subsidiary condition on the equation of state in order that a density perturbation be able to grow under its own gravitational influence.

Prior to decoupling of matter from radiation $c_s^2\gtrsim10^{-1}$, while after decoupling $c_s^2\lesssim10^{-8}$. Consequently any small initial density perturbation δ_0 is unable to grow under its own gravitational attraction until after decoupling. Since decoupling occurs after $t\sim10^3$ years, the maximum possible enhancement of δ_0 is

$$\delta_{\max}\sim\delta_0(10^{10}/10^3)^{2/3}\sim10^5\delta_0 \quad (4.11)$$

from (4.4) or (2.10).

V. THE INITIAL DENSITY PERTURBATION

Equations (4.10) and (4.11) place severe restrictions on δ_0, the initial density perturbation. $\delta_{\max}=10^5\,\delta_0$ requires δ_0 to be of order 10^{-5} to 10^{-4} [$\delta_{\text{now}}\sim10^6$, but one assumes that once $\delta(t)$ grows to ~1 nonlinear effects in the Navier-Stokes equation (2.1a) accelerate the growth rate enormously]. Thermal fluctuations which give $\delta_0\sim N^{-1/2}\sim10^{-34}$ for a galaxy seem to be hopeless. Even if the cosmic soup were to evolve through a critical point where $\delta_0\sim N^{-1/6}\sim10^{-11}$ the situation would still be hopeless. Some phenomenon must be found which can yield a δ_0 of order 10^{-4} or greater in a relatively short time period (~10^3 years).

One technique for enhancing fluctuations in a statistical system is to include a strong interparticle attraction in the interparticle force law. This attractive force appears as a negative pressure. When the statistical system is in a state such that this attractive force dominates, a condensation instability sets in which is similar to either a critical-point phenomenon or a phase transition. The application of this technique to galaxy formation using gravitation as the attractive force has been given in a rarely quoted paper by Terletsky.[6]

From a general theorem of statistical physics the mean square deviation of an arbitrary generalized coordinate is given by

$$\langle(q-\bar{q})^2\rangle=-kT\frac{\partial\bar{q}}{\partial a}\,, \quad (5.1)$$

where a is the generalized external force acting on q. Setting $q=V$ and $a=p$ and using the ideal gas law $V=NkT/p$ one finds

$$\left\langle\left(\frac{V-\bar{V}}{\bar{V}}\right)^2\right\rangle\equiv\delta_0^2=N^{-1}\,, \quad (5.2)$$

which is the standard $N^{-1/2}$ law for thermal fluctuations. Following Terletsky, if one includes gravitational forces between the ideal gas particles, the ideal gas law is modified and the new equation of state is

$$pV=NkT-V\frac{\partial U}{\partial V}\,, \quad (5.3)$$

where U is the gravitation interaction potential. Taking

$$U=-\alpha\frac{GM^2}{V^{1/3}}\,, \quad (5.4)$$

where M is the total mass of the system and α is a constant depending on the shape of V, the new equation of state (5.3) becomes

$$pV=NkT-\tfrac{1}{3}\alpha GM^2V^{-1/3}\,. \quad (5.5)$$

From (5.1) we obtain

$$\left[\left\langle\left(\frac{V-\bar{V}}{\bar{V}}\right)^2\right\rangle\right]^{1/2}=\delta_0=N^{-1/2}\left(1-\tfrac{4}{9}\alpha\frac{Gm^2n^{1/3}}{kT}N^{2/3}\right)^{-1/2}, \quad (5.6a)$$

$$\delta_0\equiv N^{-1/2}[1-(N/N_*)^{2/3}]^{-1/2}\,, \quad (5.6b)$$

where $m\equiv M/N$ is the average mass per particle and $n \equiv N/V$ is the average number density of particles. Hence, from (5.6b) it *appears* as though δ_0 can be made arbitrarily large by taking $N \sim N_*$.

However, this appearance is deceptive. Lost in the statistical analysis is all information concerning the time required for δ_0 to form. The case above is particularly instructive because it is precisely the situation which Jeans analysis (Sec. II) was designed to handle. In particular it is easy to show that the condition $N \sim N_*$ for condensation of galaxies given by (5.6b) is equivalent to the Jeans criterion (2.7) for exponential growth. Consequently, the δ_0 given by (5.6) is not the same δ_0 as the initial density perturbation occurring in the Jeans analysis, the Bonnor analysis, or Eq. (4.4a). Instead, the δ_0 of (5.6) corresponds to the *maximum* $\delta(t)$ of the analyses of Sec. II, which was shown to evolve exponentially in time in a static universe and as t^n ($n \leq \frac{4}{3}$) in an expanding universe. Thus even if one postulates the existence of very large density perturbations ($\delta_0 \gtrsim 1$) one must allow time for them to form. The alternative is to postulate the existence of large primeval density perturbations (dating from the cosmic singularity), but recent work[10,11,12] has shown that such perturbations are probably damped very rapidly in the very early universe ($t < 1$ sec).

It is useful to estimate the absolute minimum time required for any galaxy-sized density perturbation to build up to $\delta \neq 0$ from an initial $\delta_0 = 0$ state. This is independent of any particular growth mechanism (self-gravitation, thermal fluctuation, etc.).

At any particular instant of cosmic time t consider a sphere of radius $\bar{r}$ containing Mc^2 energy with the ambient energy density $\bar{\epsilon}$ given by (3.2). Let the matter inside $\bar{r}$ rush inwards at the speed of light to a new spherical configuration of radius r and energy density ϵ. Since $\epsilon \sim \bar{\epsilon}(\bar{r}/r)^3$ and $\delta \sim \Delta\epsilon/\bar{\epsilon} \sim (\epsilon/\bar{\epsilon}) - 1$ one finds

$$\Delta r \equiv \bar{r} - r \gtrsim \tfrac{1}{3}\bar{r}\delta, \quad \delta \leq 1 \tag{5.7}$$

and defining $\Delta t_{\min} \equiv \Delta r/c$ one finds

$$(\Delta t_{\min})^3 \gtrsim \frac{\delta^3}{27}\frac{\bar{r}^3}{c^3} \sim \frac{\delta^3}{36\pi}\frac{Mc^2}{\bar{\epsilon}}\frac{1}{c^3}. \tag{5.8}$$

The ambient energy density $\bar{\epsilon}$ is proportional to t^{-2} from (3.2), and one finds finally

$$(\Delta t_{\min})^3 \gtrsim \tfrac{1}{6}\delta^3 \frac{GM}{c^3} t^2, \tag{5.9}$$

where M is in grams and t is in seconds. This is an underestimate since the expansion of the universe during Δt will slow down the growth rate of δ.

By requiring $\Delta t_{\min} < t$ one finds the following relation between the size of the density perturbation δ and the cosmic time t at which δ can exist (if δ starts from the $\delta = 0$ state):

$$t > \tfrac{1}{6}\delta^3 \frac{GM}{c^3}. \tag{5.10}$$

For a galactic mass $M \sim 10^{44}$ g

$$t \gtrsim 3 \times 10^4 \delta^3 \text{ sec}, \tag{5.11}$$

so for $\delta \sim 1$, $t_{\min} \sim 10^4$ sec, which is the radiation era.

Condition (5.10) serves to eliminate the possibility of using the passage of the very early universe through a critical point to generate very large initial density perturbations. Sakharov[13] has suggested that gravitation will alter the $p = p(\epsilon)$ relation at sufficiently early cosmological times. As discussed by Novikov and Zel'dovich[14] the Sakharov equation of state can be written as

$$\epsilon = \epsilon_N - A\hbar c l_P^2 n^2 - B\hbar c l_P^4 n^{8/3}, \tag{5.12a}$$

$$p = n\frac{d\epsilon}{dn} - \epsilon = p_N - 2A\hbar c l_P^2 n^2 - \tfrac{8}{3}B\hbar c l_P^4 n^{8/3}, \tag{5.12b}$$

$$l_P \equiv \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.61 \times 10^{-33} \text{ cm}, \tag{5.12c}$$

where A and B are numerical constants of order unity, n is the baryon number density, and l_P is the Planck length. By ϵ_N we mean the best nuclear equation of state we can produce. The equation of state (5.12) has a critical point where $dp/dn = 0$ at which point the thermal fluctuations are greatly enhanced similar to (5.6). The critical density n_* at which $dp/dn\,|_{n=n_*} = 0$ is easily found to be

$$n_* \sim l_P^{-3} \sim 10^{99} \text{ cm}^{-3} \quad \text{if } \epsilon_N \sim \hbar c n^{4/3} \quad (n \to \infty), \tag{5.13a}$$

$$n_* \sim \left(\frac{\hbar/mc}{l_P}\right)^3 l_P^{-3} \sim 10^{162} \text{ cm}^{-3} \quad \text{if } \epsilon_N \sim \hbar c \left(\frac{\hbar}{mc}\right)^2 n^2, \tag{5.13b}$$

where m is the mass of the particle involved in vector-meson exchange. The first case corresponds to 1 baryon per Planck volume, which is bearable, but the second case corresponds to 10^{63} particles per Planck volume, which has more of a mystic than an objective flavor.

Condition (5.11) shows that at the time $t \sim 10^{-43}$ sec at which the condition (5.13a) occurs there is only time to form density fluctuations of order $\delta_{\max} \sim 10^{-16}$, even though naive critical-point theory ($\delta_0 \sim N^{-1/6})^5$ leads one to expect $\delta_0 \sim 10^{-11}$. Condition (4.10) then forbids the growth of such a small density perturbation until after decoupling.

VI. SYNOPSIS

The general solution of Lifshitz equations governing the growth of small density perturbations in cosmology has been derived and presented in Eqs. (3.12). Examination of these solutions shows that growth of small density perturbations by self-gravitation is possible only if the sound velocity $c_s^2 \lesssim \frac{1}{90}$ [see Eq. (4.10)], which only occurs after decoupling. Consequently, from (4.11) a maximum enhancement of $\sim 10^5$ of the original density perturbation is possible, which in turn implies that $\delta_0 \gtrsim 10^{-5}$ at decoupling. Hence, relatively large initial density perturbations are necessary for galaxy formation. Growth by self-gravitation of small density perturbations is not possible prior to decoupling, and no known phenomenon can produce $\delta_0 \gtrsim 10^{-5}$ at decoupling for a galaxy-sized system. Hence, we conclude that the growth of small density perturbations ($\delta_0 < 1$) *cannot* be considered a viable mechanism for galaxy formation.

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