Bounds on hadronic vacuum polarization

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Using the Lehmann-Symanzik-Zimmermann bound for the vertex function, we obtain an average upper bound for the ratio of hadron to muon production in e^+e^- collisions. Bounds for the hadronic contribution to the $(g_\mu - 2)$ factor and the photon propagation function are also given.

Recent experiments at MIT and SLAC[1] indicate that the ratio $R(s) = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ increases with energy up to the highest available energy $\sqrt{s} \simeq 5$ GeV. Within the framework of current thinking, this result is surprising and has led to many speculations on its origin.[2] But is it really surprising? It is not so much if we realize that for the first time we are presented with an opportunity to study the decay rate of a particle as a function of its mass $\sqrt{s}$. As s increases, because there are more channels opened up due to the growing available phase space, it should not be surprising to see that many-body channels become important which give rise to a growing value of $R(s)$. This is particularly true if the effective couplings involved are pointlike and are of comparable strengths. It is then quite conceivable that the ratio $R(s)$ keeps on rising with s until this behavior is forbidden by positivity and analyticity as discussed below.

The purpose of this note is to point out that the bound due to Lehmann, Symanzik, and Zimmermann (LSZ)[3] (which is a condition for the absence of ghosts) is relevant to the behavior of $R(s)$ at large s and can be used to establish upper bounds of the hadronic contribution to the photon propagator and the anomalous magnetic moment of the muon.

Let us write the renormalized photon propagator as

$$\Delta_{\mu\nu}(k^2) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Delta(k^2), \tag{1}$$

where $\Delta(s)$ $(k^2 = s)$ is assumed to obey the Källén-Lehmann representation

$$\Delta(s) = \frac{1}{s} - \int_{s_0}^{\infty} \frac{\sigma(s')}{s' - s - i\epsilon}\, ds' , \tag{2}$$

with $\sigma(s') \geq 0$.

From analyticity and positivity, it is easy to establish the lower bound for $\Delta(s)$. We first notice that $s\Delta(s)$ cannot tend to zero as $s \to +\infty$. This is so because in this case the Phragmén-Lindelöf theorem [we assume that in the complex direction $s\Delta(s)$ grows less than $\exp(\sqrt{|s|})$] would allow us to write an unsubtracted dispersion relation for $s\Delta(s)$ which gives $\lim_{s\to 0} s\Delta(s) < 0$, in contradiction with the condition $\lim_{s\to 0} s\Delta(s) = 1$.

Let us now define the renormalized self-energy operator $\pi(s)$ by

$$\Delta(s) = \frac{1}{s[1+\pi(s)]} , \tag{3}$$

with $\pi(0) = 0$. Since $s\Delta(s)$ cannot tend to zero as $s \to \infty$, $|\pi(s)|$ is bounded. Hence we can write a dispersion relation for $\pi(s)$ with at most one subtraction (we have assumed that between 0 and $4m_e^2$ there is no zero owing to the smallness of the fine-structure constant; in fact, the existence of such a zero would conflict with the experimental value of the Lamb shift),

$$\pi(s) = \frac{s}{\pi} \int_{s_0}^{\infty} \frac{\text{Im}\,\pi(s')}{s'(s' - s - i\epsilon)}\, ds' + \text{CDD poles}, \tag{4}$$

where CDD poles are Castillejo-Dalitz-Dyson poles and it is assumed that the zeros of the propagator are isolated points. We shall not need to use this assumption later.

By making use of the property of the Herglotz function[4] or by analyzing Eq. (4) directly together with the constraint given by Eq. (2) for negative value of s, we arrive at the inequality

$$\frac{1}{\pi} \int_{s_0}^{\infty} \frac{\text{Im}\,\pi(s')}{s'}\, ds' < 1, \tag{5}$$

which is the LSZ theorem[3] for the vertex function.

Under the assumptions that single-photon exchange dominates the cross sections for the e^+e^- annihilations into leptons and hadrons, and that the $\gamma\mu\mu$ (or γee) vertex has pointlike behavior up to a very large energy $s = N$ and some qualifications to be discussed below, we have[5,6]

$$\text{Im}\,\pi(s) = \frac{\alpha}{3}[R(s) + 2] \quad (s \gg 4m_\pi^2). \tag{6}$$

Hence the following bound is obtained:

$$\frac{a}{3\pi}\int_{s_0}^{N}\frac{R(s')}{s'}ds'<1. \tag{7}$$

This inequality gives a bound on $R(s)$ which is of comparable usefulness to that given by Cabibbo, Karl, and Wolfenstein.[7] For example, if $R(s)$ increases linearly with s, inequality (7) is saturated for $N=5000$ GeV^2. Previous phenomenological analyses[5,6] do not take into account inequality (5), which we shall use in the following.

We would like now to bound the hadronic contribution to QED quantities such as the anomalous magnetic moment of the muon and the photon propagator. As long as we can make the approximation $|1+\pi(s)|^2\simeq 1$, there is no problem, but this approximation is no longer good if the bound (5) is saturated. It is clear that no useful bound can be obtained from Eq. (2). Equation (4), which can be used to get the LSZ bound (5), can effectively be used, provided that there is no CDD pole[8] and $|1+\pi(s)|^2$ is nearly equal to unity in the low-energy region.

We now want to argue that such CDD poles are irrelevant if the right physical question is asked.[9]

As with many other quantities used in dispersion theory the functions $\Delta(s)$, $\sigma(s)$, and consequently $\pi(s)$ do not describe physical quantities until some average procedure has been achieved to take into account the finite resolution of the measurement apparatus.

For this purpose, let us define

$$\tilde{G}(s)=\frac{1}{\delta}\int_{s}^{s+\delta}G(s')ds'$$

as the average of $G(s)=s\Delta(s)$ over an energy interval $(s, s+\delta)$. The function $\tilde{G}(s)$ is also analytic in the cut s plane with a cut from $s-\delta$ to ∞. For δ sufficiently small, we also have $\tilde{G}(0)=1$ since $G(s)$ is a slowly varying function for $s\approx 0$. Because the absorptive part of $G(s)$ is always negative and can have at most isolated zeros on the cut, $\text{Im}\tilde{G}(s)$ has no zero on the cut, hence $\tilde{G}(s)$ has no zero anywhere. That the zeros of $G(s)$, if they exist, are isolated points can be easily proved. This is due to the fact that $\sigma(s)>\sigma_2(s)$, where the subscript 2 refers to a two-particle intermediate state contribution; since $\sigma_2(s)=\rho_2(s)|F_2(s)|^2$, where $\rho_2(s)$ is the two-body phase space and $F_2(s)$ is an analytic function in the cut s plane, the zeros of $\sigma_2(s)$, which are those of $F_2(s)$, must be isolated. One can therefore ignore the CDD poles in Eq. (4) bearing in mind that the function $1+\tilde{\pi}(s)$ is now defined as the inverse of $\tilde{G}(s)$ which from now on for simplicity will be denoted by $1+\pi(s)$ and $G(s)$, respectively.

Having avoided CDD poles, we now make use of inequality (5) to obtain bounds on $\pi(s)$ for s spacelike or timelike in the range of a few GeV^2. Separating the low- and high-energy contribution to $\pi(s)$, we write

$$\pi(s)=\frac{s}{\pi}\int_{s_0}^{\Lambda}\frac{\text{Im}\pi(s')}{s'(s'-s-i\epsilon)}ds+\frac{s}{\pi}\int_{\Lambda}^{\infty}\frac{\text{Im}\pi(s')}{s'(s'-s-i\epsilon)}ds'$$

for $s\ll\Lambda$. The first integral can be calculated using experimental data on the ratio $R(s)$ up to an energy Λ, and the second term gives a contribution which can be bounded by using (5):

$$\pi(s)>\frac{s}{\pi}\int_{s_0}^{\Lambda}\frac{\text{Im}\pi(s')}{s'(s'-s)}ds'+\frac{s}{\Lambda-s},\quad s<0 \tag{8a}$$

$$\pi(s)<\frac{s}{\pi}\int_{s_0}^{\Lambda}\frac{\text{Im}\pi(s')}{s'(s'-s-i\epsilon)}ds'+\frac{s}{\Lambda-s},\quad 0<s<\Lambda. \tag{8b}$$

It should be stressed that the above bounds for $\pi(s)$ are not strict and can be improved. This is due to the fact that Λ in Eq. (8) should be replaced by some mean value $\bar{\Lambda}$ which for a rising $R(s)$ lies far above Λ.

The hadronic contribution to the anomalous magnetic moment of the muon is given by

$$a_\mu^{\text{had}}=\frac{a}{\pi}\int_{4m_\pi^2}^{\Lambda}ds'\,\sigma_h(s')K(s')+\frac{a}{3\pi}m_\mu^2\int_{\Lambda}^{\infty}\frac{\sigma_h(s')}{s'}ds', \tag{9}$$

where $K(s')$ is a well-known function which is approximately equal to $\frac{1}{3}m_\mu^2/s'$ for $s'\geq\Lambda\gg m_\mu^2$, and $\sigma_h(s')$ is the hadronic contribution to the spectral function $\sigma(s')$.

The first integral in Eq. (9) can be evaluated from experimental data on $\sigma(e^+e^-\to$ hadrons up to the highest available energy Λ. To get an upper bound for the second term let us consider the function $1+\pi(s)$ for s sufficiently large compared to the electron mass to minimize the electron-positron pair contribution to $\pi(s)$ (it turns out that for $s\gg 4m_e^2$, $s\approx 1$ GeV^2, the electron-positron pair contribution is negligible),

$$1+\frac{s}{\pi}\int_{s_0}^{\Lambda}\frac{\sigma(s')}{s'+s}ds'+\frac{s}{\pi}\int_{\Lambda}^{\infty}\frac{\sigma(s')ds'}{s'+s}$$

$$\leq\left[1-\frac{s}{\pi}\int_{s_0'}^{\Lambda}\frac{\text{Im}\pi(s')}{s'+s}ds'-\frac{s}{\Lambda+s}\right]^{-1}.$$

Since

$$\int_\Lambda^\infty \frac{\sigma(s')}{s'+s}ds' \geq \frac{\Lambda}{\Lambda+s}\int_\Lambda^\infty \frac{\sigma(s')}{s'}ds' ,$$

and $\sigma_h(s') < \sigma(s')$, we arrive at

$$\int_\Lambda^\infty \frac{\sigma_h(s')}{s'}ds' \leq \int_\Lambda^\infty \frac{\sigma(s')}{s'}ds' \leq \frac{\Lambda+s}{\Lambda^2} \tag{10}$$

after making use of the fact that the difference

$$\int_{s_0}^\Lambda \frac{ds'\sigma(s')}{s'+s} - \int_0^\Lambda \frac{ds'\,\mathrm{Im}\pi(s')}{s'(s'+s)}$$

is of the order $O(\alpha^2)$ and hence can be neglected as compared with the right-hand side of (10).

Equation (10) now can be used to bound the high-energy contribution to the muon anomalous magnetic moment:

$$a_\mu^{\rm had} \leq \frac{\alpha}{\pi}\int_{4m_\pi^2}^\Lambda ds'\,K(s')\sigma_h(s') + \frac{\alpha}{3\pi}\frac{m_\mu^2}{\Lambda}. \tag{11}$$

Recent experimental evaluation[11] of the first term gives $(6.8\pm0.5)\times10^{-8}$ while for Λ sufficiently large, for example $\Lambda = 25\ \mathrm{GeV}^2$, the second term amounts to 3.0×10^{-7}. As already stressed, this bound can be improved much more since for a rising $R(s)$, Λ in (11) could be replaced by a much higher value which is the mean value of s in the interval (Λ,∞). The bound (11) does not depend upon experimental data on $\sigma(s)$ beyond Λ, nor on any experimental value of the hadronic correction to the photon propagator at spacelike transfer.

We note that once this hadronic correction [i.e., the function $G(s)$] is known experimentally at sufficiently large spacelike transfer, the bound on $a_\mu^{\rm had}$ can be improved as has been pointed out by Bell and de Rafael.[12]

Similarly an improved bound for the propagator in the low-energy region can be obtained straightforwardly from its measurement at high energy.[13]

Note added. After completion of this work, a similar consideration was given by J. Bjorken and B. Ioffe, SLAC report (unpublished).

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[1] See B. Richter, in *Proceedings of the XVII International Conference on High Energy Physics, London, 1974,* edited by J. R. Smith (Rutherford Laboratory, Chilton, Didcot, Berkshire, England, 1974), p. IV-37-55.

[2] See the review talk by J. Ellis, Ref. 1, p. IV-20.

[3] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento, 2, 425 (1955).

[4] K. Nishijima, *Fields and Particles* (Benjamin, New York, 1969), p. 451.

[5] N. Cabibbo and G. Karl, Phys. Lett. 52B, 91 (1974). In contrast with the result obtained in this reference, it is not possible for us to have a resonance effect for a rising $R(s)$.

[6] R. Gatto, Phys. Lett. B51, 371 (1974).

[7] N. Cabibbo, G. Karl, and L. Wolfenstein, Phys. Lett. B51, 387 (1974).

[8] The possible existence of CDD pole or zero has been first pointed out in L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. 101 453 (1956).

[9] For a similar situation in the forward dispersion relation, see G. Grunberg, T. N. Pham, and Tran N. Truong, Phys. Rev. D 10, 3829 (1974).

[10] C. Bouchiat and L. Michel, J. Phys. 22, 121 (1961). L. Durand III, Phys. Rev. 128, 441 (1962).

[11] B. Lautrup, A. Peterman, and E. de Rafael, Phys. Rep. 3C, 193 (1972); A. Bramon, E. Etim, and M. Greco, Phys. Lett. 39B, 514 (1972).

[12] J. S. Bell and E. de Rafael, Nucl. Phys. B11, 611 (1969).

[13] For similar results for the forward dispersion relation, see T. N. Pham and Tran N. Truong, Phys. Rev. D 8, 3980 (1973).