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Scalar particle creation in an anisotropic universe*

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The problem of quantized scalar field creation in an anisotropic spatially homogeneous background universe is reexamined from a Schrödinger-picture point of view. For each mode a complete set of orthonormal wave functions, ψ_N, is obtained using the method of Salusti and Zirilli. These wave functions are valid at all times even if there is an initial cosmological singularity and depend only on the solution of the classical equation of motion. The wave functions are fixed completely by requiring the classical solution to have positive-frequency WKB form when the universe reaches the stage of adiabatic expansion. These wave functions are eigenfunctions of a conserved number operator which has the usual particle interpretation in the adiabatic regime. An intitial state near the singularity is chosen as a superposition of the wave functions, ψ_N, and the particle number in the adiabatic regime is calculated. For plane-wave initial states, which follow the classical behavior near the singularity, the final particle number depends only on the parameters of the initial wave packet. For an initial state which instantaneously diagonalizes the Hamiltonian, an (arbitrary) initial time must be chosen. If the mode in question is in the adiabatic regime at that time almost no particle creation occurs. If it is not adiabatic, creation occurs and becomes infinite if the initial time is taken to be that of the singularity. This creation is a consequence of the failure of particle number to be well defined in this regime. Comparisons with other particle-creation studies are made.

I. INTRODUCTION

Scalar particle creation in a classical background cosmology has been studied by several authors.[1–6] In this paper we shall repeat this calculation for an arbitrary homogeneous, anisotropic, but topologically 3-torus background spacetime. We shall ignore changes in the background metric due to the stress-energy tensor of created quanta.[7,8] We shall also ignore questions of regularization and renormalization of the stress-energy tensor which have been discussed by Zel'dovich and Starobinsky,[2] Parker and Fulling,[4] and others.[9]

We shall consider here the dependence of particle-creation results on the choice of initial conditions and shall suggest three classes of initial states which yield interesting information for a background universe which has an initial singularity and then expands forever. For the opposite problem of a universe which is collapsing[2,3] the results of this paper can be used to interpret the state of the system during the late stages of the collapse. The methods used here are quite different from those of Refs. 1–4 and 5a.

For 3-torus topology of the spacelike hypersurfaces in the background spacetime, the classical and second-quantization wave equations are easily separable into equations for the spatial and time dependence.[2,10] Each mode (i.e., Fourier component[10]) of the scalar field satisfies a classical equation of motion for a harmonic oscillator with time-dependent mass and frequency.[2]

Second quantization as shown in Sec. II yields a Schrödinger equation which is that for a quantized harmonic oscillator with time-dependent mass and frequency. This equation has been studied by several authors[11,12] and the wave functions are given in Ref. 11. The methods of Salusti and Zirilli[12] may be used effectively to construct the wave functions even in the case of the frequency going to zero (with the time-dependent mass transformed away).

In Sec. III we show that the wave functions are exactly expressible at all times in terms of the solutions of the classical equation of motion. They form a complete orthonormal set and each is characterized by an integer N which is an eigenfunction of a time independent number operator. The particular solution of the classical equation is chosen so that in an adiabatic regime the number operator is just the usual one for the harmonic oscillator and the classical solution has the WKB form.[13] If the time coordinate is chosen so that the oscillator has unit mass, the adiabatic regime is controlled by the condition that the logarithmic time rate of change of the frequency be much less than the frequency itself.[14,3]

Once the complete set of wave functions is obtained and defined uniquely by adiabatic boundary conditions, remaining questions concern interpretation only. We present here two classes of initial states defined as superpositions of the above wave functions and discuss the number of quanta observed at infinity in each case. In Sec. V we discuss plane-wave initial states[14,5] which reflect the classical behavior near the initial singularity characterized by the harmonic-oscillator frequency for each mode going to zero. The final particle number depends only on parameters of the initial wave packets.

The earlier particle creation results of Berger[5a] are reproduced in Sec. IV by considering as initial states those which diagonalize the Hamiltonian at some fixed time (which is not that of the singularity). Here the final particle number depends on the choice of time at which the diagonalization is made and appears to reflect the inappropriateness of the particle definition at times near a singularity. The results of Zel'dovich and Starobinsky[2] and Hu[3] may be reproduced by representing the wave functions found here as superpositions of the states which diagonalize the Hamiltonian at some time.[15] The special case of the flat Kasner universe is discussed in Sec. IV.[2,4]

In Sec. VI we show that the results are independent of a time coordinate transformation which does not change spacetime slicing but may differ for two fields related by a conformal factor. A summary is given in Sec. VII.

II. THE WAVE EQUATION

The equations of motion for the massive scalar field in a specified background spacetime

$$ds^2 = -N^2dt^2 + g_{ij}\,dx^i dx^j \tag{2.1}$$

(where N, g_{ij} are functions of t only) may be found from the Hamiltonian density[16]

$$\mathcal{H}_{\rm msf} = \tfrac{1}{2}g^{-1/2}N\pi^2 + \tfrac{1}{2}g^{1/2}N(g^{ij}\phi_{,i}\phi_{,j} + M^2\phi^2)\,, \tag{2.2}$$

where $\phi(x)$ is a scalar field of mass M and π is its conjugate momentum. The spacetime (2.1) is assumed to have 3-torus topology by requiring $0 \leq x, y, z \leq 2\pi$; $g \equiv \det g_{ij}$, where g_{ij} is the metric of the spacelike hypersurface orthogonal to the t direction and g^{ij} is its inverse.

It is easy to show that each mode of the scalar field ϕ satisfies the classical equation of motion[2]

$$\ddot{q} + \frac{\dot{m}}{m}\dot{q} + \omega^2 q = 0 \tag{2.3a}$$

for

$$m(t) \equiv g^{1/2}N^{-1}, \quad \omega^2(t) \equiv N^2(g^{ij}k_ik_j + M^2)\,, \tag{2.3b}$$

where $q(t)$ is the field amplitude for the mode characterized by the integers k_i obtained as separation constants. Equation (2.3) may be regarded as the equation of motion obtained from the Hamiltonian

$$H = \frac{1}{2m(t)}p^2 + \tfrac{1}{2}m(t)\omega^2(t)q^2 \tag{2.4}$$

for the mode k_i, $i = 1, 2, 3$ where $m(t)$ and $\omega^2(t)$ are given above [Eq. (2.3b)] and p is the momentum conjugate to q. (For convenience, the indices k_i are suppressed on p and q.)

To quantize this system, we impose for each mode the canonical equal-time commutation relation ($\hbar = 1 = c$)

$$[p, q] = -i\,, \tag{2.5}$$

where the q's and p's from different modes commute. We define a wave function $\psi_{\vec{k}}(q, t)$ for each mode $\vec{k}$ to represent the amplitude for a given q at time t as a solution of the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi_{\vec{k}} = H\psi_{\vec{k}}\,, \tag{2.6}$$

where H (for the mode $\vec{k}$) is now the operator

$$H \equiv -\frac{1}{2m(t)}\frac{\partial^2}{\partial q^2} + \tfrac{1}{2}m(t)\omega^2(t)q^2\,. \tag{2.7}$$

III. THE WAVE FUNCTIONS

Equation (2.6) is that for a time-dependent mass and frequency harmonic oscillator. A complete set of orthonormal solutions is well known[11] and may be expressed in terms of a solution of the classical equation of motion (2.3).

We find that there exists a complete set of solutions to Eq. (2.6) characterized by the quantum number N (where $\vec{k}$ is suppressed):

$$\psi_N(q, t) = \beta^{*N/2}2^{-N/2}(2\pi)^{-1/4}(N!)^{-1/2} \times \beta^{-(N+1)/2}H_N(q/\sqrt{2}\,|\beta|)\exp(im\dot{\beta}q^2/2\beta)\,, \tag{3.1}$$

where H_N is the Hermite polynomial of order N and $\beta(t)$ is a solution of

$$\ddot{\beta} + \frac{\dot{m}}{m}\dot{\beta} + \omega^2\beta = 0 \tag{3.2}$$

such that

$$\beta^*\dot{\beta} - \beta\dot{\beta}^* = i/m\,. \tag{3.3}$$

For convenience, in the following we shall perform a time coordinate transformation $t \to \tau = f(t)$ such that $m(\tau) = 1$, i.e., β (and q) satisfy

$$\frac{d^2\beta}{d\tau^2} + \omega^2(\tau)\beta = 0 \tag{3.4}$$

and

$$\beta^*\frac{d\beta}{d\tau} - \beta\frac{d\beta^*}{d\tau} = i \tag{3.5}$$

rather than Eqs. (3.2) and (3.3). We shall show that the results are independent of this transformation. In terms of the metric coefficients this transformation gives

$$\omega^2(\tau) = g(g^{ij}k_ik_j + M^2). \tag{3.6}$$

If the Hamiltonian (2.4) for $m(\tau) = 1$ possesses an adiabatic limit characterized by[3-5,14]

$$\frac{d\omega}{d\tau} \ll \omega^2, \tag{3.7}$$

i.e., a limit when the frequency time dependence may be neglected, the wave functions ψ_N [Eq. (3.1)] may be completely specified by requiring

$$\lim_{\omega\text{ adiabatic}} \psi_N(q,\tau) = \Phi_N(\omega, q)\exp\left[-i(N+\tfrac{1}{2})\int \omega(\tau)\,d\tau\right], \tag{3.8}$$

where $\Phi_N(\omega, q)$ is the N-quantum harmonic-oscillator wave function for fixed frequency ω. In this regime $\beta(\tau)$ must have the form

$$\beta(\tau) = (2\omega)^{-1/2}\exp\left[i\int \omega(\tau)\,d\tau\right], \tag{3.9a}$$

with

$$\frac{d\beta}{d\tau} = i\omega\beta. \tag{3.9b}$$

The wave functions (3.1) may be generated by the method of Salusti and Zirilli[12] by assuming harmonic-oscillator spatial (i.e., q) dependence to obtain equations for time-dependent coefficients. This method yields creation and annihilation operators $A^\dagger, A$ where

$$A = -i\frac{d\beta}{d\tau}\hat{q} + \beta(\tau)\frac{\partial}{\partial q}, \tag{3.10}$$

and $A^\dagger$ is its Hermitian conjugate. These operators have the properties

$$\psi_N = (N!)^{-1/2}(A^\dagger)^N\psi_0, \tag{3.11}$$

where ψ_N is given by Eq. (3.1) (for $m = 1$ in the following unless otherwise noted),

$$A\psi_0 = 0, \tag{3.12}$$

and

$$[A, A^\dagger] = 1. \tag{3.13}$$

If there is an adiabatic limit such that inequality (3.7) holds, then

$$\lim_{\omega\text{ adiabatic}} A = a \tag{3.14}$$

(and its Hermitian conjugate), where a and $a^\dagger$ are the usual harmonic-oscillator creation and annihilation operators for the fixed frequency ω. The wave functions (3.1) are at all times eigenfunctions of a number operator

$$\hat{N} \equiv A^\dagger A, \tag{3.15}$$

i.e.,

$$\hat{N}\psi_N = N\psi_N. \tag{3.16}$$

From Eq. (3.14) we see that in the adiabatic limit $\hat{N}$ reduces to the usual harmonic-oscillator number operator. We shall later use this fact in the interpretation of particle number.

IV. VACUUM INITIAL STATES

In this section we follow the treatment of Zel'dovich[6] and assume that there is a regime $\tau \leq \tau_0$ such that the inequality (3.7) holds so that the usual harmonic-oscillator states may be constructed. We choose as the vacuum wave function the state Ψ_0 defined by

$$\tfrac{1}{2}\omega(\tau_0)\Psi_0 = H(\tau_0)\Psi_0, \tag{4.1}$$

where the Hamiltonian $H(\tau_0)$ is the operator

$$H(\tau) \equiv -\frac{1}{2}\frac{\partial^2}{\partial q^2} + \tfrac{1}{2}\omega^2(\tau)\hat{q}^2 \tag{4.2}$$

evaluated at $\tau = \tau_0$. Thus Ψ_0 is a harmonic-oscillator ground state for the frequency $\omega(\tau_0)$.

Since the ψ_N of Eq. (3.1) form a complete orthonormal set for all τ, we expand Ψ_0 in the ψ_N evaluated at $\tau = \tau_0$,

$$\Psi_0 = \sum_N a_N\psi_N(q, \tau_0), \tag{4.3}$$

and solve for the a_N. We find

$$a_{2N+1} = 0 \tag{4.4a}$$

and

$$a_{2N} = \left(\frac{\omega_0}{2}\right)^{1/4}\exp\left(\frac{i}{2}\int^{\tau_0}\omega\,d\tau\right)(2N!)^{1/2}(N!)^{-1}2^{-N}\beta^{*N}\beta^{-N-1/2}\left(\frac{1}{2}\omega_0 + \frac{i}{2}\frac{\dot\beta^*}{\beta^*}\right)^{-1/2}\left[(2|\beta|^2)^{-1}\left(\frac{1}{2}\omega_0 + \frac{i}{2}\frac{\dot\beta^*}{\beta^*}\right)^{-1} - 1\right]^N, \tag{4.4b}$$

where $\omega_0 \equiv \omega(\tau_0)$ and β and $\dot\beta \equiv d\beta/d\tau$ are evaluated at $\tau = \tau_0$.

It is easy to show, using the relation

$$\sum_{N=0}^{\infty}\frac{(2N)!}{N!N!4^N}x^N = (1-x)^{-1/2},$$

that Ψ_0 is normalized, i.e.,

$$\sum_{N=0}^{\infty}|a_N|^2 = 1, \tag{4.5}$$

and to calculate the expected value of the number operator $\hat{N}$ in the state Ψ_0,

$$\langle \hat{N} \rangle \equiv \int dq\, \Psi_0^* \hat{N} \Psi_0$$
$$= \tfrac{1}{2}(\omega_0 |\beta|^2 + |\dot{\beta}|^2/\omega_0 - 1), \quad (4.6)$$

where all quantities are defined as in Eq. (4.4).[17] Since in the adiabatic regime $\hat{N}$ becomes the usual number operator, $\langle \hat{N} \rangle$ is interpreted to mean the expected number of quanta in the adiabatic limit for an initial vacuum at τ_0.

We now evaluate Eq. (4.6) for the following two cases:

Case 1. Inequality (3.7) is valid for all $\tau \geq \tau_0$—i.e., the system is always adiabatic. This case is generally applicable for modes with frequency high compared to the background universe expansion rate.[3,5,6,8] For case 1, β retains its adiabatic form [Eqs. (3.9)] at τ_0 so that (in lowest order in the small quantity $\dot{\omega}/\omega^2$)

$$\langle \hat{N} \rangle = 0 + O(\dot{\omega}/\omega^2). \quad (4.7)$$

Thus if the system is always adiabatic (almost) no particles are created.[18]

Case 2. Inequality (3.7) is not valid at τ_0 and, in fact, $\lim_{\tau \ll \tau_0} \omega = 0$. This case is applicable near a singularity in the background cosmology.[19]

For $\omega \to 0$

$$\beta \approx \frac{1}{2b} + ib\tau, \quad (4.8)$$

where b is a real constant which is a feature of the exact solution of Eq. (3.4) in a form having the correct Wronskian (3.5). In lowest order in the small quantity ω, the expected particle number becomes

$$\langle \hat{N} \rangle = \frac{b^2}{\omega_0} + O(1), \quad (4.9)$$

where $\omega_0 = \omega(\tau_0)$. If, for the thus far arbitrary τ_0, we choose $\tau_0 = \tau_{-\infty}$, the time of the initial singularity (where $\omega = 0$), we find that $\langle \hat{N} \rangle$ is infinite.

To summarize: For a given choice of τ_0 (either chosen arbitrarily or the time such that $\tau \leq \tau_0$ really is an adiabatic regime) particles in those modes which are adiabatic are not created to any great extent, while those in nonadiabatic modes are created in numbers inversely proportional to $\omega(\tau_0)$.

It is appropriate to mention here the special case of the Kasner universe[20] which is really flat space.[2,4,21,22] The metric has the form

$$ds^2 = -dt^2 + t^2 dx^2 + dy^2 + dz^2. \quad (4.10)$$

With the transformation $\tau = \ln t$, we find that the field amplitude q satisfies the equation

$$\frac{d^2 q}{d\tau^2} + [k_1{}^2 + (k_2{}^2 + k_3{}^2 + M^2)e^{2\tau}]\, q = 0. \quad (4.11)$$

The function $\beta(\tau)$, also a solution of Eq. (4.11), is taken to be

$$\beta(\tau) = (\pi/4i)^{1/2} e^{-\pi k_1/2} H^{(1)}_{ik_1}(\kappa e^{\tau}), \quad (4.12)$$

where $H_\nu^{(i)}$ is a νth-order Hankel function of the ith kind and $\kappa \equiv (k_2{}^2 + k_3{}^2 + M^2)^{1/2}$. The constant factors and choice of Bessel function are made to ensure properties (3.5) and (3.9) for $\beta(\tau)$.

This model has no physical singularity[19] and $\lim_{\tau \to -\infty} \omega(\tau) = k_1$, a constant. We easily find that the measure of adiabaticity $\omega^{-2} d\omega/d\tau$ has a maximum value of $(\frac{4}{27})^{1/2} k_1{}^{-1}$ at $\tau = \ln(\sqrt{2} k_1/\kappa)$. For 3-torus topology, with $k_1 = 1, 2, \ldots$, we would therefore expect very little particle creation. Evaluating Eq. (4.6) for $\langle \hat{N} \rangle$ using Eq. (4.12) for β at $\tau_0 = -\infty$, we find

$$\langle \hat{N} \rangle = (e^{2\pi k_1} - 1)^{-1}, \quad (4.13)$$

which is at most 0.002 for the mode $k_1 = 1$. (Of course, for Euclidean topology, there is an infrared divergence.)

The nonzero creation is a consequence of the noncovariance (i.e., dependence on spacetime slicing) of the vacuum state Ψ_0 of Eq. (4.1).[23,24]

V. PLANE-WAVE INITIAL STATES

Rather than force an adiabatic regime $\tau \leq \tau_0$, i.e., an instantaneous diagonalization when the spacetime has a singularity at $\tau = \tau_{-\infty}$, we construct in this section initial states which reflect features of the behavior near the singularity.[25] Since $\lim_{\tau \to \tau_{-\infty}} \omega(\tau) = 0$, we choose as initial states wave functions $\psi_{-\infty}$ which are solutions of the Schrödinger equation

$$i \frac{\partial \psi_{-\infty}}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi_{-\infty}}{\partial q^2}, \quad (5.1)$$

the limit of Eqs. (2.6) and (2.7) for $\tau \to \tau_{-\infty}$ [and of course $t \to \tau = f(t)$ such that $m(\tau) = 1$].

We characterize these initial states which will be plane-wave packets by an initial position q_0 and momentum p_0 such that the expectation values follow the classical equation of motion [limit as $\tau \to \tau_{-\infty}$ of Eq. (3.1)]

$$d^2 q/d\tau^2 = 0. \quad (5.2)$$

That is, we require

$$\langle \hat{q} \rangle = q_0 + p_0 \tau \quad (5.3)$$

and

$$\langle \hat{p} \rangle \equiv \langle -i\partial/\partial q \rangle = p_0 \quad (5.4)$$

in the state $\psi_{-\infty}$. The expectation value $\langle \hat{q} \rangle$ satisfies the classical equation of motion

$$\frac{d^2 \langle \hat{q} \rangle}{d\tau^2} + \omega^2(\tau) \langle \hat{q} \rangle = 0 \quad (5.5)$$

at all times. Thus $\langle \hat{q} \rangle$ must have the form

$$\langle\hat{q}\rangle = q_0 Z_1(\tau) + p_0 Z_2(\tau) \tag{5.6}$$

where

$$\lim_{\tau\to\tau_{-\infty}} Z_1 = 1, \quad \lim_{\tau\to\tau_{-\infty}} Z_2 = \tau,$$

and

$$\lim_{\tau\gg\tau_{-\infty}} \begin{cases} Z_1 = \dfrac{b\sqrt{2}}{\sqrt{\omega}}\cos\left(\int \omega\, d\tau + \phi\right), \\ Z_2 = \dfrac{b^{-1}}{\sqrt{2\omega}}\sin\left(\int \omega\, d\tau + \phi\right). \end{cases} \tag{5.7}$$

The regime $\tau \gg \tau_{-\infty}$ is the adiabatic regime and b and ϕ are constant features of the exact solution of Eq. (5.5). The initial wave function $\psi_{-\infty}$ is fixed completely by the requirements (5.3) and (5.4) and the requirement that

$$\psi_{-\infty}(p_0 = q_0 = 0) = \lim_{\tau\to\tau_{-\infty}} \psi_0(q,\tau), \tag{5.8}$$

where ψ_0 is the wave function (3.1) for $N = 0$. We find

$$\psi_{-\infty}(q,\tau,p_0,q_0) = [1/b^2 + 2i\tau]^{-1/2}(2/\pi b^2)^{1/4}\exp[i p_0 q - i p_0^2\tau/2 - (q - q_0 - p_0\tau)^2/(1/b^2 + 2i\tau)]. \tag{5.9}$$

We can show[26] that for all p_0, q_0, the $\psi_{-\infty}$ form an overcomplete family of states and span the Hilbert space of solutions of Eq. (5.1).

To calculate quantities of interest, we expand $\psi_{-\infty}$ in terms of the ψ_N of Eq. (3.1) evaluated near the singularity,

$$\psi_{-\infty} = \sum_{N=0}^{\infty} a_N \lim_{\tau\to\tau_{-\infty}} \psi_N(q,\tau). \tag{5.10}$$

We find

$$a_N = \exp[i(N+\tfrac{1}{2})\pi/4](N!)^{-1/2}(-i)^N(4b^2)^{-N/2}(ip^0 + 2b^2q^0)^N \exp\left(-\frac{p_0^2}{8b^2} + \frac{i}{2}p_0q_0 - \frac{1}{2}b^2q_0^2\right) \tag{5.11}$$

and

$$\langle\hat{N}\rangle = \sum_N N|a_N|^2 = p_0^2/4b^2 + b^2q_0^2. \tag{5.12}$$

Here $\langle\hat{N}\rangle$ represents the final number of particles given an initial wave packet characterized by p_0, q_0 near the singularity. We remark here that except for the constant b, $\langle\hat{N}\rangle$ in Eq. (5.12) is independent of the specific form of the background metric—i.e., for an initial state $\psi_{-\infty}$ of Eq. (5.9), $\langle\hat{N}\rangle$ quanta [Eq. (5.12)] are present for $\tau\gg\tau_{-\infty}$ in any universe where $\lim_{\tau\to\tau_{-\infty}}\omega = 0$ and the adiabatic regime exists.

VI. FIELDS RELATED BY A CONFORMAL FACTOR AND TIME COORDINATE CHANGES

We first consider the field amplitude $q(t)$ satisfying Eq. (2.3a) where $m(t)$ and $\omega(t)$ are given by Eqs. (2.3b). Let $t\to T(t)$. The Hamiltonian (2.4), H, becomes for the new time variable T

$$\tilde{H}(p,q) = H\frac{dt}{dT}. \tag{6.1}$$

To eliminate $m(t)$ as in Eq. (3.4),[27] if we have initially that t is proper time (i.e., $N = 1$), requires that $dt/dT = g^{1/2}$. (This is exactly what one obtains from the coordinate transformation which has $N\to g^{1/2}$.) Using Eq. (2.4), Eq. (6.1) may be rewritten

$$\tilde{H}(p,q,T) = \tfrac{1}{2}p^2 + g(g^{ij}k_ik_j + M^2)q^2, \tag{6.2}$$

as required.

For $m(t)$ equal to 1, it is easy to show that rather than Eq. (3.1), we require

$$\psi_N(q,t) = \beta^{*N/2}\beta^{-(N+1)/2}2^{-N/2}(2\pi)^{-1/4}(N!)^{-1/2} \times H_N(q/\sqrt{2}|\beta|)\exp(i\dot{\beta}q^2/2\beta), \tag{6.3}$$

where the overdot denotes d/dT and $\beta(T)$ is a solution of Eq. (3.4). The transformation $dt/dT = g^{1/2}$ just gives Eq. (3.1) with $\dot{\beta}$ replaced by $d\beta/dT$ and $m = 1$ which are the wave functions (6.3).

The initial state of Eq. (4.1) becomes for $m = g^{1/2}$

$$\Psi_0(q,t_0) = \pi^{-1/4}(m_0\bar{\omega}_0)^{1/4}\exp\left[-\tfrac{1}{2}m_0\bar{\omega}_0q^2 + \tfrac{1}{2}i\int^{T_0}\bar{\omega}(t)dt\right], \tag{6.4}$$

where the subscript 0 denotes evaluation at t_0 and $\bar{\omega} = (g^{ij}k_ik_j + M^2)^{1/2}$. But for $m = g^{1/2}$, $m\bar{\omega} = \omega$ where $\omega = g^{1/2}(g^{ij}k_ik_j + M^2)^{1/2}$ is the frequency used in Eqs. (6.2) and (4.1). Further,

$$\int^{t_0}\bar{\omega}(t)dt = \int^{T_0}\omega(T)dT \tag{6.5}$$

since $dt/dT = g^{1/2}$. Thus $\Psi_0(q,t_0) = \Psi_0(q,T_0)$ of Eq. (4.1). Since the wave functions and initial states are the same whether t or T is used as time variable, all results must be independent of this coordinate transformation.[28]

We now wish to compare with several authors[1-4,29] who discuss particle creation in terms of the conformally invariant part $\chi(t)$ of the scalar field q (where we have for convenience specialized to a single mode). Following Refs. 2, 3, and 4b we define

$$\chi \equiv g^{+1/6} q . \tag{6.6}$$

If, rather than $dt/dT = g^{1/2}$, we perform $dt/d\eta = g^{1/6}$, χ satisfies an equation of the form (3.4) with the overdot denoting $d/d\eta$ and

$$\bar{\omega}^2 = g^{1/3}\omega^2 + \tfrac{5}{36} g^{-2}(dg/d\eta)^2 - \tfrac{1}{6} g^{-1} d^2g/d\eta^2 , \tag{6.7}$$

where $\omega = (g^{ij}k_ik_j + M^2)^{1/2}$. In the adiabatic regime, the last two terms in Eq. (6.7) may be neglected with respect to ω^2. In this regime

$$\beta(T) = [2\tilde{\omega}(T)]^{1/2} \exp\left[i\int \tilde{\omega}(T)dT \right], \tag{6.8}$$

while

$$\chi(\eta) = [2\bar{\omega}(\eta)]^{-1/2} \exp\left[i\int \bar{\omega}(\eta)d\eta \right] . \tag{6.9}$$

Since $\tilde{\omega}(t) = g^{1/2}\omega(t)$ and $\bar{\omega}(\eta) \approx g^{1/6}\omega(t)$, the exponentials in Eqs. (6.8) and (6.9) both reduce to the same $\exp[i\int \omega(t)dt]$. Since the remaining terms differ only by functions whose time derivatives may be neglected in this limit, results in the adiabatic limit will be independent of whether χ or β is used.

The interval of t for which the adiabatic regime is valid does, however, differ for χ and β since $\bar{\omega}$ and $\tilde{\omega}$ differ by time-dependent factors whose time derivatives may be important in certain regimes. As an extreme example consider the massless scalar field in an isotropic background universe. Then

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) . \tag{6.10}$$

In this case for an average mode number $\bar{k}$

$$\omega = a^{-1}\bar{k} , \tag{6.11}$$

while

$$\tilde{\omega} = a^2\bar{k} \text{ and } \bar{\omega} = \bar{k} . \tag{6.12}$$

Since $\bar{\omega}$ is a constant at all times,[30] the adiabatic condition is identically satisfied, we have the usual flat spacetime field equation, and no particle creation will occur for the field χ. Since $\tilde{\omega}$ is not constant and is in fact small in the limit $a \to 0$ one could[31] obtain nonzero creation for the field q. Thus Eq. (4.6) for $\langle \hat{N} \rangle$ gives a nonzero result for creation of massless scalar quanta in a conformally flat spacetime as compared with Refs. 2, 3, 4, and 29.

VII. DISCUSSION

In this paper we have attempted to show explicitly the dependence of the concept of particle creation on the choice and interpretation of the initial states of the system. We have allowed the following options for a background universe which has an initial singularity at $\tau_{-\infty}$ and an adiabatic regime at $\tau_{+\infty}$:

(1) Require the system to be in an eigenstate of the operator $\hat{N}$ as defined in Eq. (3.15). Then, by definition, there is no particle creation. An examination of the ψ_N of Eq. (3.1) shows, however, that the particle nature of the system only exists in the adiabatic regime. Thus while no quanta are created in these states, the particlelike structure of the quanta emerges only in the adiabatic regime.

(2) Choose as an initial state $\psi_{-\infty}(q, \tau, p_0, q_0)$ as defined in Eq. (5.9). This initial state most closely follows the classical behavior. Here too the particle creation occurs in the sense of appearance of particlelike characteristics in the adiabatic regime from an initially nonparticle state. In this case the interpretation of the wave functions near the singularity as plane wave packets is made explicit. Since the $\psi_{-\infty}$ for all p_0, q_0 form an overcomplete family of states,[26] these initial states are completely general.

(3) Assume that the frequency $\omega(\tau)$ is a constant $\omega(\tau_0)$ for all $\tau \leq \tau_0$. Choose as an initial state the ground state ($N = 0$) for the Hamiltonian (4.2) for ω evaluated at τ_0 and assumed fixed. If the adiabatic condition $(\omega^{-2}d\omega/d\tau)_{\tau=\tau_0} \ll 1$ is satisfied, then (almost) no particle creation occurs. The nonzero terms are of higher order in the small quantity $\omega^{-1}(\tau_0)$. These terms do, however, produce an ultraviolet divergence[1,2,4] which we shall not consider here. If the adiabatic condition is not satisfied at τ_0 (which will occur only for sufficiently long-wavelength modes), we find $\langle \hat{N} \rangle \sim \omega^{-1}(\tau_0)$ which becomes infinite for $\tau_0 = \tau_{-\infty}$. This creation is a consequence of the failure of the particle description in this regime—i.e., the fact that the $N = 0$ eigenvalue of $\hat{N}$, ψ_0, from Eq. (3.1) does not in this regime have the form of Ψ_0 [Eq. (4.1)] which we used as the initial state.

Of the three options, the plane-wave initial states appear to most accurately reflect the classical behavior of the system and so might best serve to isolate purely quantum-mechanical effects which might be present in universes with mode mixing.[32] The ψ_N eigenstates are also interesting and have in fact been used by Parker and Fulling to discuss adiabatic regularization of the stress-energy tensor.[33]

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[9]For a brief historical discussion see Ref. 4b and the references therein.

[10]To conform with the method we shall use to solve the Schrödinger equation resulting from second quantization we must consider the modes $\pm n$ to represent odd- and even-parity solutions for the spatial dependence as in Ref. 5b rather than positive- and negative-momentum solutions as in Refs. 5a and 1–4. The results obtained are the same if the $\pm n$ modes are taken together—i.e., if we consider the products of wave functions and the sums of operators for both $\pm n$.

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[13]This concept of adiabatic particle number has been made more precise by Parker in Ref. 1 and Parker and Fulling in Ref. 4a.

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[16]This form may be easily obtained from the Lagrangian density in, e.g., Ref. 1. One must check that the field equations are correct.

[17]The number of quanta at time t, $\langle \hat{N}'(t)\rangle = \frac{1}{2}(\omega|\beta|^2 + |\dot{\beta}|^2\omega^{-1} - 1)$, as found by Hu (Ref. 3) and implicitly by Zel'dovich and Starobinsky (Ref. 2) can be obtained by expanding the wave function (3.1) for $N=0$ in the time-independent harmonic-oscillator states which diagonalize the Hamiltonian (2.7) at each instant of time. That is to say, we transform from the N representation of Eq. (3.1) to an N' representation using instantaneous eigenstates. We then compute $\langle \hat{N}'\rangle$. This shows clearly that Refs. 2 and 3 treat the problem of an initial vacuum in the initial adiabatic regime of a collapsing background universe which evolves to some $\langle \hat{N}'\rangle \neq 0$ state at a later time t where $\langle \hat{N}'\rangle$ is interpreted in terms of reference harmonic-oscillator states at that time.

[18]See Refs. 1 and 4 for a more rigorous discussion of this point.

[19]In the interesting cases of Kasner and Friedmann universes, it is true that $\omega \to 0$ as the initial singularity is approached. The Kasner universe which is really flat spacetime has at zero proper time $\omega \to$ constant. An examination of the curvature invariants shows that this universe has no physical (but does have a coordinate) singularity. By comparing the conditions for $\omega \to 0$ with those for $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \to \infty$, one can show that if $\omega \to 0$, the curvature invariant will be infinite but that the reverse need not be true.

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[23]Fulling, Parker, and Hu find no particle creation in this case (Ref. 4b) even though they obtain the same solution (4.12) (or rather its complex conjugate since their χ corresponds to the β^* used here) by using the covariant 4-dimensional classical wave equation for the scalar field $\phi(\tau, x)$ [see Eq. (2.2)]. They show that $H^{(2)}_{ik_1}(\kappa e^\tau)$ can be expanded in terms of positive frequency at all values of τ where positive frequency is defined with respect to the usual flat-space coordinates $\hat{t} = t\cosh x$, $\hat{x} = t\sinh x$ where t, x are the coordinates of Eq. (4.10). In terms of the Kasner coordinates t, x, however, $H^{(2)}_{ik_1}(\kappa e^\tau)$ picks up a negative-frequency part [due to the fact that for $k_1 \lesssim 1$, $\omega^{-2}d\omega/d\tau$ is not always $\ll 1$ near $\tau = -\infty (t=0)$]. The limiting conditions (3.9) on $\beta(\tau)$ and Parker and Fulling's adiabatic particle definition both select Eq. (4.12) as the appropriate solution of Eq. (4.11).

[24]Boulware (Ref. 22) finds no particle creation for this case by considering the complex extension which covers flat spacetime rather than only that portion covered by the Kasner coordinates.

[25]See Refs. 5b and 14, and E. P. T. Liang, Phys. Rev. D 5, 2458 (1972).

[26]See Ref. 5b and J. R. Klauder, J. Math. Phys. 4, 1055 (1963); 4, 1058 (1963).

[27]Here we shall use $\beta(t)$ and $q(t)$ interchangeably since they satisfy the same classical equation of motion. In the quantum treatment, of course, q becomes an operator while β is a c number.

[28]This statement makes no claims about the independence of results under coordinate transformations which change the spacetime slicing. In fact, the nonzero particle creation in the flat Kasner case (Sec. IV) suggests that at least the choice of the vacuum state is slicing-dependent.

[29]Parker, Ref. 1, makes an assumption on the definition of adiabatic regime which at least for that purpose is equivalent to discussing the conformally invariant part of the scalar field.

[30]Actually, we must consider the last two terms of the right-hand side of Eq. (6.7). These are in fact eliminated by using the conformally invariant field equation for q as seen in Ref. 2. In Ref. 1 these terms are neglected by assuming that the exact solution for $a=t^{1/2}$, $\beta(T)=(2a^2\bar{k})^{-1/2}\exp(i\int^T a^2\bar{k}\,dt)$, is always adiabatic even though $\dot{\omega}>\omega^2$ as $a\to 0$ (with t, T defined as before).

[31]We refer here to the fact that the frequencies $\tilde{\omega}$ and $\bar{\omega}$ are qualitatively different, so they should yield different results in any calculation where $d\tilde{\omega}/dT$ cannot be neglected.

[32]See B.-L. Hu, Phys. Rev. D 8, 1048 (1973), and Ref. 3; B.-L. Hu, S. A. Fulling, and L. Parker, Phys. Rev. D 8, 2377 (1973).

[33]See Ref. 4a. With the rigorous definition of the adiabatic behavior of the classical solution $\beta(t)$ from Ref. 4a rather than Eqs. (3.9) and taking the $\pm n$ modes together as discussed in Ref. 10, the A's of Eq. (3.10) are just the time-independent operators A_k of Eq. (2.8) of Ref. 4a. Thus the ψ_N of Eq. (3.1) are the eigenfunctions of the number operator $\hat{N}\equiv A^\dagger A$ or (in the Heisenberg picture) the states $|N_A\rangle$ of Ref. 4a. The agreement of $\beta(t)$ with its generalized WKB asymptotic expansion indicates the closeness of the ψ_N to the usual particle states.