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Quark model for K_{l3} form factors*

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A nonrelativistic quark model for the K_{l3} form factors is presented which provides in a straightforward manner an explanation of their observed characteristics in terms of quark masses.

From Lorentz invariance alone one can conclude that

$$\langle \pi^-(K')|V^\mu_{4-i5}(0)|K^0(K)\rangle = \frac{1}{(2\pi)^3}[f_+(t)(K+K')^\mu + f_-(t)(K-K')^\mu], \quad (1)$$

where $t=(K-K')^2$ and V^μ_{4-i5} is the vector current operative in weak strangeness-changing decays.[1] In the SU(3) limit V^μ_{4-i5} is conserved and $\mu_\pi = \mu_K$ so that one has

$$f_+^{\mathrm{SU(3)}}(t) = f_\pi^{\mathrm{SU(3)}}(t), \quad (2a)$$

$$f_-^{\mathrm{SU(3)}}(t) = 0, \quad (2b)$$

where $f_\pi(t)$ is the pion electromagnetic form factor. On the basis of the Ademollo-Gatto theorem[2] one can therefore anticipate that $f_+(0)\simeq f_+^{\mathrm{SU(3)}}(0)=1$, but to say much more than this one must turn to a dynamical theory. There is, in fact, a vast literature on the theory of the K_{l3} form factors[3] dominated by treatments based on a combination of dispersion relations and the current algebra.

We wish to report here on the results of a study of $f_\pm(t)$ in the quark model.[4] Our investigations were motivated by a picture in which the quarks are relatively light confined objects. We shall see that the observed characteristics of the K_{l3} form factors have a remarkably simple interpretation in terms of SU(3)-breaking quark masses.

We define a mock meson $|\tilde{M}\rangle$ as being a state of a free quark and antiquark with the wave function of the real meson $|M\rangle$. That is, $|\tilde{M}\rangle$ is the instantaneous state one obtains by turning off the forces which bind the quarks. Since $|M\rangle$ and $|\tilde{M}\rangle$ are characterized by the same quantum numbers, matrix elements and mock matrix elements have analogous expansions in terms of the various possible Lorentz amplitudes. In particular,

$$\langle \tilde{\pi}^-(\tilde{K}')|V^\mu_{4-i5}(0)|\tilde{K}^0(\tilde{K})\rangle = \frac{1}{(2\pi)^3}[\tilde{f}_+(\tilde{t})(\tilde{K}+\tilde{K}')^\mu + \tilde{f}_-(\tilde{t})(\tilde{K}-\tilde{K}')^\mu], \quad (3)$$

where $\tilde{t}=(\tilde{K}-\tilde{K}')^2$. Since $|\tilde{\pi}^-\rangle$ and $|\tilde{K}^0\rangle$ are composed of free quarks and antiquarks, with

$$V^\mu_{4-i5} = \bar{s}\gamma^\mu u, \quad (4)$$

we can easily calculate $\tilde{f}_\pm(\tilde{t})$. We shall see that these straightforwardly deduced form factors of mock mesons are an excellent representation of the real K_{l3} form factors.

We begin by constructing the mock meson states. For small $\vec{K}$ and for nonrelativistic internal motions, if M is made of a type-α quark and a type-β antiquark, then

$$|\tilde{M}(\vec{K})\rangle = (2\tilde{\mu})^{1/2}\int d^3p\, f_M(\vec{p})\chi^M_{s\bar{s}}\left| q_\alpha\left(\frac{m_\alpha}{\tilde{\mu}}\vec{K}+\vec{p},s\right)\bar{q}_\beta\left(\frac{m_\beta}{\tilde{\mu}}\vec{K}-\vec{p},\bar{s}\right)\right\rangle, \quad (5)$$

where $f_M(\vec{p})$ and $\chi^M_{s\bar{s}}$ are the momentum and spin wave functions of the physical meson M, and $\tilde{\mu}=m_\alpha+m_\beta$. Note that $\int d^3p|f_M(p)|^2=1$.

Consider Eq. (3) for $\mu=m$, where $m=1$, 2, or 3. Then we have

$$(2\pi)^3\langle\tilde{\pi}^-(\vec{K}')|V^m_{4-i5}(0)|\tilde{K}^0(\vec{K})\rangle = f_+(\tilde{t})(K+K')^m + \tilde{f}_-(\tilde{t})(K-K')^m \quad (6)$$

$$= (4\tilde{\mu}_\pi\tilde{\mu}_K)^{1/2}\int d^3p\, f^*_\pi\left(\vec{p}+\frac{m_d}{\tilde{\mu}_\pi}\vec{K}'-\frac{m_d}{\tilde{\mu}_K}\vec{K}\right)f_K(\vec{p})J^m, \quad (7)$$

where

$$J^m = \left[\frac{p+K'-(m_d/\tilde{\mu}_K)K}{2m_u}+\frac{p+(m_s/\tilde{\mu}_K)K}{2m_s}\right]^m. \quad (8)$$

Since

$$f^*_\pi\left(\vec{p}+\frac{m_d}{\tilde{\mu}_\pi}\vec{K}'-\frac{m_d}{\tilde{\mu}_K}\vec{K}\right) \simeq f^*_\pi(\vec{p}) + \frac{\partial f^*_\pi}{\partial p^n}\left(\frac{m_d}{\tilde{\mu}_\pi}K'-\frac{m_d}{\tilde{\mu}_K}K\right)^n \quad (9)$$

for $\vec{K}$ and $\vec{K}'$ near zero, if one defines

$$I_{\pi K} \equiv \int d^3p f_\pi^*(\vec{p}) f_K(\vec{p}), \tag{10}$$

$$B_{\pi K} \equiv \tfrac{1}{3} \int d^3p\, p \left(\frac{\partial f_\pi^*}{\partial p} f_K - f_\pi^* \frac{\partial f_K}{\partial p} \right) \tag{11}$$

and uses the fact that

$$\int d^3p \frac{\partial f_\pi^*}{\partial p} f_K \left(\frac{p^m p^n}{p^2} \right) = -\tfrac{1}{2} \delta^{mn} (I_{\pi K} - B_{\pi K}) \tag{12}$$

then one finds by comparison with (7) that for $x \equiv m_s / m_d$

$$\tilde{f}_+(t_m) = \left(\frac{1+x}{2} \right)^{1/2} \left[\left(\frac{3x+1}{4x} \right) I_{\pi K} + \frac{x-1}{4x} B_{\pi K} \right], \tag{13}$$

$$\tilde{f}_-(t_m) = -\tfrac{3}{4} \left(\frac{1+x}{2} \right)^{1/2} \left[\left(\frac{x-1}{x} \right) I_{\pi K} + \frac{x+3}{3x} B_{\pi K} \right], \tag{14}$$

where $t_m = (\tilde{\mu}_K - \tilde{\mu}_\pi)^2$, corresponding to $\vec{K} = \vec{K}' = 0$. Since in the symmetry limit $I_{\pi K} \to 1$, $B_{\pi K} \to 0$, and $x \to 1$, one may easily verify that $\tilde{f}_+$ satisfies the Ademollo-Gatto theorem.

To evaluate (13) and (14) explicitly we must consider the two overlap integrals $I_{\pi K}$ and $B_{\pi K}$. The usual quark-model assumption is that $I_{\pi K} = 1$, i.e., $f_\pi(p) = f_K(p)$. This would in turn imply that $B_{\pi K} = 0$. Although this is the assumption that we shall employ here, we point out that the information we possess regarding these wave functions argues against it. In particular, both $\psi_\pi(0)$ and $\psi_K(0)$, the wave functions of the relative coordinate ρ of the π and K "atoms" at $\rho = 0$, are known well enough from the decays $\pi \to \mu\nu$ and $K \to \mu\nu$ to allow us to conclude that[5]

$$\frac{|\psi_K(0)|}{|\psi_\pi(0)|} \simeq 2, \tag{15}$$

so that $I_{\pi K}$ is necessarily less than unity. On the other hand, the success of the assumption that the overlap integrals are unity [consider, for example, the decay $\omega \to \pi\gamma$ (see Ref. 6), where $|\psi_\omega(0)| \simeq 4|\psi_\pi(0)|$] implies that $I_{\pi K}$ is not much less than unity. We shall respond to this situation by discussing at each step the effect relaxation of the assumption $I_{\pi K} = 1$ would have on our results.

With $I_{\pi K} = 1$ and $B_{\pi K} = 0$ in (13) and (14) one can easily find $\tilde{f}_\pm(t_m)$ as a function of x. We display the resulting values for $1 \leq x \leq 2$ in Fig. 1. This range of x corresponds to the condition that $m_d \geq (m_s - m_d) \simeq \tfrac{1}{2}(m_\phi - m_\rho)$. In particular, if the quarks are very massive then $x \simeq 1$, if $m_d \simeq \tfrac{1}{2} m_\rho$ and $m_s \simeq \tfrac{1}{2} m_\phi$ then $x \simeq 1.3$, while if one requires that the quark model correctly predicts the ratio of the magnetic moment of the Λ^0 to that of the proton then $x \simeq 1.4$.

We think the last of these possibilities is particularly attractive in that it simultaneously (1) allows the quarks to be pointlike, (2) provides an understanding of the large anomalous magnetic moment of the proton, and (3) corresponds in a natural way to the hadron spectrum. In fact, the explicit values $m_d \simeq 0.34$ GeV and $m_s \simeq 0.48$ GeV, which give $x \simeq 1.4$, have recently been proposed in a number of contexts.[7] With $x \simeq 1.4$ we find here that

$$\tilde{f}_+(t_m) = 1.02, \quad \tilde{\xi}(\tilde{t}_m) \equiv \frac{\tilde{f}_-(t_m)}{\tilde{f}_+(t_m)} = -0.23. \tag{16}$$

Before comparing these values with the experimental ones we must resolve a small ambiguity. Since both $f_\pm$ and $\tilde{f}_\pm$ are functions of momentum transfer, we must decide how to make the correspondence between mock momentum transfer $\tilde{t}$ and real momentum transfer t. Our choice is the most direct and unique possible: $\tilde{t} \to t$. Since in the case at hand $\tilde{t}$ is small ($t_m \simeq 0.02$ GeV^2), this is for all practical purposes the usual quark-model correspondence to quantities at zero momentum transfer. The experimental data give[8]

$$f_+(t_m) = 0.97 \pm 0.04, \quad \xi(t_m) = -0.17 \pm 0.05. \tag{17}$$

The agreement is clearly good. Since, furthermore,

$$I_{\pi K} = \int d^3\rho\, \psi_\pi^*(\rho) \psi_K(\rho) \tag{18}$$

and

$$B_{\pi K} = \tfrac{1}{3} \int d^3\rho\, \rho \left(\psi_\pi^* \frac{\partial \psi_K}{\partial \rho} - \frac{\partial \psi_\pi^*}{\partial \rho} \psi_K \right), \tag{19}$$

we can see that the combination of (15) and the fact that the wave functions are normalized means that $I_{\pi K} < 1$ and $B_{\pi K} < 0$. Relaxing the assumption $I_{\pi K} = 1$ would therefore make $\tilde{f}_+(t_m)$ smaller and

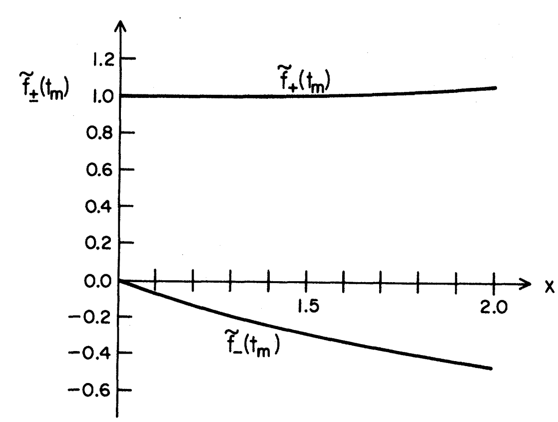

FIG. 1. The form factors $f_\pm(t_m)$ as functions of x.

$\tilde{\xi}(t_m)$ more positive, bringing the predictions (16) into even closer agreement with the experimental values.

One can also understand the momentum-transfer dependence of $f_\pm(t)$ by this technique. Corresponding to the usual expansion

$$f_\pm(t) \simeq f_\pm(0)\left(1+\lambda_\pm \frac{t}{m_\pi^2}\right) \tag{20}$$

we write

$$\tilde{f}_\pm(\tilde{t}) \simeq \tilde{f}_\pm(0)\left(1+\tilde{\lambda}_\pm \frac{\tilde{t}}{m_\pi^2}\right). \tag{21}$$

To find $\lambda_\pm$ we simply extend the expansion (9) to higher order in $\vec{K}$ and $\vec{K}'$, finding

$$\frac{\lambda_\pm}{m_\pi{}^2} = \frac{1}{12(1+x)}\left[\frac{J_{\pi K} - \left(\frac{x+1\mp 2}{4x}\right)(J_{\pi K} - C_{\pi K})}{I_{\pi K} - \left(\frac{x+1\mp 2}{4x}\right)(I_{\pi K} - B_{\pi K})}\right], \tag{22}$$

where

$$J_{\pi K} \equiv \int d^3p \frac{\partial f_\pi^*}{\partial p}\frac{\partial f_K}{\partial p} \tag{23a}$$

$$= \int d^3\rho\, \rho^2 \psi_\pi^* \psi_K \tag{23b}$$

and

$$C_{\pi K} = \tfrac{1}{5}\int d^3p\, p\left(\frac{\partial^2 f_\pi^*}{\partial p^2}\frac{\partial f_K}{\partial p} - \frac{\partial f_\pi^*}{\partial p}\frac{\partial^2 f_K}{\partial p^2}\right) \tag{24a}$$

$$= \tfrac{1}{5}\int d^3\rho\, \rho^3\left(\psi_\pi^* \frac{\partial \psi_K}{\partial \rho} - \frac{\partial \psi_\pi^*}{\partial \rho}\psi_K\right). \tag{24b}$$

If $I_{\pi K}=1$, we have $J_{\pi K} = \langle \rho^2\rangle_\pi = 4\langle r^2\rangle_\pi$ (see Ref. 9) and $C_{\pi K}=0$. It follows that $\lambda_+ = \lambda_-$ and that

$$\lambda_+ = \frac{\mu_\pi^2}{3(1+x)}\langle r^2\rangle_\pi . \tag{25}$$

For $x=1.4$ we therefore find that[10]

$$\lambda_+ = 0.033 \pm 0.006 \tag{26}$$

in excellent agreement with the experimental value[11]

$$\lambda_+ = 0.029 \pm 0.002 . \tag{27}$$

Relaxation of the assumption $I_{\pi K}=1$ tends to decrease λ_+. Once again this would bring the prediction (26) into even closer agreement with experiment.

Mock mesons, it would seem, are an amazingly accurate representation of real mesons.

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[3]We refer the interested reader to the review article by L. M. Chounet, J. M. Gaillard, and M. K. Gaillard, Phys. Rep. 4C, 199 (1972).

[4]There have been other investigations of the K_{l3} form factors in the quark model; these previous treatments have used the Bethe-Salpeter equation and are based on quite different physical assumptions. See M. Bohm and D. Rein, Nucl. Phys. B11, 61 (1969); and D. Rein, Z. Phys. 259, 205 (1973).

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[8]Since λ_+ is a well-established number, $f_+(0)$ can be extracted from $\Gamma(K\to\pi e\nu)$ [the quoted error in $f_+(0)$ is due almost entirely to the uncertainty in $\sin\theta$]. For $\xi(0)$ we have used the compilation of the Particle Data Group, Phys. Lett. 50B, 1 (1974); by taking the error-weighted average of their best fit to the $K_{\mu 3}^+$ data with their best fit to the $K_{\mu 3}^0$ data we arrive at the number quoted in the text. Considering, however, that there are still some inconsistencies in the data, the quoted error of ± 0.05 should perhaps be enlarged.

[9]There has been some degree of confusion on this point [see, for example, J. J. J. Kokkedee, *The Quark Model* (Benjamin, New York, 1969), section 10-3]. The wave function discussed by van Royen and Weisskopf (Ref. 5), which is the Fourier transform of $f_M(\vec{p})$ defined in (5), is the wave function of the relative coordinate of the quark-antiquark system. For the pion one therefore has $\langle\rho^2\rangle_\pi \equiv \int d^3\rho\,\rho^2|\psi_\pi(\rho)|^2 = 4\langle r^2\rangle_\pi$, where $\langle r^2\rangle_\pi$ is the mean square radius determined by the pion electromagnetic form factor. With this understood, one can actually "deduce" $\langle r^2\rangle_\pi$: With $\int d^3\rho|\psi_\pi(\rho)|^2=1$ and $|\psi_\pi(0)|$ known, any one-parameter form for $\psi(\rho)$ is completely determined, so that $\langle r^2\rangle_\pi = \frac{1}{4}\langle\rho^2\rangle_\pi$ may be calculated. By considering a variety of such possibilities for $\psi(\rho)$ one finds values for $\langle r^2\rangle_\pi^{1/2}$ which cluster around 0.7 fm, in good agreement with the measured rms radius of the pion (see Ref. 10).

[10]We have taken $\langle r^2\rangle_\pi^{1/2} = 0.7 \pm 0.1$. See, for example, C. N. Brown *et al.*, Phys. Rev. Lett. 26, 991 (1971).

[11]This is the error-weighted average of λ_+ as measured by all methods compiled by the Particle Data Group, Phys. Lett. 50B, 1 (1974).