Relativistic treatment of the quark-confinement potential

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(Received 14 May 1975)

We discuss the spectrum resulting from a relativistic treatment of a linear confinement potential. We employ Klein-Gordon and Dirac wave equations regarding the linear potential as a Lorentz scalar.

With the recent discovery of several new particles[1] interest in hadron spectroscopy has been revived. Several attempts have been made to study the hadron spectrum using the nonrelativistic Schrödinger equation with a linear potential.[2,3] A linear potential is suggested by the gauge theory of quark confinement[4] and by exactly soluble 2-dimensional QED.[5] The resulting spectroscopy seems to be quite satisfactory and even has significant advantages over earlier schemes, in the case of ordinary mesons and baryons.[3] Because of this success it is important to go beyond the nonrelativistic approach, which must be regarded as a first approximation for a complicated hadronic system.

A complete treatment should incorporate both relativistic and quantum effects, and in addition, requires a full understanding of the underlying dynamics of the quarks. The well-known complexity encountered in bound-state problems in relativistic field theory suggests that the complete solution of this problem is rather remote at the present time. Nevertheless one might hope to gain some insight by examining crude approximations which include some of the above effects.

In this paper we will investigate the relativistic effects for a linear potential model leaving aside the more difficult question of quantum effects. We will work within the framework of relativistic, Dirac-type or Klein-Gordon-type wave equations to determine the hadron spectrum. The approach is similar in spirit to the relativistic treatment of the hydrogen atom.[6] However, if we just replace the Coulomb potential, α/r, the time component of a 4-vector, by the quark-confining linear potential,[7] the $q\bar{q}$ system will not have any bound states. This is because the time component of a 4-vector changes sign in going from the positive-energy components to the negative-energy components. Thus, the linear potential, which grows with distance, will give strong attraction for the positive energy, i.e., particle, component but repulse the antiparticle component leading to a finite amplitude at infinity. Another way of viewing this is that in a relativistic equation the "effective" potential rises initially as r grows, reaches a maximum, and eventually goes to $-\infty$ at $r=\infty$. Thus instead of bound states one has "resonances" with a finite transition amplitude into a quark-antiquark continuum. One might interpret this as the presence of decay channels but then an additional mechanism, such as automatic pair creation, would be needed in order to prevent quark escape. This is contrary to the original motivation for a linear potential.

This type of difficulty is easily avoided by treating the linear potential as a Lorentz scalar which has the same sign for both particle and antiparticle components of a given field.[8] Another slightly more complicated possibility which would also yield bound states for a linear potential is to use a 3-vector potential $\vec{A}=k\vec{r}$ with a minimal coupling. Current theoretical approaches to the linear potential are too crude to single out any one of these various possibilities. Thus we treat the Lorentz scalar case in this paper leaving the others for later investigation.

When we compare the resulting excitation spectrum with that of the nonrelativistic case we find that the first several levels are very similar, but that significant changes occur in higher levels and in the interpretation of the parameters. We should remark that in the usual nonrelativistic reduction of the Dirac or Klein-Gordon equation, the crucial assumption is that the field energy is much smaller than the rest mass. This is not generally satisfied for the linear potential which is strong for large distances. Our results show that only for the first few low-lying levels is the expectation value of the potential small compared to the rest mass. This contrasts with the case of the hydrogen atom where the expectation value of the Coulomb potential is small compared to the electron rest mass for all levels.[6] From this latter example we also learn that certain features of the wave function are very sensitive to whether or not a relativistic treatment is employed. Regardless of the large-distance form of the potential, if, at small distances, there is a component which takes a Coulomb-type form, α/r, the exact relativistic wave function will have a mild singularity $\propto (1/r)^{\alpha^2/2}$ at $r=0$.[6] This prevents a smooth connection to the nonrelativistic wave function in this region. Thus physical calculations which are sensitive to the exact small r behavior, such as various decays, will be strongly affected by relativistic considerations.

Without a detailed calculation it is hard to do more than speculate concerning the quantum effects we neglect. At one extreme, we can appeal to 2-dimensional QED, which is exactly soluble,[9] to make some useful guesses. In that theory the particle spectrum of the $q\bar{q}$ system consists of a single massive vector boson, even though the interaction between quarks is precisely the linear potential. The physical origin of the absence of higher excited states is that the vacuum polarization and shielding are complete in the sense that as soon as energy is available an additional $q\bar{q}$ boson bound state of the given mass is produced; the original quarks are then completely shielded from one another.

This phenomenon could persist in three dimensions. Vacuum polarization may again be associated with a linear potential. For instance, the familiar argument[10] using the cluster decomposition theorem to suggest a linear potential requires that all communicating gluons between quarks separated at large distances be massive (otherwise there is a long-range interaction which violates the form of cluster decomposition used.) One possible mechanism for generating gluon mass in non-Abelian gauge theories is via vacuum polarization as suggested by Schwinger.[9] Polarization and shielding could again be sufficiently complete that only a very few particles actually appear in the resonance spectrum.

On the other hand, in strong coupling analogs of atomic physics and in dual resonance models (such as the Veneziano model in which quark containment is absolute— there being no threshold for quark production) an infinite sequence of resonance levels persists[11]; the higher resonances acquire increasingly large widths eventually merging into the continuum but the process is gradual rather than sudden. Memory of the linear potential could possibly be retained even for very large separation despite the increasing probability of decay to continuum states (of perhaps a linearly increasing number of particles). Further investigation is needed to decide which approach is the more viable theoretically. Phenomenologically a dual type approach has strong support.

We now turn to a presentation of our relativistic wave equation calculations. We set up a framework for both Klein-Gordon-type and Dirac-type equations and compare these results with one another and with the nonrelativistic calculation.

Consider for the moment the two-spinless-quark (i.e., quark-antiquark) wave function component of a meson bound state defined as[12]

$$\langle 0 | T(\phi_{\bar{q}}(x')\phi_q(x)) | M\rangle = \psi(r) e^{-iP\cdot R}\,, \tag{1}$$

where R, r are defined as the usual center-of-mass and relative coordinates,

$$r = x - x'$$
$$R = \tfrac{1}{2}(x + x')\,, \tag{2}$$

and P is the cm momentum. Suppose that one quark field feels only an effective potential due to the other quark (the potential may, of course, be influenced by higher pair states in $|M\rangle$) and that this potential depends on the relative coordinate r. If we take this potential V to be a Lorentz scalar, either of the quark fields will satisfy a Klein-Gordon equation of the form

$$[\Box_x + (m+V)^2]\,\phi_q(x) = 0\,. \tag{3}$$

m, here, is the "renormalized" quark mass i.e., it includes any possible constant term in the potential V. If we operate with $\Box_x$ on Eq. (1), and neglect the equal-time commutator, we obtain

$$\left[\left(\frac{\partial}{\partial r_\mu} - \tfrac{1}{2} iP_\mu\right)^2 + (m+V)^2\right]\psi(r) = 0\,. \tag{4}$$

We assume that $V(r)$ may be effectively treated as an equal-time potential depending only on the spatial part of r,

$$|\vec{r}\,|^2 = r^2 - \frac{(P\cdot r)^2}{P^2}\,. \tag{5}$$

In this case ψ can also be taken as a function of $\vec{r}$, $P_\mu(\partial/\partial r_\mu)\psi(\vec{r}) = 0$, and we have

$$\left[\tfrac{1}{4}M^2 + \left(\frac{\partial}{\partial \vec{r}}\right)^2 - (m+V)^2\right]\psi(\vec{r}) = 0\,. \tag{6}$$

This equation can be interpreted as the Klein-Gordon equation for a particle with energy $M/2$, mass m, moving in a central scalar potential.

A similar procedure in the case of a spin-$\frac{1}{2}$ quark feeling a spin-independent potential due to the presence of the second spin-$\frac{1}{2}$ quark leads to a Dirac-type equation,

$$[\tfrac{1}{2}M\gamma_0 + i\vec{\mathrm{r}}\cdot\vec{\nabla} - (m+V)]_{\alpha\delta}\psi_{\delta\beta}(\vec{r}) = 0\,, \tag{7}$$

where $\psi_{\delta\beta}$ is defined by

$$\langle 0|T(\bar{q}_\beta(x')q_\delta(x))|M\rangle = \psi_{\delta\beta}(r)e^{-iP\cdot R}\,. \qquad (8)$$

In this approximation the Dirac equation involves only one of the spinor indices of ψ and thus a very specific form of spin-orbit coupling involving the spin of only one quark. Nonetheless it should allow some estimate of the magnitude of such effects. Note that the spin of the meson states is obtained by coupling the $\frac{1}{2}$ unit of spin to the total angular momentum, $j \equiv l \pm \frac{1}{2}$ (this coupling does not affect the energy levels and thus is omitted in the notation of the relevant tables). This is clearly not a realistic model of a meson (unless, perhaps, one quark is very much heavier than the other) but should, at least, indicate the general magnitude of spin-orbit effects due to our purely linear potential.

We use Eqs. (4) and (7) with $V=k|\vec{r}|$ to calculate the several spectra of interest:

(i) that appropriate to the $\psi(3.1)$ and $\psi(3.7)$ being the ground state and first excited state of the $L=0$ spectrum. Here we ignore the effects of the short-range potential (e.g., the Coulomb-type $1/|\vec{r}|$ term), which are expected to be small in the heavy-charm-quark system. (Of course, for a scalar potential constant terms in the potential can be absorbed in the definition of the mass.) We find that $k=0.137$ GeV^2 and $m=1.12$ GeV fit the ψ and ψ' masses. Results are summarized in Table I.

(ii) that appropriate to $\rho(0.77)$ and $\rho(1.25)$ being the first two levels. [We recall[3] that spectroscopy using the $\rho'(1.25)$ as the first excited state $L=0$ ρ meson has some possible advantage over that using the $\rho'(1.6)$ as first excited state.] Here the short-range forces are not likely to be small due to the light quark mass. Thus we give spectra for a typical light quark mass, $m=0.15$ GeV, and choose k to give the right spacing (0.48 GeV). The required k value 0.07 GeV^2 is essentially independent of quark mass for moderate masses. As is apparent from Table II, which summarizes the results, one requires short-range attractive effects of the order of 0.3 GeV in order to obtain the observed ρ mass. This is consistent with the considerations of Ref. 3. A more detailed discussion of the short-range effects will be given shortly.

(iii) that appropriate to the $\rho(0.77)$ and $\rho(1.6)$ being the first two $L=0$ resonances. Again using a quark mass $m=0.15$, we find that $k=0.20$ GeV^2 gives the right spacing between the first two levels (Table II). However, we need a rather strong short-range attraction, of order of 1 GeV, to obtain the relatively small ρ mass. Without such attraction the first level would lie at ~1.8 GeV. The spectrum levels for $L>1$ in cases (ii) and (iii) are not given but may fairly precisely be reconstructed by using the approximate degeneracy discussed in (b) below.

The solutions presented for cases (i) and (ii) above must be regarded as no more than typical due to the uncertain value of the "renormalized" quark mass m [see (f) below for further discussion]. The short-range attractive energy shift must be large, however, so long as positive m values are employed. Negative m values tend to yield peculiar spectrum spacings.

The method we use to solve the Dirac-type or Klein-Gordon-type equations [Eqs. (4) and (7)] is the standard numerical method of solving differential equations with appropriate boundary conditions for the bound-state wave function. The Dirac results for cases (i), (ii), and (iii) appear in

TABLE I. Klein-Gordon results for the energy levels (in GeV) of the ψ,ψ' system, $k=0.137$ GeV^2; $m=1.12$ GeV.

n	$L=0$	$L=1$	$L=2$	$L=3$
1	3.1	3.47	3.73	3.98
2	3.71	3.95	4.17	4.39
3	4.17	4.38	4.56	4.73
4	4.54	4.72	4.90	5.05
5	4.89	5.04	5.2	5.35
6	5.19	5.34	5.49	5.62
7	5.48	5.62	5.76	5.88
8	5.75	5.87	6.0	6.12
9	6.0	6.11	6.23	6.35
10	6.24	6.35	6.46	6.57
11	6.45	6.56	6.67	6.78
12	6.68	6.78	6.88	
13	6.88			

TABLE II. Klein-Gordon results for the energy levels (in GeV) of the ρ system, using (a) $\rho'(1.25)$ and (b) $\rho'(1.6)$ as a first excitation.

(a) $k=0.07$ GeV^2; $m=0.15$ GeV		(b) $k=0.21$ GeV^2; $m=0.15$ GeV	
$L=0$	$L=1$	$L=0$	$L=1$
1.13	1.4	1.81	2.28
1.61	1.79	2.64	2.98
1.96	2.12	3.27	3.54
2.25	2.39	3.78	4.0
2.52	2.63		
2.74	2.84		
2.95	3.05		
3.15	3.24		
3.34	3.43		

Tables III, IV, and V. The following features of these results are noteworthy:

(a) For a given choice of parameters and a given L value, the Klein-Gordon and Dirac equations yield essentially identical spectra. The $\vec{L}\cdot\vec{S}$ splitting coming from the linear scalar potential is negligible. $\vec{L}\cdot\vec{S}$ splitting, if significant, must arise from short-range potential effects not included here.

(b) There seems to be approximate degeneracy, analogous to that for the 3-dimensional harmonic oscillator; the degeneracy becomes more exact for the higher levels. This is due to the fact that both Dirac and Klein-Gordon equations [Eqs. (4) and (7)] have the same form as the nonrelativistic Schrödinger equation for a 3-dimensional harmonic oscillator, apart from some constant and linear terms in $|\vec{r}|$, which are not important for large distance.

(c) The $L=1$ states are above the midway points between the $L=0$ states, though not quite as much as in the nonrelativistic calculation.

(d) For the first few levels, there is no significant difference between the relativistic and nonrelativistic calculations of the spectra—even for the ρ spectra involving light quarks. After the first few levels, the relativistic spectra slowly become denser than those of the nonrelativistic system. For example, by the 6th level the nonrelativistic result is about 200 MeV above the relativistic result. In terms of the phenomenology of Ref. 3 this would imply a missing state in the $L=0$ ρ-like spectrum at about 1.89 GeV. The next states at 2.17 and 2.27 GeV (we use an attractive shift of all $L=0$ masses in Table II of 0.36 GeV) would appear to be identifiable with the $\rho(2.1)$ and $\rho(2.3)$.

TABLE III. Dirac results ($\omega=-1, j=l-\omega/2$) for the energy levels of the ψ,ψ' system, $k=0.137$ GeV^2; $m=1.12$ GeV.

E (GeV)			
$l=0, j=\frac{1}{2}$	$l=1, j=\frac{3}{2}$	$l=2, j=\frac{5}{2}$	$l=3, j=\frac{7}{2}$
3.103	3.442	3.725	3.973
3.7	3.946	4.170	4.377
4.158	4.36	4.551	4.732
4.545	4.72	4.89	5.053
4.886	5.043	5.198	5.346
5.196	5.339	5.481	5.619
5.48	5.614	5.747	5.876
5.747	5.871	5.996	6.118
5.997	6.114	6.232	6.348
6.233	6.345	6.457	6.568
6.459	6.566	6.673	6.778
6.674	6.777	6.879	
6.881			

TABLE IV. Dirac results ($\omega=1, j=l-\omega/2$) for the energy levels of the ψ,ψ' system, $k=0.137$ GeV^2; $m=1.12$ GeV.

E (GeV)			
$l=1, j=\frac{1}{2}$	$l=2, j=\frac{3}{2}$	$l=3, j=\frac{5}{2}$	$l=4, j=\frac{7}{2}$
3.47	3.757	4.006	4.23
3.965	4.194	4.403	4.597
4.374	4.56	4.753	4.926
4.731	4.95	5.07	5.507
5.053	5.21	5.361	5.768
5.347	5.493	5.633	6.015
5.621	5.756	5.883	6.249
5.878	6.005	6.128	6.473
6.12	6.24	6.358	6.687
6.35	6.465	6.577	6.892
6.57	6.679	6.787	
6.781	6.886		

(e) The $L=0$ wave function at the origin [with the relativistic normalization $\int \phi^*(\vec{r})\phi(\vec{r})d^3r = 2m/E$] is constant independent of the levels, just like the nonrelativistic case. Regardless of the normalization, the short-range attractive energy shift (calculated in the perturbation approximation of Ref. 3) may be expected to increase (by $\approx 40\%$) in going from the 1st to the 5th level instead of being completely level independent.

We reemphasize that the short-range potential form will also have a substantial effect upon the exact form of the wave function at the origin which in the nonrelativistic approximation controls the leptonic widths. However, in the relativistic case, a Coulomb-type potential introduces a mild singularity ($|\vec{r}|^{-\alpha}$) at small $|\vec{r}|$. There is then no simple relation between the leptonic decay widths and the wave function at the origin.

(f) If we ignore momentarily the ambiguity of a

TABLE V. Dirac results for the energy levels of the $L=0$ ρ system, using (a) ρ' (1.25) and (b) ρ' (1.6) as first excitation.

(a) $k=0.07$ GeV^2; $m=0.15$ GeV E (GeV)	(b) $k=0.21$ GeV^2; $m=0.15$ GeV E (GeV)
1.09	1.725
1.59	2.61
1.95	3.24
2.25	3.76
2.5	4.21
2.74	4.61
2.95	4.96
3.15	5.325
	5.65
	5.95

constant term in the potential, we may compare the parameters appropriate to the relativistic equations with the corresponding quantities in the nonrelativistic case, where the wave equation is given by

$$\left(E_{nr}+\frac{1}{m}\vec{\nabla}_r^{\,2}-k'|\vec{r}\,|\right)\psi(|\vec{r}\,|)=0\,, \tag{9}$$

with resonance mass given by

$$M=2m+E_{nr}\ .$$

For the first few levels, where presumably the $V^2(r)$ term in the relativistic equation, Eq. (6), is not important, we should compare $k'/2$ of Eq. (9) with k of Eq. (6). The results are shown in Table VI. It shows that for the ρ spectra the parameters are quite sensitive to the relativistic effect, whereas for the more massive charm quarks there is little change in the parameters as well as in the resulting spectrum. We again remind the reader that in the relativistic case it turns out that the level *spacing* is quite insensitive to the mass value used and depends only on k.

We now discuss the effect of a possible constant term in the potential. In the relativistic equation with scalar potential, the constant term is indistinguishable from the mass term. Hence the energy levels will be unchanged if we redefine the quark mass without changing k. In contrast a constant potential term in the nonrelativistic equation can be reabsorbed only by changing both the mass and the potential strength k.

(g) It is interesting to note that in the nonrelativistic treatment of the $\rho(0.77)$, $\rho(1.25)$ spectrum,[3] the choice of the parameters was suspicious because of the large difference in the k value required for it as compared to the $\psi(3.1)$, $\psi(3.7)$ system. The difference is much reduced in the relativistic case. This seems to make the interpretation that the linear potential originates from the effect of color gluons, which are SU(4) singlets, more plausible.

One may gain some feeling for these parameters by examining the large-quantum-number limit of the energy levels, for both Klein-Gordon and Dirac equations, using the WKB approximation. Since in this semiclassical limit there should not be any difference between Dirac and Klein-Gordon equations, we work with the Klein-Gordon equation for simplicity. Here the effective radial momentum is given by

$$p_r^{\,2}=\left[\frac{M^2}{4}-(k|\vec{r}\,|+m)^2\right]+\frac{(l+\frac{1}{2})^2}{|\vec{r}\,|}\ . \tag{10}$$

The quantum condition

$$\int_{r_0}^{r_1}p_r\,dr=(n+\tfrac{1}{2})\pi\ ,$$

where r_0 and r_1 are the roots of $p_r=0$, implies in the limit l large and m small

$$M^2\to 16k(n+\tfrac{1}{2}l)\ .$$

This means the leading trajectory is linear asymptotically. In the corresponding nonrelativistic case one obtains

$$M^2=\left[\left(\frac{k^2}{m}\right)^{1/3}\left(\frac{3\pi}{2}\right)^{2/3}\right]^2 n^{4/3}\ .$$

Similar nonlinear asymptotic trajectories result from a WKB treatment for both the relativistic and nonrelativistic treatments of a $k|\vec{r}\,|^2$ potential. Only the relativistically treated linear potential yields a linear trajectory, with the class of potentials of the form $k|\vec{r}\,|^\gamma$.

In conclusion a study of the quark-confinement problem using a linear potential, within the framework of relativistic Dirac and Klein-Gordon equations, indicates that the resulting spectrum does not differ very much from that of the non-

TABLE VI. Comparison of parameters appropriate to the relativistic equations with the corresponding quantities in the nonrelativistic case.

	$\psi(3.1)-\psi(3.7)$	$\rho(0.77)-\rho(1.25)$	$\rho(0.77)-\rho(1.63)$
		Nonrelativistic[a]	
m (GeV)	$\simeq$1.23	$\simeq$0.163	$\simeq$0.163
$k'/2$ (GeV^2)	0.11	0.029	0.065
		Relativistic	
m (GeV)	1.12	(0.15)[b]	(0.15)
k (GeV^2)	0.137	0.07	0.213

[a] Taken from Ref. 3.

[b] Parenthesis indicates that quark mass is relatively arbitrary.

relativistic treatment for the first few low-lying states. Substantial differences do appear for high-lying levels. Moreover, the relativistically treated linear potential yields an asymptotically linear Regge trajectory whereas the nonrelativistic treatment does not. We also find that the Dirac equation spectrum levels depend essentially only on the orbital angular momentum, indicating that spin-orbit coupling arising from a purely linear potential is negligible. The above is in contrast with the sensitivity of the local properties of the wave function to relativistic effects, especially at small distances. Thus one would expect important alterations in the calculations of various decay rates for the bound states. A full consideration of decay processes requires complete knowledge of the potential at short distances. A linear potential is expected to be a good approximation when the distance between quarks is large. However, at small distances asymptotic freedom suggests that the interaction could well be quite different—most probably a Coulomb-type $1/|\vec{r}|$ potential. As in the hydrogen atom case, this has a drastic effect on short-distance properties of the relativistic wave function and hence can potentially alter substantially decay rates calculated using a nonrelativistic treatment.[13] These considerations we leave to a future publication.

We would like to thank Professor R. Cutkosky, Professor R. Willey, Professor L. Wolfenstein, and Dr. Y. A. Chao for helpful discussions.

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*Work supported by the National Science Foundation.
†Work supported by ERDA.

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