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Spin-$\frac{1}{2}$ quantum field theory in Schwarzschild space*

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The theory of a quantized Dirac field in the maximal analytic extension of the Schwarzschild metric is presented. The wave equation and its properties under the continuous and discrete symmetries are reviewed, the Green's function is constructed, and the vacuum state so defined is shown to be stable. The excitations of the system are exhibited and the reduction formulas for the states are given.

I. INTRODUCTION

In a recent publication,[1a] the scalar field was quantized in the maximal analytic extension of Schwarzschild and Rindler spaces. The Rindler space quantization yielded precisely the usual quantization when expressed in terms of the Minkowski space to which the Rindler space is equivalent. The Schwarzschild space is more complicated because the time dependence of the metric within the past and future event horizons is real; nevertheless, it was possible to find a quantization condition that yielded a stable vacuum the energy of which as measured from infinity is a minimum.[1b] The purpose of this paper is to exhibit a similar development for the spin-$\frac{1}{2}$ field. As expected, the development goes through just as before, with precisely the same conclusions: It is possible to define a global stable vacuum despite the time variation of the space-time in the interior region.

In Sec. II, the Dirac equation and parallel transport in curvilinear spaces with a set of local Lorentz frames is briefly reviewed, and the stress-energy tensor for a Dirac field is derived, while Sec. III is devoted to the specialization to the Kruskal space-time and the derivation of the radial equation. These sections are, essentially, a review of the existing literature and are included for completeness and to establish notations, conventions, etc. The angular and spin eigenfunctions of the total angular momentum and parity are also exhibited. The properties of the solutions to the radial wave equation in the four space-time regions and the continuity conditions across the event horizons are discussed in Sec. IV. The Green's function is constructed, its symmetries exhibited, and the wave functions and reduction formulas are exhibited in Sec. V. The stability of the vacuum is briefly discussed in Sec. VI.

II. THE DIRAC EQUATION IN CURVED SPACE-TIME

The Dirac equation in curved space-time has been extensively discussed; for a more complete treatment the reader is referred to the articles of Brill and Wheeler,[2] and of Kibble[3] and the references given there.

The Dirac equation is generally given in flat, Minkowski space-time; it is a field equation for a spinor field defined relative to Minkowski coordinates. Locally, one may always introduce such coordinates, and define the Dirac spinors or whatever other field components one has relative to that basis. In general, it is convenient to introduce a set of orthonormal basis vectors, $\vec{e}_a$, at every point of space-time, with inner product

$$\vec{e}_a \cdot \vec{e}_b = \eta_{ab}\,, \tag{2.1}$$

where the sign conventions of Misner, Thorne, and Wheeler[4] will be used, $-\eta_{00}=\eta_{kk}=1$ and $\eta_{ab}=0$, $a\neq b$. The basis vectors may be chosen independently at each point and, further, there must be invariance under Lorentz transformations of the choice at each point,

$$\vec{e}_a(x) = \Lambda_a{}^b(x)\,\vec{e}_b(x)\,,$$

where (2.2)

$$\Lambda_a{}^b\,\Lambda_{a'}{}^{b'}\,\eta_{bb'} = \eta_{aa'}\,.$$

The infinitesimal version of the Lorentz transformation (discrete transformations are discussed in Sec. III) is

$$\Lambda_a{}^b(x) \simeq \delta_a{}^b + \delta\omega_a{}^b(x)\,,$$

where (2.3)

$$\delta\omega_{ab}(x) + \delta\omega_{ba}(x) = 0\,.$$

Under these transformations, the spinors (or other fields) transform just as do the corresponding fields in flat space-time:

$$\psi(x) \to \psi'(x) = e^{i\omega_{ab}(x)S^{ab}/2}\psi(x)\,,$$

the infinitesimal version of which is

$$\psi'(x) \simeq \psi(x) + \tfrac{1}{2}\,i\,\delta\omega_{ab}(x)S^{ab}\,\psi(x)\,,$$

where S^{ab} is the generator of Lorentz transformations on the spinor.

The orthonormal basis vectors may be expressed in terms of the more usual coordinate basis vectors[4] $\partial/\partial x^\mu$,

$$\vec{e}_a(x) = e_a{}^\mu(x)\frac{\partial}{\partial x^\mu}, \tag{2.5}$$

in terms of the *vierbein* components $e_a{}^\mu(x)$. The covariant derivative of $\vec{e}_a$ is

$$\begin{aligned}\vec{\nabla}_\nu \vec{e}_a &= e_a{}^\mu{}_{;\nu}\frac{\partial}{\partial x^\mu}\\ &= e_a{}^\mu{}_{;\nu} e^b{}_\mu \vec{e}_b\\ &\equiv -\omega_a{}^b{}_\nu \vec{e}_b\\ &= \omega^b{}_{a\nu}\vec{e}_b,\end{aligned} \tag{2.6}$$

where

$$f^\mu{}_{;\nu} \equiv \partial_\nu f^\mu + \Gamma_{\lambda\nu}^{\mu} f^\lambda,$$

$$\Gamma_{\lambda\nu}^{\mu} \equiv g^{\mu\sigma}\tfrac{1}{2}(g_{\lambda\sigma,\nu} + g_{\nu\sigma,\lambda} - g_{\lambda\nu,\sigma}),$$

and the covariant derivative of a vector, $\vec{A} = A^a\vec{e}_a$, is

$$\vec{\nabla}_\nu \vec{A} = (A^a{}_{,\nu} - A^b\omega_b{}^a{}_\nu)\vec{e}_a,$$

that is, (2.7)

$$(\vec{\nabla}_\nu \vec{A})^a = (\partial_\nu \delta^a{}_b + \omega^a{}_{b\nu})A^b$$

is the covariant derivative of a vector expressed in terms of the components relative to the orthonormal basis, $\vec{e}_a$. A vector is, of course, just a special case of a general spinor, and the covariant derivative may be immediately generalized to

$$\vec{\nabla}_\nu \psi = (\partial_\nu + \tfrac{1}{2} iS^{ab}\omega_{ab\nu})\psi, \tag{2.8}$$

which is covariant under local Lorentz transformations of the basis vectors $\vec{e}_a$ with respect to which the spinors ψ are defined. The matrix S^{ab} is the generator of infinitesmal Lorentz transformations for ψ.

In the Dirac theory it is desirable, if not essential, to employ both sets of basis vectors, the usual coordinate basis and the local orthonormal basis; hence the covariant derivative must be generalized to include the parallel transport of ordinary vectors, Lorentz vectors (defined relative to the local frames), and the Lorentz spinors. Consider a spinor $f^{a\mu}_{\ \lambda}$; the covariant derivative is then defined as

$$\begin{aligned}(\vec{\nabla}_\nu f)^{a\mu}_{\ \lambda} \equiv{}& \partial_\nu f^{a\mu}_{\ \lambda} + \Gamma_{\sigma\nu}^{\mu} f^{a\sigma}_{\ \lambda} - \Gamma_{\lambda\nu}^{\sigma} f^{a\mu}_{\ \sigma}\\ &+ \omega^a{}_{b\nu} f^{b\mu}_{\ \lambda} + \tfrac{1}{2} iS^{cd}\omega_{cd\nu} f^{a\mu}_{\ \lambda}\end{aligned} \tag{2.9}$$

and transforms as $\hat{f}^{a\mu}_{\ \lambda\nu}$ under both coordinate transformations and rotations of the local frames. Note that under this definition, $g_{\mu\nu}$, $e_a{}^\mu$, and any Dirac matrix γ^a have vanishing covariant derivatives. The latter property is somewhat complicated: For Dirac spinors, $S^{ab} = \frac{1}{2}\sigma^{ab}$, $\bar{\psi} = \psi^\dagger\beta$, and $\beta\sigma^{ab\dagger}\beta = \sigma^{ab}$; thus,

$$\vec{\nabla}_\nu \bar{\psi} = \partial_\nu \bar{\psi} - \tfrac{1}{2} i\bar{\psi}S^{ab}\omega_{ab\nu} \equiv \bar{\psi}(-\overleftarrow{\nabla}_\nu), \tag{2.10}$$

and the covariant derivative of a bispinor Γ^a is

$$\vec{\nabla}_\nu(\Gamma^a) = \partial_\nu\Gamma^a + \omega^a{}_{b\nu}\Gamma^b + \tfrac{1}{2} i[S^{cd}, \Gamma^a]\,\omega_{cd\nu}, \tag{2.11}$$

which vanishes for $\Gamma^a = \gamma^a$.

The commutator of the covariant derivatives yields the curvature tensor in a coordinate basis; here

$$\begin{aligned}([\vec{\nabla}_\mu, \vec{\nabla}_\nu]f)_a{}^\lambda ={}& R^\lambda{}_{\sigma\mu\nu} f_a{}^\sigma - R^b{}_{a\mu\nu} f_b{}^\lambda\\ &+ \tfrac{1}{2} iS^{cd}R_{cd\mu\nu} f_a{}^\lambda,\end{aligned}$$

where (2.12)

$$R^\lambda{}_{\sigma\mu\nu} \equiv \Gamma_{\sigma\nu,\mu}^{\lambda} - \Gamma_{\sigma\mu,\nu}^{\lambda} + \Gamma_{\tau\mu}^{\lambda}\Gamma_{\sigma\nu}^{\tau} - \Gamma_{\tau\nu}^{\lambda}\Gamma_{\sigma\mu}^{\tau}$$

and

$$\begin{aligned}R_{ba\mu\nu} &\equiv \omega_{ba\nu,\mu} - \omega_{ba\mu,\nu} - \omega_{bc\mu}\omega^c{}_{a\nu} + \omega_{bc\nu}\omega^c{}_{a\mu}\\ &= e_{b\lambda}e_a{}^\sigma R^\lambda{}_{\sigma\mu\nu},\end{aligned}$$

with the latter equality following from the vanishing of the covariant derivatives of $e^{a\tau}$ and the possibility of converting the Lorentz index a to a tensor index τ. It is now straightforward to write the Dirac action in curved space; the Minkowski action is

$$W = -\int d^4x\,\bar{\psi}\left(\frac{1}{2i}\gamma^a\overleftrightarrow{\partial}_a + m\right)\psi, \tag{2.13}$$

where units in which $\hbar = c = 1$ are used; $\overleftrightarrow{\partial}_a \equiv \vec{\partial}/\partial x^a - \overleftarrow{\partial}/\partial x^a$ and this may be made covariant by converting the ordinary derivative to a covariant derivative and including a factor of $(-g)^{1/2} \equiv (-\det g)^{1/2}$ to make the integrand a density,

$$W = -\int d^4x(-g)^{1/2}\left(\bar{\psi}\,\frac{1}{2i}\,\gamma^a\overleftrightarrow{\nabla}_a + m\right)\psi, \tag{2.14}$$

where $\overleftrightarrow{\nabla}_a \equiv e_a{}^\mu\overleftrightarrow{\nabla}_\mu$ and the ordering of the factors is immaterial because $[\vec{\nabla}_\mu, \gamma^a] = 0$, with $\vec{\nabla}_\mu$ now being considered as an operator acting on everything to its right (or left for $\overleftarrow{\nabla}_\mu$). This action is unique modulo curvature terms and the only possible term linear in the curvature is $R\bar{\psi}\psi$ which must vanish in the absence of other matter. (Possible $\bar{\psi}\gamma^a\gamma_5\nabla_a\psi$ and $\bar{\psi}\gamma_5\psi$ terms may be eliminated by standard transformations of the field.[5])

The field equation is then obtained by varying $\bar{\psi}$,

$$\left(\gamma^a\,\frac{1}{i}\,\nabla_a + m\right)\psi = 0 \tag{2.15}$$

while the variations of the *vierbein* fields, which are not dynamical, yield the stress-energy tensor

$$\delta_e W = \int d^4x\,\delta e^a{}_\mu\,\mathcal{T}^\mu{}_a. \tag{2.16}$$

However, the action is identically invariant under local Lorentz transformations, hence if $\delta e^a{}_\mu = \delta\omega^a{}_b e^b{}_\mu$ the variation must vanish, or

$$\mathcal{T}^{ab} - \mathcal{T}^{ba} = 0 \qquad (2.17)$$

and invariance under coordinate transformations,

$$\delta e^a{}_\mu = \delta x^\tau e^a{}_{\mu,\tau} + \delta x^\tau{}_{,\mu} e^a{}_\tau$$
$$= \delta x^\tau{}_{;\mu} e^a{}_\tau - \omega^a{}_{b\tau} e^b{}_\mu \delta x^\tau ,$$

implies

$$\mathcal{T}^\mu{}_{\tau;\mu} = 0 . \qquad (2.18)$$

In order to calculate the variation, the identities

$$\delta\Gamma_{\lambda\sigma}^{\mu} = \tfrac{1}{2} g^{\mu\nu}(\overleftrightarrow{\nabla}_\sigma \delta g_{\lambda\tau} + \overleftrightarrow{\nabla}_\lambda \delta g_{\sigma\nu} - \overleftrightarrow{\nabla}_\nu \delta g_{\lambda\sigma})$$

and (2.19)

$$\delta\omega_{ab\tau} = -\tfrac{1}{2}\overleftrightarrow{\nabla}_\tau(\delta e_{a\mu} e_b{}^\mu - e_a{}^\mu \delta e_{b\mu}) + \tfrac{1}{2} e_a{}^\lambda e_b{}^\sigma(\overleftrightarrow{\nabla}_\sigma \delta g_{\lambda\tau} - \overleftrightarrow{\nabla}_\lambda \delta g_{\sigma\tau})$$

are useful and lead to

$$\delta_e W = + \int d^4x \left\{ \delta e^a{}_\mu e_a{}^\mu \mathcal{L} + \delta e^a{}_\mu e_a{}^\lambda e_b{}^\mu \bar\psi \left[\left(\gamma^b \frac{1}{2i}\overleftrightarrow{\nabla}_\lambda + \frac{1}{2i}\overleftrightarrow{\nabla}_\lambda \gamma^b\right)\right]\psi - \tfrac{1}{4}\bar\psi\{\gamma^c, S^{ef}\}\psi e_c{}^\nu \delta\omega_{ef\nu} \right\}$$
$$= \int d^4x\, \delta e_{a\mu} \left\{ e^{a\mu}\mathcal{L} + \left(e_b{}^\mu \bar\psi\gamma^b \frac{1}{2i}\overleftrightarrow{\nabla}_\lambda \psi e^{a\lambda}\right) - \tfrac{1}{4}\, e_b{}^\mu \overleftrightarrow{\nabla}_c \left[\bar\psi((\{\gamma^c, S^{ab}\}) + \{\gamma^a, S^{bc}\} + \{\gamma^b, S^{ac}\})\psi\right] \right\}. \qquad (2.20)$$

This form for $\mathcal{T}^{\mu a}$ is manifestly symmetric with the exception of the two terms in bold parentheses; with the aid of the wave equation and the relation

$$\gamma^c \overleftrightarrow{\nabla}_c S^{ab} = \frac{1}{i}\overleftrightarrow{\nabla}^a \gamma^b - \frac{1}{i}\overleftrightarrow{\nabla}^b \gamma^a + S^{ab}\gamma^c \overleftrightarrow{\nabla}_c ,$$

they combine to form a symmetric result. The last two terms cancel identically, leaving

$$\delta_e W = \int d^4x\, \delta e_{a\mu}(-g)^{1/2} \times \left\{ \bar\psi\gamma^\mu \frac{1}{2i}\overleftrightarrow{\nabla}^a \psi - \tfrac{1}{4} e_b{}^\mu \overleftrightarrow{\nabla}_c[\bar\psi\{\gamma^c, S^{ab}\}\psi] \right\}$$
$$\equiv \int d^4x\, \delta e_{a\mu}(-g)^{1/2}\bar\psi\left(\gamma^\mu \frac{1}{4i}\overleftrightarrow{\nabla}^a + \gamma^a \frac{1}{4i}\overleftrightarrow{\nabla}^\mu\right)\psi$$
$$\equiv \int d^4x\, \delta e_{a\mu} e^a{}_\nu \mathcal{T}^{\mu\nu} ,$$

where $\gamma^\mu \equiv e_a{}^\mu \gamma^a$.

The symmetry, Eq. (2.17), is manifest in the second form, while the confirmation of covariant conservation requires the relations

$$\gamma^a \frac{1}{i}\overleftrightarrow{\nabla}_a \overleftrightarrow{\nabla}_b \psi = (-m\overleftrightarrow{\nabla}_b + \gamma^a S^{cd} R_{cdab})\psi$$

and

$$0 = \left(m - \gamma^a \frac{1}{i}\overleftrightarrow{\nabla}_a\right)\left(m + \gamma^b \frac{1}{i}\overleftrightarrow{\nabla}_b\right)\psi$$
$$= (m^2 - \overleftrightarrow{\nabla}_a\overleftrightarrow{\nabla}_a - iS^{ab}S^{cd}R_{abcd})\psi ,$$

from which it is easy to show that $\mathcal{T}^{\mu\nu}$ is conserved.

If a representation of the Dirac matrices such that the γ^μ are imaginary is chosen, the Dirac equation is real and the fields may be chosen to be Hermitian. If they are not already Hermitian, let

$$\psi_1 = (\psi + \psi^\dagger)/\sqrt{2} ,$$
$$\psi_2 = (\psi - \psi^\dagger)/\sqrt{2}\, i ;$$

then

$$\bar\psi\gamma^a \frac{1}{i}\overleftrightarrow{\nabla}_a \psi = \tfrac{1}{2}(\psi_1 - i\psi_2)\beta\gamma^a \frac{1}{i}\overleftrightarrow{\nabla}_a(\psi_1 + i\psi_2)$$
$$= \tfrac{1}{2}(\psi_1 \;\; \psi_2)\beta\gamma^a \frac{1}{i}\overleftrightarrow{\nabla}_a \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$$
$$+ \text{total divergence},$$

and the action may be replaced by

$$W = -\int d^4x(-g)^{1/2}\tfrac{1}{2}\psi\beta\left(\gamma^a \frac{1}{2i}\overleftrightarrow{\nabla}_a + m\right)\psi$$

and, likewise, the stress tensor acquires an additional factor of $\frac{1}{2}$ and $\bar\psi \cdots \psi \to \psi \circ\circ\circ \psi$.

The only possible complication is with neutrinos for which $(1 - i\gamma_5)\psi = 0$; this implies that

$$\psi_2 = \gamma_5 \psi_1$$

and only one field survives: If the neutrino has only one helicity, a single Hermitian field describes both it and its antiparticle with the opposite helicity.

III. THE DIRAC EQUATION IN SCHWARZSCHILD COORDINATES

For the Schwarzschild space, the metric in Schwarzschild coordinates is given by (units in which $\hbar = c = G = 1$ are used)

$$ds^2 = -(1 - 2M/r)dt^2 + dr^2/(1 - 2M/r) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{3.1}$$

for the entire space-time. There is a coordinate singularity at $r = 2M$ and a real singularity at $r = 0$. The maximal analytic extension of the metric has been given by Kruskal,[6] and the full space-time is shown in the Kruskal diagram of Fig. 1. Region I is the ordinary exterior region, $r > 2M$, while F is the future interior region, $r < 2M$, P is the past interior region, and II is a second exterior region, every point of which is spacelike with respect to every point of I.

The metric has the same functional form in each of the four regions and t runs from $-\infty$ to $+\infty$ in each region. In region I (II), t is a timelike coordinate and the direction of increasing t is towards later (earlier) proper times. In regions F (P) r is the timelike coordinate and decreasing r is the direction of later (earlier) times; the t coordinate is the spacelike coordinate. The coordinate singularity at $r = 2M$, $t = \pm\infty$ may be removed by choosing coordinates $\tau = t \pm r^*$, where

$$r^* = r + 2M \ln(|r - 2M|/2M)\,; \tag{3.2}$$

then physical continuity requirements must be imposed across the event horizon, $r = 2M$. Alternatively, both singularities may be removed by employing Kruskal coordinates

$$\begin{aligned} u &= \pm 4Me^{r^*/4M}\cosh(t/4M)\,,\\ v &= \pm 4Me^{r^*/4M}\sinh(t/4M)\,, \end{aligned}$$

for regions I and II, and

$$\begin{aligned} u &= \pm 4Me^{r^*/4M}\sinh(t/4M)\,,\\ v &= \pm 4Me^{r^*/4M}\cosh(t/4M)\,, \end{aligned} \tag{3.3}$$

for regions F and P, and

$$ds^2 = -(2M/r)e^{-(r/2M)}(dv^2 - du^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

The two obvious choices for the basis vectors of the local frames[2] are unit vectors along the coordinate basis vectors or along Cartesian vectors oriented relative to fixed orthogonal directions (the spherical symmetry makes such a specification possible in the exterior regions). It is more convenient to orient the local frames relative to the coordinate basis; the *vierbein* components $e_a{}^\mu$ and $e_{a\mu}$ in the Schwarzschild coordinates are given in Table I with $\Phi \equiv M/r$ while those in Kruskal coordinates are given in Table II.

The affine connections in the orthonormal basis, $\omega_{ab\mu} = e_a{}^\lambda e_{b\lambda;\mu}$, may be calculated directly from the affine connections of the Schwarzschild coordinate basis, the nonvanishing elements of which are

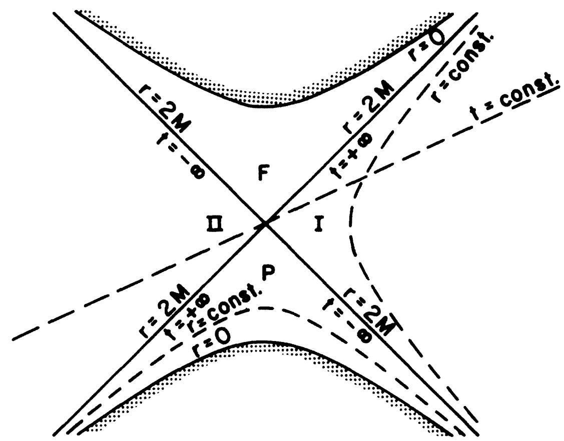

FIG. 1. Maximal analytic extension of the Schwarzschild metric.

$$\begin{aligned} \Gamma_{00}^{\ 1} &= (1 - 2\Phi)^2\Gamma_{10}^{\ 0}\\ &= -(1 - 2\Phi)^2\Gamma_{11}^{\ 1}\\ &= -\phi'(1 - 2\Phi)\,,\\ \Gamma_{33}^{\ 1} &= \sin^2\theta\, \Gamma_{22}^{\ 1}\\ &= -r\sin^2\theta(1 - 2\Phi)\,,\\ \Gamma_{13}^{\ 3} &= \Gamma_{12}^{\ 2} = 1/r\,, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \Gamma_{33}^{\ 2} &= -\sin^2\theta\, \Gamma_{23}^{\ 3}\\ &= -\sin\theta\cos\theta\,; \end{aligned}$$

the results are collected in Table III.

It is now straightforward to write down the Dirac equation, $[\gamma^a(1/i)\vec{\nabla}_a + m]\psi = 0$, as

$$\left[m + \gamma^2 \frac{1}{r\sin^{1/2}\theta}\,\frac{1}{i}\,\partial_\theta \sin^{1/2}\theta + \gamma^3 \frac{1}{r\sin\theta}\,\frac{1}{i}\,\partial_\phi + \epsilon\left(\frac{\gamma^0}{w}\,\frac{1}{i}\,\partial_t + \gamma^1\,\frac{1}{r}\,w^{1/2}\,\frac{1}{i}\,\partial_r w^{1/2} r\right)\right]\psi = 0\,,$$

for $r > 2M$ and $\epsilon = 1$ (-1) in I (II), and (3.5)

$$\left[m + \gamma^2 \frac{1}{r\sin^{1/2}\theta}\,\frac{1}{i}\,\partial_\theta \sin^{1/2}\theta + \gamma^3 \frac{1}{r\sin\theta}\,\frac{1}{i}\,\partial_\phi - \epsilon\left(\gamma^0\,\frac{1}{r}\,w^{1/2}\,\frac{1}{i}\,\partial_r w^{1/2} r - \frac{\gamma^1}{w}\,\frac{1}{i}\,\partial_t\right)\right]\psi = 0\,,$$

$$w(r) \equiv |1 - 2\Phi(r)|^{1/2}$$

for $r < 2M$ and $\epsilon = 1$ (-1) in F (P). The appearance of the ϵ's reflects the differing directions associated with increasing t.

Just as in flat space, the spherical symmetry implies the existence of a conserved angular

TABLE I. (a) *Vierbein* components $e_a{}^\mu$ for Schwarzschild coordinates. The local frames are oriented so that the axes are parallel to the coordinate axes. The upper signs refer to regions I, F while the lower signs refer to regions II, P. (b) *Vierbein* components $e_{a\mu}$ for Schwarzschild coordinates. The sign conventions are the same as for (a).

Lorentz index \ Schwarzschild coordinate index	t	r	θ	ϕ
		(a)		
0	$\pm 1/(1-2\Phi)^{1/2}$ $r>2M$	$\mp(2\Phi-1)^{1/2}$ $r<2M$	0	0
1	$\pm 1/(2\Phi-1)^{1/2}$ $r<2M$	$\pm(1-2\Phi)^{1/2}$ $r>2M$	0	0
2	0	0	$1/r$	0
3	0	0	0	$(1/r)\sin\theta$
		(b)		
0	$\mp(1-2\Phi)^{1/2}$ $r>2M$	$\pm 1/(2\Phi-1)^{1/2}$ $r<2M$	0	0
1	$\pm(2\Phi-1)^{1/2}$ $r<2M$	$\pm 1/(1-2\Phi)^{1/2}$ $r>2M$	0	0
2	0	0	r	0
3	0	0	0	$r\sin\theta$

momentum in addition to the conserved energy (momentum for $r<2M$) associated with t translation invariance. Because of the latter, one may obtain the Fourier transform of the equation with respect to t, replacing $(1/i)\partial_t$ by $-\omega$. In Cartesian coordinates, the generators of rotations on ψ are $\vec{J}=\vec{r}\times(1/i)\vec{\nabla}+\frac{1}{2}\vec{\sigma}$ which only refer to the angular variables, ψ and θ. However, to complete the parallel, the spinors must be rotated so that

$$R\hat{r}\cdot\vec{\gamma}R^{-1}=\gamma^1,\quad R\hat{\theta}\cdot\vec{\gamma}R^{-1}=\gamma^2,$$

and

$$R\hat{\psi}\cdot\vec{\gamma}R^{-1}=\gamma^3;$$

TABLE II. (a) *Vierbein* components $e_a{}^\mu$ for Kruskal coordinates. The local frames are oriented so that the axes are parallel to the coordinate axes. (b) *Vierbein* components $e_{a\mu}$ for Kruskal coordinates.

Lorentz index \ Kruskal coordinate index	v	u	θ	ϕ
		(a)		
0	f^{-1}	0	0	0
1	0	f^{-1}	0	0
2	0	0	$1/r$	0
3	0	0	0	$(1/r)\sin\theta$
		$f=[e^{-r/2M}(2M/r)]^{1/2}$		
		(b)		
0	$-f$	0	0	0
1	0	f	0	0
2	0	0	r	0
3	0	0	0	$r\sin\theta$

TABLE III. (a) Components of $\omega_{ab\mu}$ for $r>2M$ presented as an ab matrix, $(\omega_\mu)_{ab}$. The upper signs refer to regions I, F, while the lower signs refer to II, P. (b) Components of $\omega_{ab\mu}$ for $r<2M$.

	0	1	2	3
		(a)		
0	0	$\delta_\mu^t\Phi'$	0	0
1	$-\delta_\mu^t\Phi'$	0	$\mp\delta_\mu^\theta(1-2\Phi)^{1/2}$	$\mp\delta_\mu^\phi(1-2\Phi)^{1/2}\sin\theta$
2	0	$\pm\delta_\mu^\theta(1-2\Phi)^{1/2}$	0	$-\delta_\mu^\phi\cos\theta$
3	0	$\pm\delta_\mu^\phi(1-2\Phi)^{1/2}\sin\theta$	$\delta_\mu^\phi\cos\theta$	0
		(b)		
0	0	$-\Phi'\delta_\mu^t$	$\mp(2\Phi-1)^{1/2}\delta_\mu^\theta$	$\mp\sin\theta\,(2\Phi-1)^{1/2}\delta_\mu^\phi$
1	$\Phi'\delta_\mu^t$	0	0	0
2	$\pm(2\Phi-1)^{1/2}\delta_\mu^\theta$	0	0	$-\cos\theta\,\delta_\mu^\phi$
3	$\pm\sin\theta\,(2\Phi-1)^{1/2}\delta_\mu^\phi$	0	$\cos\theta\,\delta_\mu^\phi$	0

this is achieved by

$$R=e^{-i\pi\sigma^{23}/4}e^{-i(\pi/2-\theta)\sigma^{31}/2}e^{i\phi\sigma^{12}/2} \tag{3.6}$$

and the generators of rotations become

$$J_3=\frac{1}{i}\partial_\phi,\qquad J_\pm=e^{\pm i\phi}\left(\pm\partial_\theta-\cot\theta\,\frac{1}{i}\partial_\phi+\frac{1}{2}\frac{\sigma^1}{\sin\theta}\right). \tag{3.7}$$

Note that the derivatives are not covariant derivatives acting on the Dirac field, but are the generalizations of the Lie derivatives for spinor fields. It is then straightforward to verify that

$$[\vec{J},\mathfrak{D}]=0,$$

where $\mathfrak{D}$ is the Dirac equation operator given in Eqs. (3.5) and $\vec{J}$ represents the three operators displayed in Eq. (3.7). The commuting operators J_3 and J^2 do not completely characterize the angular and spin dependence because the total angular momentum includes both orbital and spin angular momentum. However, the two possible states, $j=l\pm\frac{1}{2}$ are characterized by different parities, hence they are not mixed by the free Dirac equation, nor are they by $\mathfrak{D}$. The operator k defined by Dirac to designate the two states,[7]

$$k=\beta(1+\vec{\sigma}\cdot\vec{L}),$$

which in this representation becomes

$$k=i\beta\gamma^1\left(\gamma^2\frac{1}{\sin^{1/2}\theta}\frac{1}{i}\partial_\theta\sin^{1/2}\theta+\gamma^3\frac{1}{\sin\theta}\frac{1}{i}\partial_\phi\right), \tag{3.8}$$

has eigenvalues $\pm(j+\frac{1}{2})$ and commutes with the Dirac equation $\mathfrak{D}$.

The matrices $i\beta\gamma^1\gamma^2$ and $i\beta\gamma^1\gamma^3$ anticommute with each other and commute with β, α^1, and γ^1; it is therefore convenient to take the representation of the Dirac matrices to be, in terms of a direct product $\vec{\rho}\otimes\vec{\sigma}$ of two-dimensional matrices:

$$\beta=\rho_2,\quad \alpha^1=\rho_3,\quad \alpha^2=\rho_1\sigma_3,\quad \alpha^3=\rho_1\sigma_1,$$
$$\gamma^1=i\rho_1,\quad \gamma^2=-i\rho_3\sigma_3,\quad \gamma^3=-i\rho_3\sigma_1,\quad \gamma^5=i\rho_3\sigma_2,$$

which is related to the usual representation by the unitary transformation

$$V=e^{-i\pi\rho_3/4}e^{-i\pi\rho_1/4}e^{i\pi\sigma_3/4}e^{i\pi(1-\sigma_2)(1-\rho_2)/4},$$

$$V\beta V^\dagger=\rho_3 \text{ and } V\alpha^iV^\dagger=\rho_1\sigma^i.$$

Then, the Dirac equation is real, k operates in the two-dimensional σ subspace and the eigenvectors of J_3 and k completely determine the angular and (two-component) spin dependence,

$$k\mathcal{Y}_{k'}^{m}(\theta,\phi)=k'\mathcal{Y}_{k'}^{m}(\theta,\phi),$$
$$J_3\mathcal{Y}_{k'}^{m'}(\theta,\phi)=m'\mathcal{Y}_{k'}^{m'}(\theta,\phi), \tag{3.9}$$
$$J^2\mathcal{Y}_{k'}^{m'}(\theta,\phi)=(k'^2-\tfrac{1}{4})\mathcal{Y}_{k'}^{m'}(\theta,\phi)$$

to form a complete orthonormal set in that subspace

$$\int d\theta\,d\theta\,\sin\theta[\mathcal{Y}_{k''}^{m''}(\theta,\phi)]^\dagger\mathcal{Y}_{k'}^{m'}(\theta,\phi)=\delta^{m',m''}\delta_{k',k''}, \tag{3.10}$$

$$\sum_{k',m'}\mathcal{Y}_{k'}^{m'}(\theta,\phi)[\mathcal{Y}_{\lambda'}^{m'}(\theta',\phi')]^\dagger=\delta(\phi-\phi')\frac{\delta(\theta-\theta')}{\sin\theta}.$$

Explicit formulas for the harmonics $\mathcal{Y}_{k'}^{m}$ are given in Appendix A.

The field ψ is then expanded in $\mathcal{Y}$ and $e^{-i\omega t}$,

$$\psi(x) = \sum_{k',m'} \int \frac{d\omega}{2\pi} e^{-i\omega t} \mathcal{Y}_{k'}^{m'}(\theta,\phi) \frac{\psi_{k'}(r,\omega)}{r w^{1/2}(r)} ,$$

and $\psi_{k'}(r,\omega)$ is a two-component vector in the ρ subspace which must satisfy

$$\{m - i\alpha^1(k'/r) + \epsilon[-\gamma^0 \omega w^{-1}(r) + \gamma^1 w(r)(-i)\partial_r]\}\psi_{k'}(r,\omega) = 0 ,$$

for $r > 2M$ with $\epsilon = 1$ (-1) in I (II), and (3.11)

$$\{m - i\alpha^1(k'/r) - \epsilon[\gamma^0 w(r)(-i)\partial_r + \gamma^1 \omega w^{-1}(r)]\}\psi_{k'}(r,\omega) = 0 ,$$

for $r < 2M$ with $\epsilon = 1$ (-1) in F (P).

The solutions to these equations are discussed in Sec. IV. The only remaining task here is to exhibit the transformation properties of the equations under the discrete invariances of the space-time. These invariances are severalfold. First, consider the exterior region I. The usual parity P, time reversal T, charge conjugation C, and TCP (Θ) invariances hold. Because ψ is taken to be Hermitian, charge conjugation is completely disjoint from the space-time transformations and will not be discussed here; also T and TCP are not independent, hence only TCP will be discussed with T to be composed from TCP, P, and C if desired. There is an additional symmetry, that of mapping I onto II: the two exterior regions are isomorphic, hence the reflection of u into $-u$ is also an invariance. In the exterior regions, this invariance is just the mapping of the functions of one region into those of the other, but in the interior regions the t reflection is a second parity operation. The time-reversal operation maps the interior regions into each other.

First, consider the parity operation in ordinary Minkowski space-time.[8] It is

$$P\psi(t,r,\hat{r})P^{-1} = i\beta\psi(t,r,-\hat{r}) , \quad (3.12)$$

but if the local basis oriented relative to $(\hat{r}, \hat{\theta}, \hat{\phi})$ is used, $\psi \to R^{-1}(\hat{r})\psi$ and

$$i\beta \to R(\hat{r}) i\beta R^{\dagger}(-\hat{r}) = -\beta\sigma^{31} = \sigma_1 .$$

Thus, the parity operation is

$$P_I\,\psi(t,r,\theta,\phi)P_I^{-1} = \sigma_1\psi(t,r,\pi-\theta,\phi+\pi) \quad (3.13)$$

and it is straightforward to verify that the action is invariant under this transformation in each region.

The TCP transformation may be similarly inferred,

$$R^*(\hat{r})\gamma^5 R^{\dagger}(-\hat{r}) = i\gamma^5\sigma^{31} = \rho_1\sigma_3 ,$$

and, because the transformation is antiunitary,

$$\Theta\psi(t,r,\theta,\phi)\Theta^{-1} = \rho_1\sigma_3\psi^*(-t,r,\pi-\theta,\pi+\phi) \quad (3.14)$$

in the regions I and II. In P and F, the same results hold except that the coordinate labels now refer to the interchanged regions.

In order to obtain the transformation which interchanges I and II it is easiest to consider the parity transformation in F. That transformation reverses t and may be chosen to leave ϕ and θ unchanged. Then, the transformation may be taken to be

$$P_F\psi(t,r,\theta,\phi)P_F^{-1} = \rho_2\sigma_2\psi(-t,r,\theta,\phi) , \quad (3.15)$$

under which the action is invariant in both F and P; in I and II, the action is also invariant if the regions are also interchanged. The σ_2 factor causes a change of the angular momentum-parity eigenvalue k'; however, the large components of the new wave function have the same orbital angular momentum as did the large components of the old wave function. The change in k', for a particle at rest at infinity, reflects the change in the intrinsic parity associated with the particle in the two regions (I and II); this change is necessary if the parity, P_I, is to be continuously defined over the entire space, specifically within the regions F and P.

IV. SOLUTIONS TO THE RADIAL WAVE EQUATION

It is straightforward to discuss the behavior of the various solutions to the radial equations. In terms of the representation given in Sec. III the radial equation becomes, after multiplying by $\beta = \gamma^0$,

$$\left[\left(m\rho_2 + \rho_1 \frac{k'}{r}\right) + \epsilon\left(\rho_3 w \frac{1}{i}\frac{d}{dr} - \frac{\omega}{w}\right)\right]\psi_{k'}(r,\omega) = 0 ,$$

for $r > 2M$ with $\epsilon = 1$ (-1) in I (II), and (4.1)

$$\left[\left(m\rho_2 + \rho_1 \frac{k'}{r}\right) - \epsilon\left(w \frac{1}{i}\frac{d}{dr} + \frac{\rho_3\omega}{w}\right)\right]\psi_{k'}(r,\omega) = 0 ,$$

for $r < 2M$ with $\epsilon = 1$ (-1) in F (P). For large r, $r^* \sim r$ and the two possible asymptotic behaviors are

$$e^{\pm i q(\omega) r}\begin{pmatrix} [\omega \pm q(\omega)]^{1/2} \\ i\epsilon[\omega \mp q(\omega)]^{1/2} \end{pmatrix} ,$$

hence the two independent solutions may be characterized by their behavior at infinity,

$$\psi^a_{k'}(r,\omega) \underset{r\sim\infty}{\sim} \begin{pmatrix} (\omega+q)^{1/2} \\ i\epsilon(\omega-q)^{1/2} \end{pmatrix} \left\{ \exp\left[+iq\left(r^* + \frac{2m^2M}{q}\ln r\right)\right] + O\left(\frac{1}{r}\right)\right\},$$

$$\psi^b_{k'}(r,\omega) \underset{r\sim\infty}{\sim} \begin{pmatrix} (\omega-q)^{1/2} \\ i\epsilon(\omega+q)^{1/2} \end{pmatrix} \left\{ \exp\left[-iq\left(r^* + \frac{2m^2M}{q}\ln r\right)\right] + O\left(\frac{1}{r}\right)\right\}, \quad (4.2)$$

where $q(\omega)=(\omega^2-m^2)^{1/2}$ and the branch is chosen so that $\mathrm{Im}q>0$ and the branch of $(\omega+q)^{1/2}$ is chosen so that $\mathrm{Im}(\omega+q)^{1/2}>0$ with $(\omega+q)^{1/2}(\omega-q)^{1/2}=m$.

Near $r=2M$, the solutions behave as $e^{\pm i(\omega^2)^{1/2}r^*}$, hence two other independent solutions may be chosen:

$$\psi^{(\pm)}_{k'}(r,\omega) \underset{r\gtrsim 2M}{\sim} \begin{pmatrix} [\omega\mp(\omega^2)^{1/2}]^{1/2} \\ i[\omega\pm\omega^2)^{1/2}]^{1/2} \end{pmatrix} e^{\mp i(\omega^2)^{1/2}r^*}, \quad (4.3)$$

where $\mathrm{Im}(\omega^2)^{1/2}>0$ and $\mathrm{Im}[\omega\pm(\omega^2)^{1/2}]^{1/2} \gtrless 0$. On the first sheet, the functions $\psi^{(\pm)}$ and $\psi^{(ab)}$ are analytic in the cut ω plane except that $\psi^{(-)}$ has poles along the imaginary ω axis. [As is familiar from Bessel's equation, for $\omega=\pm in/2M$ the two independent solutions behave as $(r-2M)^n$ and $(r-2M)^{-n}+\cdots+(r-2M)^n\ln(r-2M)$ rather than $(r-2M)^{\pm n}$.] As there are only two independent solutions,

$$\psi^a_{k'}(r,\omega)=\alpha^{k'}(\omega)\psi^{(-)}_{k'}(r,\omega)+\beta^{k'}(\omega)\psi^{(+)}_{k'}(r,\omega)$$

and (4.4)

$$\psi^b_{k'}(r,\omega)=\gamma^{k'}(\omega)\psi^{(-)}_{k'}(r,\omega)+\delta^{k'}(\omega)\psi^{(+)}_{k'}(r,\omega).$$

The consequences of the various symmetries are recorded in Appendix B.

In the interior regions only the behavior near $r=2M$ is of interest, and the forms are analogous to those in the exterior regions:

$$\psi^{(\pm)}_{k'}(r,\omega) \underset{r\lesssim 2M}{\sim} \begin{pmatrix} [\omega\mp(\omega^2)^{1/2}]^{1/2} \\ i[\omega\pm(\omega^2)^{1/2}]^{1/2} \end{pmatrix} e^{\mp i(\omega^2)^{1/2}r^*}. \quad (4.5)$$

The radial equation, Eq. (4.1), is invariant under (1) complex conjugation and $k'\to -k'$, $\omega\to-\omega^*$, and (2) multiplication by ρ_2, $\omega\to-\omega$, and $k'\to-k'$. The first is the Hermiticity relation while the second is a parity transformation, involving reflection of the t coordinate; the full transformations given in Appendix B are found by combining these with the first and second transformations of the spherical harmonics given in Appendix A [Eq. (A15)].

The remaining task for this section is to discuss the boundary conditions across the event horizons. This was discussed in Ref. 1 for a scalar wave, where it was shown that when a wave packet was constructed using

$$e^{-i\omega t}\,\frac{\psi^{(\pm)}_{k'}(r,\omega)}{r w^{1/2}(r)},$$

the wave propagated across the event horizon for which $\omega t+(\omega^2)^{1/2}r^*$ remained finite, that is the $t=\pm\infty$ horizons for $(\omega^2)^{1/2}=\pm\omega$, and $\psi^{(-)}$ propagates across the other horizons. Here the additional factor of $w^{-1/2}$ intervenes and, although the wrong wave vanishes across event horizon when used to form a packet, the correct wave is still not continuous. This is due to the misalignment of the local frames with respect to which the spinors are defined. Not only must continuous coordinates be used but also the local frames on each side of the event horizon must be aligned so that the unit vectors are parallel.

In order to align the local frames, first express their basis vectors $e_a{}^\mu$ in terms of continuous coordinates u and v (the angular variables θ and ϕ are already continuous as are the unit basis vectors parallel to them). Then

$$e_a{}^u=e_a{}^t\,\frac{\partial u}{\partial t}+e_a{}^r\,\frac{\partial u}{\partial r}$$

$$e_a{}^v=e_a{}^t\,\frac{\partial v}{\partial t}+e_a{}^r\,\frac{\partial v}{\partial r} \quad (4.6)$$

and the *vierbein* components inside and out are given by

$$e_a{}^\mu=\begin{matrix}\\0\\1\end{matrix}\begin{pmatrix} u & v \\ 0 & u/v \\ u/v & 0\end{pmatrix}|v|/w4M$$

on the outside (I or II) and (4.7)

$$e_a{}^\mu=\begin{matrix}\\0\\1\end{matrix}\begin{pmatrix} u & v \\ 0 & v/u \\ v/u & 0\end{pmatrix}|u|/w4M$$

on the inside (F or P). The discontinuity is here in the *vierbein* components even when expressed in terms of continuous coordinates; the time axis is rotating closer and closer to the light cone as the event horizon is approached. This can be avoided by reorienting the frames; a Lorentz transformation,

$$L_{\bar{a}}{}^{a'} = \begin{pmatrix} \cosh(t/4M) & -\sinh(t/4M) \\ -\sinh(t/4M) & \cosh(t/4M) \end{pmatrix} \tag{4.8}$$

produces

$$e_{\bar{a}}{}^{\mu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{r/4M} \tag{4.9}$$

on either side. The Lorentz transformation is a boost $-t/4M$ in the t-r plane, thus the spinors must be transformed by $e^{-i(t/4M)\sigma^{01}/2} = e^{(t/8M)\alpha^1}$, which in the representation used here is $e^{(t/8M)\rho_3}$ and, when used to form a wave packet,

$$e^{(t/8M)\alpha^1}\psi_{k'}^{(\pm)}(r,\omega)e^{-i\omega t}/w^{1/2}$$

should be continuous across the event horizons at which it does not vanish. First, let $(\omega^2)^{1/2} = \omega$, then

$$e^{(t/8M)\alpha^1}\,\frac{\psi_{k'}^{(+)}(r,\omega)}{w^{1/2}}\,e^{-i\omega t}$$

$$\sim e^{-i\omega(t+r^*)}\begin{pmatrix} 0 \\ i(2\omega)^{1/2} \end{pmatrix}\frac{e^{-t/8M}}{w^{1/2}}$$

$$= \left(\frac{|u+v|}{4M}\right)^{-i\omega 4M}\frac{(r/2M)^{1/4}}{(|u+v|/4M)^{1/2}}\begin{pmatrix} 0 \\ i(2\omega)^{1/2} \end{pmatrix} \tag{4.10}$$

from either side which is continuous for the event horizon $u=v$. Thus, similar arguments for the other 3 cases imply that for ω just above the real axis, $\omega \to \omega + i\epsilon$, $\psi^{(\pm)}$ is "continuous" across the $t=\pm\infty$ event horizon and just below, $\omega \to \omega - i\epsilon$, $\psi^{(\mp)}$ is "continuous" across the $t=\pm\infty$ event horizon. No additional factors need be included.

V. THE GREEN'S FUNCTION

With the properties of the solutions to the wave equation in the various sectors in hand, the procedure for constructing the Green's function follows exactly that given in Ref. 1a for the scalar field, thus only those aspects which are different will be discussed here.

As before, the Green's function is to be the time-ordered product of the fields, but now the fields are fermion rather than boson fields, hence

$$S(x,x') = \begin{cases} i\langle 0|\psi(x)\psi(x')|0\rangle, & x \in J^+(x') \\ -i\langle 0|\psi(x')\psi(x)|0\rangle, & x' \in J^+(x) \\ i\langle 0|\psi(x)\psi(x')|0\rangle = -i\langle 0|\psi(x')\psi(x)|0\rangle, & (x,x') \text{ spacelike,} \end{cases} \tag{5.1}$$

where ψ has been taken to be a Hermitian field. As with the scalar field, S may be expanded in spherical harmonics and Fourier transformed with respect to the t coordinate:

$$S(x,x') = \sum_{k',m'} \frac{\mathcal{Y}_{k'}^{m'}(\theta,\phi)\mathcal{Y}_{k'}^{m'\dagger}(\theta',\phi')}{rr'[w(r)w(r')]^{1/2}} \times \int_{\infty}^{\infty}\frac{dt}{2\pi}\,e^{-i\omega(t-t')}S^{k'}(r,r';\omega) \tag{5.2}$$

and, owing to the anticommutation relations,

$$\delta(n(x-x'))\{\psi(x),\psi(x')\} = \frac{\delta(x-x')}{(-g)^{1/2}}\ , \tag{5.3}$$

where n is a unit timelike vector lying in the forward light cone,

$$\beta\left(m + \frac{1}{i}\gamma^a\vec{\nabla}_a\right)S(x-x') = \delta(x-x')/(-g)^{1/2} \tag{5.4}$$

or

$$\left[w(r)(m\rho_2 + \rho_1 k'/r) + \epsilon\left(\rho_3\,\frac{1}{i}\,\frac{d}{dr^*} - \omega\right)\right]S^{k'}(r,r';\omega) = \delta(r-r')w^2(r)\,,$$

for $r > 2M$ and $\epsilon = 1$ (-1) for $r \in$ I (II), and (5.5)

$$\left[w(r)(m\rho_2 + \rho_1 k'/r) + \epsilon\left(\frac{1}{i}\,\frac{d}{dr^*} - \omega\rho_3\right)\right]S^{k'}(r,r';\omega) = \delta(r-r')w^2(r),$$

for $r < 2M$ and $\epsilon = 1$ (-1) for $r \in$ F (P); in all cases the δ functions are taken to vanish if r and r' do not lie in the same sector. For $r, r' \in$ I or II, the Green's function must be well behaved as $r \to \infty$ or

$2M$, and as $r \to \infty$ the usual positive-frequency result must obtain. From this, the usual form for the Green's function in the exterior region follows,

$$S^{k'}(r,r';\omega) = i\theta(r-r')\psi^a_{k'}(r,\omega)\psi^{(+)}_{k'}(r',\omega^*)^\dagger A_{k'}(\omega) - i\theta(r'-r)\psi^{(+)}_{k'}(r,\omega)\psi^a_{k'}(r',\omega^*)^\dagger B_{k'}(\omega)\,, \tag{5.6}$$

and the application of the wave equation, Eq. (5.5) yields

$$1 = \epsilon\rho_3[\psi^a_{k'}(r,\omega)\psi^{(+)}_{k'}(r,\omega^*)^\dagger A_{k'}(\omega) + \psi^{(+)}_{k'}(r,\omega)\psi^a_{k'}(r,\omega^*)^\dagger B_{k'}(\omega)]$$

or

$$\psi^{(+)}_{k'}(r,\omega^*)^\dagger\rho_3\psi^a_{k'}(r,\omega)\epsilon A_{k'} = 1$$

and

$$\psi^a_{k'}(r,\omega^*)^\dagger\rho_3\psi^{(+)}_{k'}(r,\omega)\epsilon B_{k'}(\omega) = 1\,,$$

hence

$$S^{k'}(r,r';\omega) = \frac{i}{2(\omega^2)^{1/2}\epsilon\alpha^{k'}(\omega)} \times [\theta(r-r')\psi^a_{k'}(r,\omega)\psi^{(+)}_{k'}(r',\omega^*)^\dagger + \theta(r'-r)\epsilon\psi^{(+)}_{k'}(r,\omega)\psi^a_{k'}(r',\omega^*)^\dagger]\,. \tag{5.7}$$

The antiunitary TCP transformation yields the negatively time-ordered product rather than the positively time-ordered product considered here; however, since that propagator is obtained by integrating the same $S^{k'}(r,r';\omega)$ from $-\infty + i\epsilon$ to $\infty - i\epsilon$ rather than $-\infty - i\epsilon$ to $\infty + i\epsilon$ as here it also yields an invariance. The choice of Hermitian fields implies that

$$S^{k'}(r,r';\omega) = -S^{-k'}(r',r,-\omega) \tag{5.8}$$

while TCP implies

$$\rho_1 S^{k'}(r,r';\omega)^*\rho_1 = S^{k'}(r,r';\omega^*) \tag{5.9}$$

and the P_F invariance which interchanges the I and II sectors implies

$$\rho_2 S^{k'}_{\mathrm{I}}(r,r';\omega)\rho_2 = S^{-k'}_{\mathrm{II}}(r,r';-\omega)\,, \tag{5.10}$$

all of which are satisfied identically by the form Eq. (5.7) with the aid of the transformation properties of Appendix B, Eqs. (B5), (B6), and (B7).

The Green's function for $x \in \mathrm{F}$ and $x' \in \mathrm{I}$, II may now be obtained by taking the Fourier transform with respect to ω and then letting x cross into F. The continuity conditions from Sec. IV yield the $e^{-i\omega t}\psi^{(+)}_{k'}(r,\omega)$ term, and the arguments

TABLE IV. The Green's function $S^{k'}(r,r';\omega)$ for r, r' in the four sectors. The Green's function related to its Fourier transform

$$S^{k'}(r,t;r',t') = \int_{-\infty-i\epsilon}^{\infty+i\epsilon}\frac{d\omega}{2\pi}\,e^{-i\omega(t-t')}\,S^{k'}(r,r';\omega)\,.$$

	$r' \in \mathrm{I}$	$r' \in \mathrm{P}$	$r' \in \mathrm{II}$	$r' \in \mathrm{F}$
$r \in \mathrm{I}$	$\frac{i}{2(\omega^2)^{1/2}\alpha^{k'}_{\mathrm{I}}(\omega)}[\theta(r-r')\psi^a_{k'}(r,\omega)\psi^{(+)}_{k'}(r',\omega^*)^\dagger + \theta(r'-r)\psi^{(+)}_{k'}(r,\omega)\psi^a_{k'}(r',\omega^*)^\dagger]$	$\frac{i\theta(\omega)}{2\omega\alpha^{k'}_{\mathrm{I}}(\omega+i\epsilon)}\psi^a_{k'}(r,\omega+i\epsilon)\psi^{(+)}_{k'}(r',\omega-i\epsilon)^\dagger$	0	$+\frac{i\theta(-\omega)}{2(-\omega)\alpha^{k'}_{\mathrm{I}}(\omega-i\epsilon)}\psi^a_{k'}(r,\omega-i\epsilon)\psi^{(+)}_{k'}(r',\omega+i\epsilon)^\dagger$
$r \in \mathrm{P}$	$\frac{i\theta(-\omega)}{2(-\omega)\alpha^{k'}_{\mathrm{I}}(\omega-i\epsilon)}\psi^{(+)}_{k'}(r,\omega-i\epsilon)\psi^a_{k'}(r',\omega+i\epsilon)^\dagger$	$+\frac{i}{2\omega}[\theta(r-r')\psi^{(-)}_{k'}(r,\omega)\psi^{(+)}_{k'}(r',\omega^*)^\dagger - \theta(r'-r)\psi^{(+)}_{k'}(r,\omega)\psi^{(-)}_{k'}(r',\omega^*)^\dagger]$	$\frac{i\theta(\omega)}{2\omega\alpha^{k'}_{\mathrm{II}}(\omega+i\epsilon)}\psi^{(+)}_{k'}(r,\omega+i\epsilon)\psi^a_{k'}(r',\omega-i\epsilon)^\dagger$	$-\frac{i}{2\omega}\left[\theta(\omega)\frac{\beta^{k'}_{\mathrm{II}}(\omega+i\epsilon)}{\alpha^{k'}_{\mathrm{II}}(\omega+i\epsilon)}\psi^{(+)}_{k'}(r,\omega+i\epsilon)\psi^{(+)}_{k'}(r',\omega-i\epsilon)^\dagger + \theta(-\omega)\frac{\beta^{k'}_{\mathrm{I}}(\omega-i\epsilon)}{\alpha^{k'}_{\mathrm{I}}(\omega-i\epsilon)}\psi^{(+)}_{k'}(r,\omega-i\epsilon)\psi^{(+)}_{k'}(r',\omega+i\epsilon)^\dagger\right]$
$r \in \mathrm{II}$	0	$-\frac{i\theta(-\omega)}{2(-\omega)\alpha^{k'}_{\mathrm{II}}(\omega-i\epsilon)}\psi^a_{k'}(r,\omega-i\epsilon)\psi^{(+)}_{k'}(r',\omega+i\epsilon)^\dagger$	$\frac{-i}{2(\omega^2)^{1/2}\alpha^{k'}_{\mathrm{II}}(\omega)}[\theta(r-r')\psi^a_{k'}(r,\omega)\psi^{(+)}_{k'}(r',\omega^*)^\dagger - \theta(r'-r)\psi^{(+)}_{k'}(r,\omega)\psi^a_{k'}(r',\omega^*)^\dagger]$	$\frac{-i\theta(\omega)}{2\omega\alpha^{k'}_{\mathrm{II}}(\omega+i\epsilon)}\psi^a_{k'}(r,\omega+i\epsilon)\psi^{(+)}_{k'}(r',\omega-i\epsilon)^\dagger$
$r \in \mathrm{F}$	$\frac{i\theta(\omega)}{2\omega\alpha^{k'}_{\mathrm{I}}(\omega+i\epsilon)}\psi^{(+)}_{k'}(r,\omega+i\epsilon)\psi^a_{k'}(r',\omega-i\epsilon)^\dagger$	$\frac{i}{2\omega}\left[\theta(\omega)\frac{\beta^{k'}_{\mathrm{I}}(\omega+i\epsilon)}{\alpha^{k'}(\omega+i\epsilon)}\psi^{(+)}_{k'}(r,\omega+i\epsilon)\psi^{(+)}_{k'}(r',\omega-i\epsilon)^\dagger + \theta(-\omega)\frac{\beta^{k'}_{\mathrm{II}}(\omega-i\epsilon)}{\alpha^{k'}_{\mathrm{II}}(\omega-i\epsilon)}\psi^{(+)}_{k'}(r,\omega-i\epsilon)\psi^{(+)}_{k'}(r',\omega+i\epsilon)^\dagger\right]$	$\frac{i\theta(-\omega)}{2(-\omega)\alpha^{k'}_{\mathrm{II}}(\omega-i\epsilon)}\psi^{(+)}_{k'}(r,\omega-i\epsilon)\psi^a_{k'}(r',\omega+i\epsilon)^\dagger$	$-\frac{i}{2\omega}[\theta(r-r')\psi^{(-)}_{k'}(r,\omega)\psi^{(+)}_{k'}(r',\omega^*)^\dagger - \theta(r'-r)\psi^{(+)}_{k'}(r,\omega)\psi^{(-)}_{k'}(r',\omega^*)^\dagger]$

given in Ref. 1a indicate that there should be no $e^{-i\omega t}\psi^{(-)}(r,\omega)$ solution as that would correspond to $\psi(x)$ having annihilated a particle which emerged from II rather than I; there are assumed to be no particles preexisting in II.

Thus

$$S^{k'}(r,r';\omega)=\frac{i\,\theta(\omega)}{2\omega\alpha^{k'}(\omega+i\epsilon)}\times\psi_{k'}^{(+)}(r,\omega+i\epsilon)\psi_{k'}^{a}(r',\omega-i\epsilon)^{\dagger} \tag{5.11}$$

for $r\in$F, $r'\in$I, while Hermiticity implies

$$S^{k'}(r,r';\omega)=+i\,\frac{\theta(-\omega)}{2(-\omega)\alpha^{k'}(\omega-i\epsilon)}\times\psi_{k'}^{a}(r,\omega-i\epsilon)\psi_{k'}^{(+)}(r',\omega+i\epsilon)^{\dagger} \tag{5.12}$$

for $r'\in$F, $r\in$I. Invariance under the various discrete symmetries yields the Green's functions for the points in all possible pairs of sectors of which one is exterior and the other interior; the results are recorded in Table IV, along with all the other forms for the Green's function.

From these forms, all the states and the reduction formulas may be inferred. For $t>t'$, and $x,x'\in$I, the Green's function becomes, assuming that $\alpha^{k}(\omega)$ has no zeros in the complex plane,

$$\begin{aligned}i\langle 0|\psi(x)\psi(x')|0\rangle=i\sum_{k',m'}\frac{\mathcal{Y}_{k'}^{m'}(\theta,\phi)}{rw^{1/2}(r)}&\times\int_0^\infty\frac{d\omega}{4\pi\omega}e^{-i\omega t}\Bigg\{\theta(r-r')\left[\frac{\psi_{k'}^{a}(r,\omega+i\epsilon)\psi_{k'}^{(+)}(r',\omega-i\epsilon)^{\dagger}}{\alpha^{k'}(\omega+i\epsilon)}+\frac{\psi_{k'}^{a}(r,\omega-i\epsilon)\psi_{k'}^{(+)}(r',\omega+i\epsilon)^{\dagger}}{\alpha^{k'}(\omega-i\epsilon)}\right]\\&+\theta(r'-r)\left[\frac{\psi_{k'}^{(+)}(r,\omega+i\epsilon)\psi_{k'}^{a}(r,\omega-i\epsilon)^{\dagger}}{\alpha^{k'}(\omega+i\epsilon)}+\frac{\psi_{k'}^{(+)}(r,\omega-i\epsilon)\psi_{k'}^{a}(r',\omega+i\epsilon)^{\dagger}}{\alpha^{k'}(\omega-i\epsilon)}\right]\Bigg\}\\&\times e^{i\omega t}\,\frac{\mathcal{Y}_{k'}^{m'}(\theta',\phi')^{\dagger}}{r'w^{1/2}(r')}\,,\end{aligned} \tag{5.13}$$

which may be rewritten, using

$$\psi_{k'}^{a}(r,\omega-i\epsilon)=\{[\omega\psi_{k'}^{(+)}(r,\omega+i\epsilon)/q]+\beta^{k'}(\omega+i\epsilon)^{*}\,\psi_{k'}^{a}(r,\omega+i\epsilon)\}/\alpha^{k'}(\omega+i\epsilon)$$

and

$$\psi_{k'}^{(+)}(r,\omega-i\epsilon)=[\psi_{k'}^{a}(r,\omega+i\epsilon)-\beta^{k'}(\omega+i\epsilon)\psi_{k'}^{(+)}(r,\omega+i\epsilon)]/\alpha^{k'}(\omega+i\epsilon)\,,$$

for $\omega>m$ as

$$\begin{aligned}\langle 0|\psi(x)\,\psi(x')|0\rangle&=\sum_a\langle 0|\psi(x)|a\rangle\langle a|\psi(x')|0\rangle\\&=\sum_{k',m'}\frac{\mathcal{Y}_{k'}^{m'}(\theta,\phi)}{rw^{1/2}(r)}\Bigg(\int_0^m d\omega\,\frac{e^{-i\omega t}}{4\pi\omega}\left[\frac{\psi_{k'}^{a}(r,\omega)}{\alpha^{k'}(\omega+i\epsilon)}\right]\left[\frac{\psi_{k'}^{a}(r',\omega)}{\alpha^{k'}(\omega+i\epsilon)}\right]^{\dagger}e^{i\omega t'}\\&\quad+\int_0^\infty\frac{q^2dq}{2\omega(q)}e^{-i\omega(q)t}\Bigg\{\left[\frac{\psi_{k'}^{(+)}(r,\omega+i\epsilon)}{(2\pi)^{1/2}q\alpha^{k'}(\omega+i\epsilon)}\right]\left[\frac{\psi_{k'}^{(+)}(r',\omega+i\epsilon)}{(2\pi)^{1/2}q\alpha^{k'}(\omega+i\epsilon)}\right]^{\dagger}\\&\quad+\left[\frac{\psi_{k'}^{a}(r,\omega+i\epsilon)}{(2\pi\omega q)^{1/2}\alpha^{k'}(\omega+i\epsilon)}\right]\left[\frac{\psi_{k'}^{a}(r',\omega+i\epsilon)}{(2\pi\omega q)^{1/2}\alpha^{k'}(\omega+i\epsilon)}\right]^{\dagger}\Bigg\}\times e^{+i\omega(q)t'}\Bigg)\frac{\mathcal{Y}_{k'}^{m'}(\theta',\phi')^{\dagger}}{r'w^{1/2}(r')}\\&=\sum_{k',m'}\frac{\mathcal{Y}_{k'}^{m'}(\theta,\phi)}{rw^{1/2}(r)}\Bigg(\int_0^m d\omega\,e^{-i\omega t}\left[\frac{\psi_{k'}^{a}(r,\omega)}{(4\pi\omega)^{1/2}\delta^{k'}(\omega+i\epsilon)}\right]\left[\frac{\psi_{k'}^{a}(r',\omega)}{(4\pi\omega)^{1/2}\delta^{k'}(\omega+i\epsilon)}\right]^{\dagger}e^{i\omega t'}\\&\quad+\int_0^\infty\frac{q^2dq}{2\omega(q)}e^{-i\omega t}\Bigg\{\left[\frac{\psi_{k'}^{(-)}(r,\omega+i\epsilon)}{(2\pi)^{1/2}q\delta^{k'}(\omega+i\epsilon)}\right]\left[\frac{\psi_{k'}^{(-)}(r',\omega+i\epsilon)}{(2\pi)^{1/2}q\delta^{k'}(\omega+i\epsilon)}\right]^{\dagger}\\&\quad+\left[\frac{\psi_{k'}^{b}(r,\omega+i\epsilon)}{(2\pi\omega q)^{1/2}\delta^{k'}(\omega+i\epsilon)}\right]\left[\frac{\psi_{k'}^{b}(r',\omega+i\epsilon)}{(2\pi\omega q)^{1/2}\delta^{k'}(\omega+i\epsilon)}\right]^{\dagger}\Bigg\}\times e^{i\omega t'}\Bigg)\frac{\mathcal{Y}_{k'}^{m'}(\theta',\phi')^{\dagger}}{r'w^{1/2}(r)}\,,\end{aligned} \tag{5.14}$$

where $\omega \equiv (q^2+m^2)^{1/2} = \omega(q)$ inside the q integral. The wave functions $e^{-i\omega t}\psi^a$ and $e^{-i\omega t}\psi^{(+)}$, respectively, describe waves which emerge from P and waves which come in from infinity, and each propagates to both F and future infinity, while $e^{-i\omega t}\psi^b$ and $e^{-i\omega t}\psi^{(-)}$ are the time-reversed waves.

To show that $\alpha^k(\omega')$ has no zeros in the complex plane, suppose that it did have a zero at ω'; then ψ^a would be regular both at ∞ and at $r=2M$, and

$$0<\int_{2M}^{\infty}\frac{dr}{w^2(r)}\psi^a_{k'}(r,\omega')^{\dagger}\psi^a_{k'}(r,\omega')=\frac{1}{2i\,\mathrm{Im}\omega'}\int_{2M}^{\infty}\frac{dr}{w^2(r)}(\omega'-\omega'^*)\psi^a_{k'}(r,\omega')^{\dagger}\psi^a_{k'}(r,\omega')$$

$$=\frac{-1}{2\,\mathrm{Im}\omega'}\int_{2M}^{\infty}dr\,\frac{d}{dr}[\psi^a_{k'}(r,\omega')^{\dagger}\alpha^1\psi^a_{k'}(r,\omega')]$$

$$=0\,.$$

There is a contradiction; hence $\alpha^{k'}(\omega)$ has no zeros in the complex plane. Along the real axis, $|\alpha|^2=|\beta|^2+[q/(\omega^2)^{1/2}]>0$ for $\mathrm{Re}|\omega|>m$ and $\alpha^*\propto\beta$ for $|\omega|<m$, hence there can be no zeros along the real axis either. In order for the Green's function to satisfy its equation along the event horizons, it must be true that the "wrong" solutions ($t-r^*$ as the I, F border is approached) vanish. This is determined by the $\omega\sim 0$ behavior of the Green's function, which in turn is fixed by the behavior of $\alpha^{k'}(\omega)$ for small ω. As $\omega\to 0$, ψ^a remains finite but $\psi^{(-)}$ vanishes as $\sqrt{\omega}$, hence $\alpha^{k'}(\omega)$ must diverge as $1/\sqrt{\omega}$, when inserted into the Green's function; this implies that the "wrong" solutions vanish as $1/(t\pm r^*)$ and the Green's function is indeed a solution.

For $\omega<m$, the states may be labeled by k', m', and ω'; then

$$\langle 0|\psi(x)|k',m',\omega',\ \mathrm{out}\rangle=\frac{\mathcal{Y}^{m'}_{k'}(\theta,\phi)\psi^a_{k'}(r,\omega'+i\epsilon)e^{-i\omega' t}}{r\,w^{1/2}(r)(4\pi\omega')^{1/2}\delta^{k'}(\omega'+i\epsilon)}\equiv\psi(x;k',m',\omega',\mathrm{out})$$
$$=\mathcal{Y}^{m'}_{k'}(\theta,\phi)e^{-i\omega' t}\,\psi(r;k',\omega',\mathrm{out})\,,$$

while the in state is given by (5.15)

$$\langle 0|\psi(x)|k',m',\omega',\mathrm{in}\rangle=\frac{\mathcal{Y}^{m'}_{k'}(\theta,\phi)\psi^a_{k'}(r,\omega'+i\epsilon)}{r\,w^{1/2}(r)(4\pi\omega')^{1/2}\alpha^{k'}(\omega+i\epsilon)}\equiv\psi(x;k',m',\omega',\mathrm{out})$$
$$=\mathcal{Y}^{m'}_{k'}(\theta,\phi)e^{-i\omega' t}\,\psi(r;k',\omega',\mathrm{in})$$

and only differs from the out state by a phase because the particle does not have enough energy to escape to infinity. The scattering states $\omega'>m$ are defined by

$$\langle 0|\psi(x)|k',m',q',1,1,\mathrm{out}\rangle=\frac{\mathcal{Y}^{m'}_{k'}(\theta,\phi)e^{-i\omega' t}\psi^{(-)}_{k'}(r,\omega'+i\epsilon)}{r\,w^{1/2}(r)(2\pi\omega' q')^{1/2}\delta^{k'}(\omega'+i\epsilon)}\equiv\psi(x;k',m',q',1,1,\mathrm{out})$$
$$\equiv\mathcal{Y}^{m'}_{k'}(\theta,\phi)e^{-i\omega' t}\psi(r;k',q',1,1,\mathrm{out})$$

for the state which represents a particle emerging to infinity in I, and (5.16)

$$\langle 0|\psi(x)|k',m',q',2,1,\mathrm{out}\rangle\equiv\langle 0|\psi(x)|k',m',\omega',\mathrm{out}\rangle\left(\frac{2}{q'}\right)^{1/2}$$

$$=\frac{\mathcal{Y}^{m'}_{k'}(\theta,\phi)e^{-i\omega' t}\psi^b_{k'}(r,\omega'+i\epsilon)}{r\,w^{1/2}(r)(2\pi\omega' q')^{1/2}\delta^{k'}(\omega'+i\epsilon)}\equiv\psi(x;k',m',2,1,\mathrm{out})$$
$$=\mathcal{Y}^{m'}_{k'}(\theta,\phi)e^{-i\omega' t}\psi(r;k',q',2,1,\mathrm{out})$$

for the state which represents a particle propagating into the future event horizon. In both equations, $\omega'=(q'^2+m^2)^{1/2}$ and the common index 1 indicates that the states are localized in I rather than II. The states are normalized so that

$$\langle\cdots q''|\cdots q'\rangle=2\omega'\delta(q''-q')/q'^2$$

and (5.17)

$$\langle\cdots\omega''|\cdots\omega'\rangle=\delta(\omega''-\omega')$$

and, from the anticommutation relations, the radial wave functions satisfy the completeness and orthonormality relations

$$\int_0^m d\omega'[\psi(r;k',\omega',\text{out})\psi(r';k',\omega',\text{out})^\dagger+\psi(r;-k',\omega',\text{out})^*\psi(r';-k',\omega',\text{out})^T]$$
$$+\sum_{j=1}^{2}\int_0^\infty \frac{q'^2dq'}{2\omega'}[\psi(r;k',q',j,1,\text{out})\psi(r';k',q',j,1,\text{out})^\dagger$$
$$+\psi(r;-k',q',j,1,\text{out})^*\psi(r';-k',q',j,1,\text{out})^T]=\delta(r-r')w/r^2\,, \quad (5.18)$$

where the Hermiticity property, Eq. (A15), has been used to write $\mathcal{Y}_{k'}^{m'*}$ in terms of $\mathcal{Y}_{-k'}^{-m'}$, and

$$\int_{2M}^\infty \frac{r^2dr}{w}\,\psi(r;k',\omega',\text{out})^\dagger\psi(r,k',\omega''\,\text{out})=\delta(\omega'-\omega'')\,,$$

$$\int_{2M}^\infty \frac{r^2dr}{w}\,\psi(r;k',q',j',1,\text{out})^\dagger\psi(r;k',q'',j'',1,\text{out})=[2\omega'\delta(q'-q'')/q'^2]\,\delta_{j'j''}\,,$$

while the corresponding integrals of $\psi_{-k'}^T\psi_{k'}$ vanish, as do the integrals of $\psi(r;k',\omega',\text{out})^\dagger$ $\times\psi(r;k',q',j,1,\text{out})$. Similar arguments yield definitions for the wave functions in region II,

$$\psi(r;k',\omega',\text{out})=\frac{\psi_{k'}^a(r,\omega'-i\epsilon)}{(4\pi|\omega|)^{1/2}\delta^{k'}(\omega'-i\epsilon)rw^{1/2}(r)}\,,\quad \omega'<0$$
$$\psi(r;k',q',j,-1,\text{out})=\frac{\delta_j^1\psi_{k'}^{(-)}(r,\omega'-i\epsilon)}{rw^{1/2}(r)(2\pi|\omega'|q')^{1/2}\delta^{k'}(\omega'-i\epsilon)}+\frac{\delta_j^2\psi_{k'}^a(r,\omega'-i\epsilon)}{rw^{1/2}(r)(2\pi|\omega'|q')^{1/2}\delta^{k'}(\omega'-i\epsilon)}\,, \quad (5.19)$$
$$\omega'=-(q'^2+m^2)^{1/2}$$

which are exactly analogous to those in region I. Then, the reduction formulas are

$$\langle k',m',\omega',\text{out}|=\lim_{t\to\epsilon\infty}\int_{2M}^\infty \frac{r^2dr\,d\Omega\,\psi(x;k',m',\omega',\text{out})^\dagger}{w(r)}\langle 0|\psi(x)\,,$$

where $\epsilon=\pm1$, $\epsilon\omega'>0$, and (5.20)

$$\langle k',m',q,j,\epsilon,\text{out}|=\lim_{t\to\epsilon\infty}\int_{2M}^\infty \frac{r^2dr\,d\Omega}{w(r)}\,\psi(x;k',m',q',j,\epsilon,\text{out})^\dagger\langle 0|\psi(x)\,,$$

where the integrations are in region I (II) for $\epsilon=1$ (-1). For the outgoing waves at infinity, the reduction formulas are correct as they stand even in an interacting theory. The reduction formulas for the waves propagating into the event horizon are not valid in the presence of interactions because the particles do not become asymptotically free; they continue to interact with other particles as they cross the horizon, hence the propagation of particles in the interior region must be considered.

By taking one point in an exterior region and the other in an interior region, the matrix elements of the field in the interior regions may be obtained. The results, for x in F, are

$$\langle 0|\psi(x)|k',m',\omega',\text{out}\rangle=\frac{\mathcal{Y}_{k'}^{m'}(\theta,\phi)\psi_{k'}^{(+)}(r,\omega'(1+i\epsilon))e^{-i\omega't}}{rw^{1/2}(r)(4\pi|\omega'|)^{1/2}}\,. \quad (5.21)$$

The reduction formula is then

$$\langle k',m',\omega',\text{out}|=\lim_{r\to0}\int_{-\infty}^\infty dt\,d\Omega\,r^2w(r)e^{i\omega't}\psi_{k'}^{(+)}(r,\omega'(1+i\epsilon))^\dagger\langle 0|\psi(x)\,.$$

Some modification of these formulas is required in the case of neutrinos; their mass is zero (if not, no modification is required or allowed) and the neutrino states are eigenvalues of $(1-i\gamma_5)$. The matrix $-i\gamma_5$ when applied to the zero-mass equation leaves it invariant with $k'\to-k'$; the resultant relations between the wave functions and the field matrix elements and reduction formulas are given in Appendix B.

VI. STABILITY OF THE VACUUM

Just as with the spin-zero field, it is now easy to discuss the stability of the vacuum,

$$\delta \ln\langle 0|0\rangle = i\int_{\text{all space-time}} d^4x\, \delta e^a{}_\mu(x)\langle 0|\mathcal{T}^\mu{}_a(x)|0\rangle \tag{6.1}$$

under a variation of M, in Schwarzschild coordinates,

$$\delta e^a{}_\mu = \begin{cases} \dfrac{\delta M}{r}[-\delta^a_0\delta^t_\mu\,\epsilon/w(r) + \delta^a_1\delta^r_\mu\,\epsilon/w^3(r)], & x\in \text{I (II)} \\ \dfrac{\delta M}{r}[\delta^a_0\delta^r_\mu\,\epsilon/w^3(r) - \delta^a_1\delta^t_\mu\,\epsilon/w(r)], & x\in \text{F (P)}, \end{cases} \tag{6.2}$$

hence

$$\begin{aligned} \delta\ln\langle 0|0\rangle &= i\int_{\text{I+II+P+F}} d\Omega\,\frac{r^2dr\,dt}{w^2(r)}\,\frac{\delta M}{r}\,\langle 0|\tfrac{1}{2}\bar\psi(x)\left(\gamma^0\frac{1}{2i}\overleftrightarrow{\nabla}_0 - \gamma^1\frac{1}{2i}\overleftrightarrow{\nabla}_1\right)\psi(x)|0\rangle \\ &= i\int_{\text{I+II}} d\Omega\int_{2M}^\infty \frac{r^2dr}{w^2(r)}\left(\frac{\delta M}{r}\right)\int_{-\infty}^\infty dt\,\langle 0|\tfrac{1}{2}\psi(x)(-\alpha^1 i\overleftrightarrow{\nabla}_1 - m\beta + i\gamma^1 k/r)\psi(x)|0\rangle \\ &\quad + i\int_{\text{F+P}} d\Omega\int_0^{2M}\frac{r^2dr}{w^2(r)}\,\frac{\delta M}{r}\int_{-\infty}^\infty dt\,\langle 0|\tfrac{1}{2}\psi(x)(-i\overleftrightarrow{\nabla}_0 + m\beta - i\gamma^1 k/r)\psi(x)|0\rangle\,, \end{aligned} \tag{6.3}$$

which, when expressed in terms of the complete sum over states, is manifestly imaginary, hence $\delta\ln\langle 0|0\rangle$ is imaginary and $\langle 0|0\rangle$ remains of magnitude 1.

An energy operator may be defined in terms of the stress-energy tensor density $\mathcal{T}^{\mu\nu}\equiv r^2\sin\theta\, T^{\mu\nu}$ and the timelike (in I and II) Killing vector field δ^μ_t; however, the contribution from II is then negative. Instead, take the spacelike surface $t=\text{const}$ passing through I and II; the normal to the surface is $e_0{}^\mu$ and the energy density in I and II may be taken to be

$$\epsilon\mathcal{T}^\mu{}_t \equiv \epsilon(u)\,\mathcal{T}^\mu{}_t\,,$$

which is not conserved,

$$\begin{aligned} \partial_\mu(\epsilon\mathcal{T}^\mu{}_t) &= \mathcal{T}^\mu{}_t\,\partial_\mu\epsilon(u) \\ &= v\delta(u)[\mathcal{T}^u{}_u(v,0+) + \mathcal{T}^u{}_u(v,0-)]\,, \end{aligned} \tag{6.4}$$

but which has vanishing divergence on any surface passing through $v=u=0$. Thus, $\epsilon\mathcal{T}^\mu{}_t$ yields a "conserved" quantity for any spacelike surface which is restricted to the exterior regions. This operator measures the sum of the energies as seen by observers in I and II.

If the surface is taken to be a $t=\text{constant}$ surface,

$$\begin{aligned} H &= -\int d\sigma_\mu\,\epsilon\,\mathcal{T}^\mu{}_t \\ &= -\int_{\text{I+II}} d\Omega\int_{2M}^\infty r^2dr\,T^t{}_t \\ &= \int_{\text{I+II}} d\Omega\int_{2M}^\infty \frac{r^2dr}{w}\,\frac{\epsilon}{2}\,\psi\left(\frac{-1}{i}\overleftrightarrow{\partial}_t\right)\psi\,. \end{aligned} \tag{6.5}$$

If this quantity is evaluated between single-particle states,

$$\langle k',m',a|P_t|k',m',a\rangle - \langle k',m',a|k',m',a\rangle\langle 0|P_t|0\rangle = \int_{\text{I+II}} d\Omega\int_{2m}^\infty \frac{r^2dr}{w}\,\epsilon\psi(x,k',m',a)^\dagger\left(\frac{-1}{2i}\overleftrightarrow{\partial}_t\right)\psi(x,k',m',a) \tag{6.6}$$

but only positive (negative) frequencies appear in I (II) where ϵ is 1 (-1); hence this is positive-definite and the energy as measured by P_t is positive. That is, the vacuum is the lowest energy state.

These results should not be at all surprising. The vacuum states have been defined in terms of what does (not) appear in the external regions;

hence the initial and final vacua are just those states which respectively develop into and from the state with no particles in the exterior region; the stability proof provides assurance that no subtleties have intervened. In the exterior region, the energy is well defined and positive; that energy is being used to characterize the system.

APPENDIX A

The operators which define the spherical harmonics are, in the chosen representation of the Dirac matrices,

$$J_3=\frac{1}{i}\partial_\phi,\quad J_\pm=e^{\pm i\phi}\left(\pm\partial_\theta-\cot\theta\frac{1}{i}\partial_\phi+\frac{1}{2}\frac{\sigma_2}{\sin\theta}\right),$$
$$k=\frac{\sigma_3}{\sin^{1/2}\theta}\frac{1}{i}\partial_\theta\sin^{1/2}\theta+\frac{\sigma_1}{\sin\theta}\frac{1}{i}\partial_\phi. \tag{A1}$$

The spherical harmonics $\mathcal{Y}_{k'}^{m'}$ must satisfy

$$J_3\mathcal{Y}_{k'}^{m'}(\hat{r})=m'\mathcal{Y}_{k'}^{m'}(\hat{r}), \tag{A2}$$

$$J_\pm\mathcal{Y}_{k'}^{m'}(\hat{r})=[j(j+1)-m'(m'\pm1)]^{1/2}\mathcal{Y}_{k'}^{m'\pm1}(\hat{r}), \tag{A3}$$

and

$$k\mathcal{Y}_{k'}^{m'}(\hat{r})=k'\mathcal{Y}_{k'}^{m'}(\hat{r}),\quad k'=\pm(j+\tfrac{1}{2}). \tag{A4}$$

The solution to the azimuthal equation (A2) is

$$\mathcal{Y}_{k'}^{m'}(\hat{r})=\frac{e^{im'\phi}}{(2\pi)^{1/2}}f_k^{m'}(\theta). \tag{A5}$$

The operators $J_\pm$ may be written as

$$J_\pm=e^{\pm i\phi}\left[\frac{(\tan\frac{1}{2}\theta)^{\mp\sigma_2/2}}{\sin^{\mp J_3}\theta}\times(\pm\partial_\theta)\sin^{\mp J_3}\theta(\tan\tfrac{1}{2}\theta)^{\pm\sigma_2/2}\right]. \tag{A6}$$

Then

$$J_+\mathcal{Y}_k^j=0$$

for a state of total angular momentum $j(j+1)$, hence

$$f_{k'}^j(\theta)=\sin^j\theta(\tan\tfrac{1}{2}\theta)^{-\sigma_2/2}\psi_{k'}A_{k'} \tag{A7}$$

where $\psi_{k'}$ is a unit two-component spinor upon which σ_2 acts and $A_{k'}$ is a normalization factor. The spinor $\psi_{k'}$ is determined by applying the operator k,

$$k\mathcal{Y}_{k'}^{j}(\hat{r})=-i(j+\tfrac{1}{2})\sigma_3\left[\cot\theta-\frac{\sigma_2}{\sin\theta}\right]\mathcal{Y}_{k'}^{m'}(\hat{r})$$
$$=-i(j'+\tfrac{1}{2})\frac{e^{ij\phi}}{(2\pi)^{1/2}}\sin^j\theta(\tan\tfrac{1}{2}\theta)^{-\sigma_2/2}\times\sigma_3\left[(\tan\tfrac{1}{2}\theta)^{-\sigma_2}(\cot\theta-\frac{\sigma_2}{\sin\theta})\right]\psi_{k'}A_{k'}, \tag{A8}$$

but the factor in square brackets is equal to $-\sigma_2$, hence

$$k\mathcal{Y}_{k'}^{m'}=(j+\tfrac{1}{2})\frac{e^{ij\phi}}{(2\pi)^{1/2}}\sin^j\theta(\tan\tfrac{1}{2}\theta)^{-\sigma_2/2}\sigma_1\psi_{k'}A_{k'}, \tag{A9}$$

and $\psi_{k'}$ must be chosen to be an eigenvector of σ_1,

$$\psi_{k'}=\frac{1}{\sqrt{2}\,|k'|}\begin{pmatrix}|k'|\\ k'\end{pmatrix}, \tag{A10}$$

and the normalization factor may be calculated from

$$\int d\Omega\,\mathcal{Y}_{k''}^{j\dagger}\mathcal{Y}_{k'}^{j}=\int_{-1}^{1}d(\cos\theta)\sin^{2j}\theta\times\psi_{k''}^\dagger(\tan\tfrac{1}{2}\theta)^{-\sigma_2}\psi_{k'}A_{k''}^*A_{k'}$$
$$=\delta_{k'k''}\int_{-1}^{1}d(\cos\theta)\sin^{2j-1}\theta|A_{k'}|^2$$
$$=\delta_{k'k''}|A_{k'}|^2\frac{2^{2j}[(j-\frac{1}{2})!]^2}{(2j)!}, \tag{A11}$$

hence

$$A_{k'}=\frac{[(2j)!]^{1/2}}{2^j(j-\frac{1}{2})!}.$$

The result of repeated application of the lowering operator J_- is then

$$\mathcal{Y}_{k'}^{m'}(\hat{r})=\frac{e^{im'\phi}}{(2\pi)^{1/2}}\left[\frac{(j+m')!}{(j-m')!}\right]^{1/2}\frac{(\tan\frac{1}{2}\theta)^{\sigma_2/2}}{\sin^{m'}\theta}\times\left(\frac{\partial}{\partial\cos\theta}\right)^{j-m'}\frac{\sin^{2j}\theta(\tan\frac{1}{2}\theta)^{-\sigma_2}}{2^j(j-\frac{1}{2})!}\psi_{k'}. \tag{A12}$$

The factor upon which $\partial/\partial\cos\theta$ acts is a polynomial of order $2j$ in $\cos\theta$; hence $\mathcal{Y}_{k'}^{m'}$ vanishes for $m'<-j$ as it must. The harmonic for $m'=-j$ is then

$$\mathcal{Y}_{k'}^{-j}=(-1)^{j-1/2}\frac{e^{-ij\phi}}{(2\pi)^{1/2}}[(2j)!]^{1/2}\times\frac{\sin^j\theta(\tan\frac{1}{2}\theta)^{\sigma_2/2}}{2^j(j-\frac{1}{2})!}\sigma_2\psi_{k'}, \tag{A13}$$

which may then be operated on with J_+ to obtain the alternative form

$$\mathcal{Y}_{k'}^{m'}=(-1)^{m'+1/2}\frac{e^{+im'\phi}}{(2\pi)^{1/2}}\left[\frac{(j-m')!}{(j+m')!}\right]^{1/2}\frac{(\tan\frac{1}{2}\theta)^{-\sigma_2}}{\sin^{-m'}\theta}\times\left(\frac{\partial}{\partial\cos\theta}\right)^{j+m'}\frac{\sin^{2j}\theta(\tan\frac{1}{2}\theta)^{\sigma_2}}{2^j(j-\frac{1}{2})!}\sigma_2\psi_{k'}. \tag{A14}$$

From these forms the relations

$$\begin{aligned}
&\mathcal{Y}_{k'}^{m'}(\hat{r})^* = -i(-1)^{m'-1/2}(k'/|k'|)\mathcal{Y}_{-k'}^{-m'}(\hat{r}),\\
&\sigma_2\mathcal{Y}_{k'}^{m'}(\hat{r}) = -i(k'/|k'|)\mathcal{Y}_{-k'}^{m'}(\hat{r}),\\
&\sigma_1\mathcal{Y}_{k'}^{m'}(-\hat{r}) = i(k'/|k'|)(-1)^{j-1/2}\mathcal{Y}_{k'}^{m'}(\hat{r})\\
&\qquad \equiv i(-1)^l\mathcal{Y}_{k'}^{m'}(\hat{r}),
\end{aligned} \tag{A15}$$

where l is the orbital angular momentum of the large components in a Cartesian basis, and

$$\sigma_3\mathcal{Y}_{k'}^{m'}(-\hat{r})^* = (k'/|k'|)(-1)^{j-m'}\mathcal{Y}_{k'}^{-m'}(\hat{r})$$

are readily derived; the first, third, and fourth equations are respectively the transformations under complex conjugation, parity, and TCP while the second equation is the result of applying the other three; it is also the transformation required for the interchange of regions I and II.

APPENDIX B

The representation of the Dirac matrices is $\vec{\rho}\otimes\vec{\sigma}$,

$$\begin{aligned}
&\beta = \rho_2, \quad \gamma^1 = i\rho_1, \quad \gamma^2 = -i\rho_3\sigma_3, \quad \gamma^3 = -i\rho_3\sigma_1,\\
&\alpha^1 = \rho_3, \quad \alpha^2 = \rho_1\sigma_3, \quad \alpha^3 = \rho_1\sigma_1,\\
&\gamma^5 = \alpha^1\alpha^2\alpha^3\\
&\quad = i\rho_3\sigma_2,
\end{aligned} \tag{B1}$$

and the radial equation is in Schwarzschild coordinates,

$$\{(m\rho_2 + \rho_1 k'/r) + [\epsilon/w(r)][(-i\rho_3 d/dr^*) - \omega]\}\psi_{k'}(r,\omega) = 0,$$

in I (II) with $\epsilon = 1\ (-1)$, and (B2)

$$[(m\rho_2 + \rho_1 k'/r) - [\epsilon/w(r)](id/dr^* + \omega\rho_3)]\psi_{k'}(r,\omega) = 0,$$

in F (P) with $\epsilon = 1\ (-1)$. The solutions have the asymptotic behaviors, for complex ω,

$$\begin{aligned}
\psi_{k'}^a(r,\omega) \underset{r\sim\infty}{\sim} &\begin{pmatrix} f_m^+(\omega) \\ i\epsilon f_m^-(\omega)\end{pmatrix}\\
&\times\exp\left[+iq(\omega)\left(r^* + \frac{2m^2M}{q^2}\ln\frac{r}{2M}\right)\right],\\
\psi_{k'}^b(r,\omega) \underset{r\sim\infty}{\sim} &\begin{pmatrix} f_m^-(\omega) \\ i\epsilon f_m^+(\omega)\end{pmatrix}\\
&\times\exp\left[-iq(\omega)\left(r^* + \frac{2m^2M}{q^2}\ln\frac{r}{2M}\right)\right],
\end{aligned} \tag{B3}$$

$$\begin{aligned}
f_m^\pm(\omega) &\equiv [\omega \pm q(\omega)]^{1/2} = \pm i f_m^\pm(-\omega^*)^*\\
&= f_m^\mp(\omega^*)^*, \quad \mathrm{Im} f_m^\pm(\omega) \gtrless 0\\
q(\omega) &\equiv (\omega^2 + m^2)^{1/2} = -q(\omega^*)^*, \quad \mathrm{Im} q(\omega) > 0
\end{aligned}$$

and in each region

$$\psi_{k'}^{(\pm)}(r,\omega) \underset{r\sim 2M}{\sim} \begin{pmatrix} f^\mp(\omega) \\ if^\pm(\omega)\end{pmatrix}\exp[\mp i(\omega^2)^{1/2}r^*]. \tag{B4}$$

$e^{-i\omega t}\psi_{k'}^{(\pm)}(r,\omega)$ continues across an event horizon with no factors if $\omega t \pm (\omega^2)^{1/2}r^*$ remains finite; otherwise there is no continuity condition.

The symmetries of the wave functions are as follows.

Hermiticity:

$$\begin{aligned}
&[\psi_{k'}^a(r,\omega)]^* = -i\psi_{-k'}^a(r,-\omega^*),\\
&[\psi_{k'}^b(r,\omega)]^* = +i\psi_{-k'}^b(r,-\omega^*),\\
&[\psi_{k'}^{(\pm)}(r,\omega)]^* = \pm i\psi_{-k'}^{(\pm)}(r,-\omega^*)
\end{aligned} \tag{B5}$$

for r in all regions.

TCP:

$$\begin{aligned}
&\rho_1[\psi_{k'}^a(r,\omega)]^* = -i\epsilon\psi_{k'}^a(r,\omega^*),\\
&\rho_1[\psi_{k'}^b(r,\omega)]^* = -i\epsilon\psi_{k'}^b(r,\omega^*),\\
&\rho_1[\psi_{k'}^{(\pm)}(r,\omega)]^* = -i\psi_{k'}^{(\pm)}(r,\omega^*)
\end{aligned} \tag{B6}$$

for $r\in$ I, II and

$$\rho_1[\psi_{k'}^{(\pm)}(r,\omega)]^*|_{\mathrm{F}} = -i\psi_{k'}^{(\pm)}(r,\omega^*)|_{\mathrm{P}}.$$

Parity P_{I} has no implications for the radial wave functions. Parity P_{F} implies

$$\begin{aligned}
&\rho_2\psi_{k'}^a(r,\omega)|_{\mathrm{I}} = -i\psi_{-k'}^a(r,-\omega)|_{\mathrm{II}},\\
&\rho_2\psi_{k'}^b(r,\omega)|_{\mathrm{I}} = +i\psi_{-k'}^b(r,-\omega)|_{\mathrm{II}},\\
&\rho_2\psi^{(\pm)}(r,\omega)|_{\mathrm{I}} = [i\omega/(\omega^2)^{1/2}]\psi_{-k'}^{(\pm)}(r,-\omega)|_{\mathrm{II}},\\
&\rho_2\psi_{k'}^{(\pm)}(r,\omega) = i[\omega/(\omega^2)^{1/2}]\psi_{-k'}^{(\pm)}(r,-\omega)
\end{aligned} \tag{B7}$$

within each region for $r < 2M$ ($r\in$ P, F). The four solutions in the exterior regions are related:

$$\begin{aligned}
&\psi_{k'}^a(r,\omega) = \alpha^{k'}(\omega)\psi_{k'}^{(-)}(r,\omega) + \beta^{k'}(\omega)\psi_{k'}^{(+)}(r,\omega),\\
&\psi_{k'}^b(r,\omega) = \gamma^{k'}(\omega)\psi_{k'}^{(-)}(r,\omega) + \delta^{k'}(\omega)\psi_{k'}^{(+)}(r,\omega),\\
&\psi_{k'}^{(-)}(r,\omega) = [\epsilon(\omega^2)^{1/2}/q(\omega)][\delta^{k'}(\omega)\psi_{k'}^a(r,\omega)\\
&\qquad - \beta^{k'}(\omega)\psi_{k'}^b(r,\omega)],\\
&\psi_{k'}^{(+)}(r,\omega) = [\epsilon(\omega^2)^{1/2}/q(\omega)][\alpha^{k'}(\omega)\psi_{k'}^b(r,\omega)\\
&\qquad - \gamma^{k'}(\omega)\psi_{k'}^a(r,\omega)].
\end{aligned} \tag{B8}$$

The Wronskian is $\psi_{k'}^1(r,\omega^*)^\dagger\rho_3\psi_{k'}^2(r,\omega) = \text{constant}$,

$$\begin{aligned}
&\psi_{k'}^{(b)}(r,\omega^*)^\dagger\rho_3\psi_{k'}^a(r,\omega) = -\psi_{k'}^a(r,\omega^*)^\dagger\rho_3\psi_{k'}^b(r,\omega)\\
&\qquad = 2q(\omega),\\
&\psi_{k'}^{(+)}(r,\omega^*)^\dagger\rho_3\psi_{k'}^{(-)}(r,\omega) = -\psi_{k'}^{(-)}(r,\omega^*)^\dagger\rho_3\psi_{k'}^{(+)}(r,\omega)\\
&\qquad = 2(\omega^2)^{1/2},
\end{aligned}$$

and (B9)

$$\delta^{k'}(\omega^*)^*\alpha^{k'}(\omega) - \gamma^{k'}(\omega^*)^*\beta^{k'}(\omega) = q(\omega)/(\omega^2)^{1/2}.$$

Owing to the various symmetries the coefficient functions satisfy

$$(\alpha,\delta)^{k'}(\omega)^* = (\alpha,\delta)^{-k'}(-\omega^*),$$
$$(\beta,\gamma)^{k'}(\omega)^* = -(\beta,\gamma)^{-k'}(-\omega^*),$$
$$(\alpha,\beta,\gamma,\delta)^{k'}(\omega)^* = \epsilon(\alpha,\beta,\gamma,\delta)^{k'}(\omega^*),$$
$$(\alpha,\delta)^{k'}(\omega) = \epsilon(\alpha,\delta)^{-k'}(-\omega),$$
$$(\beta,\gamma)^{k'}(\omega) = -\epsilon(\beta,\gamma)^{-k'}(-\omega),$$

where $\epsilon = 1\ (-1)$ in I (II) and

$$(\alpha,\beta)_{\text{II}}^{-k'}(-\omega) = -[\omega/(\omega^2)^{1/2}](\alpha,\beta)_{\text{I}}^{k'}(\omega),$$
$$(\gamma,\delta)_{\text{II}}^{-k'}(-\omega) = [\omega/(\omega^2)^{1/2}](\gamma,\delta)_{\text{I}}^{k'}(\omega). \qquad \text{(B10)}$$

For real ω, $\omega \to \omega \pm i\epsilon$,

$$\psi_{k'}^{(a,b)}(r,\omega\pm i\epsilon) = \begin{cases} \psi_{k'}^{(b,a)}(r,\omega\mp i\epsilon), & \omega > m \\ \psi_{k'}^{(a,b)}(r,\omega\mp i\epsilon), & |\omega| < m \\ -\psi_{k'}^{(b,a)}(r,\omega\mp i\epsilon), & \omega < -m \end{cases}$$

$$\psi_{k'}^{(+)}(r,\omega\pm i\epsilon) = (\omega/|\omega|)\psi_{k'}^{(-)}(r,\omega\mp i\epsilon),$$

$$(\gamma,\delta)^{k'}(\omega\pm i\epsilon) = (\beta,\gamma)^{k'}(\omega\mp i\epsilon), \quad |\omega| > m$$

and (B11)

$$(\alpha,\delta)^{k'}(\omega\pm i\epsilon) = (\omega/|\omega|)(\beta,\gamma)^{k'}(\omega\mp i\epsilon), \quad |\omega| < m$$

$$|\alpha^{k'}(\omega\pm i\epsilon)|^2 - |\beta^{k'}(\omega\pm i\epsilon)|^2 = |\delta^{k'}(\omega\pm i\epsilon)|^2 - |\gamma^{k'}(\omega\pm i\epsilon)|^2 = \frac{\epsilon q(\omega\pm i\epsilon)}{[(\omega\pm i\epsilon)^2]^{1/2}}, \quad |\omega| > m.$$

The states are

$$|k', m', \omega', \text{out (in)}\rangle, \quad |k', m', q, 1, \epsilon, \text{out (in)}\rangle$$

and

$$|k', m', q, 2, \epsilon, \text{out (in)}\rangle \equiv |k', m', \epsilon(q^2+m^2)^{1/2}, \text{out (in)}\rangle\left(\frac{2}{q}\right)^{1/2},$$

where $-\infty < \omega' < \infty$, $0 \le q < \infty$, and $\epsilon = \pm 1$. The $|\omega'$ out (in)$\rangle$ states represents particles propagating into P from (out of F into) an exterior region; if $\omega' > 0$ ($\omega' < 0$) the particle may be found in I (II) but not II (I). The $|\ldots, q, 1, \epsilon, \text{out (in)}\rangle$ state represents a particle which escapes to (comes from) spatial infinity in I (II) for $\epsilon = 1\ (-1)$, while the $|\ldots, q, 2, \epsilon, \text{out (in)}\rangle$ states represent particles which have enough energy to escape to infinity but which to into F from (go from P into) I (II) for $\epsilon = 1\ (-1)$.

The matrix elements of $\psi(x)$ between the vacuum and these states are

$$\langle 0|\psi(x)|k', m', a\rangle = \frac{\mathcal{Y}_k^{m'}(\theta,\phi)e^{-i\omega' t}}{r w^{1/2}(r)(4\pi|\omega'|)^{1/2}} f^a(r) \equiv \psi(x; k', m' a), \qquad \text{(B12)}$$

where f^a is given in Table V. There,

TABLE V. Values of the function f^a which appear in Eq. (B12) for the state $|a\rangle$ in each of the regions of the Kruskal space-time.

	$\lvert k', m', \omega', \text{out}\rangle$	$\lvert k', m', \omega', \text{in}\rangle$	$\lvert k', m', q', 1, \epsilon, \text{out}\rangle q'/(2\lvert\omega'\rvert)^{1/2}$	$\lvert k', m', q', 1, \epsilon, \text{out}\rangle q'/(2\lvert\omega'\rvert)^{1/2}$
$x \in$ F	$\psi_{k'}^{(+)}(r,\omega')$	$[\theta(\omega')(\beta^{k'}/\alpha^{k'})_{\text{I}}(\omega') + \theta(-\omega')(\beta^{k'}/\alpha^{k'})_{\text{II}}(\omega')]\psi_{k'}^{(+)}(r,\omega')$	0	$\{[\theta(\omega')/\alpha_{\text{I}}^{k'}(\omega')] + [\theta(-\omega')/\alpha_{\text{II}}^{k'}(\omega')]\}\psi_{k'}^{(+)}(r,\omega')(q'/\lvert\omega'\rvert)$
$x \in$ I	$[\theta(\omega')/\delta_{\text{I}}^{k'}(\omega')]\psi_{k'}^{b}(r,\omega')$	$[\theta(\omega')/\alpha_{\text{I}}^{k'}(\omega')]\psi_{k'}^{a}(r,\omega')$	$[\theta(\omega')/\delta_{\text{I}}^{k'}(\omega')]\psi_{k'}^{(-)}(r,\omega')(q'/\omega')$	$[\theta(\omega')/\alpha_{\text{I}}^{k'}(\omega')]\psi_{k'}^{(+)}(r,\omega')(q'/\omega')$
$x \in$ II	$[\theta(-\omega')/\delta_{\text{II}}^{k'}(\omega')]\psi_{k'}^{b}(r,\omega')$	$[\theta(-\omega')/\alpha_{\text{II}}^{k'}(\omega')]\psi_{k'}^{a}(r,\omega')$	$[\theta(-\omega')/\alpha_{\text{II}}^{k'}(\omega')]\psi_{k'}^{(-)}(r,\omega')(q'/\lvert\omega'\rvert)$	$[(-\omega')/\alpha_{\text{II}}^{k'}(\omega')]\psi_{k'}^{(+)}(r,\omega')(q'/\lvert\omega'\rvert)$
$x \in$ P	$[\theta(\omega')(\gamma^{k'}/\delta^{k'})_{\text{I}}(\omega') + \theta(-\omega')(\gamma^{k'}/\delta^{k'})_{\text{II}}(\omega')]\psi_{k'}^{(-)}(r,\omega')$	$\psi_{k'}^{(-)}(r,\omega')$	$\{[\theta(\omega')/\delta_{\text{I}}^{k'}(\omega')] + [\theta(-\omega')/\delta_{\text{II}}^{k'}(\omega')]\}\psi_{k'}^{(-)}(r,\omega')(q'/\lvert\omega'\rvert)$	0

$\omega' = \epsilon(q'^2 + m^2)^{1/2}$ if it is not otherwise defined and when it appears as the argument of an analytic function (ψ, α, β, γ, or δ), the function is to be evaluated at $\omega' + i\epsilon$ for $\omega' > 0$ and at $\omega' - i\epsilon$ for $\omega' < 0$. The wave functions of the states with negative (positive) ω' vanish in I (II).

The reduction formulas are then

$$\langle k', m', q', 1, \epsilon, \text{out (in)}| = \lim_{t\to\pm\epsilon\infty} \int_{2M}^{\infty} \frac{r^2 dr}{w(r)} \psi^\dagger (x; k', m', q', 1, \epsilon, \text{out (in)}) \langle 0|\psi(x),$$

where the integration is over I (II) for $\epsilon = 1\,(-1)$, and (B13)

$$\langle k', m', \omega', \text{out (in)}| = \lim_{r\to 0} \int_{-\infty}^{\infty} dt\, d\Omega\, r^2 w(r) \psi^\dagger(x, k', m', \omega', \text{out (in)}) \langle 0|\psi(x),$$

and the integration is in F (P) for the out (in) state. If the particles are not interacting, the ω' states can be created in I or II; the reduction formula is

$$\langle k', m', \omega', \text{ out (in)}| = \lim_{t\to\pm\omega'\infty} \int_{2M}^{\infty} \frac{r^2 dr\, d\Omega}{w'(r)} \psi^\dagger (x; k', m', \omega', \text{out (in)}) \langle 0|\psi(x), \tag{B14}$$

and the integration is over I (II) for $\omega' > 0$ ($\omega' < 0$).

In the case of zero mass, there is a further invariance,

$$\rho_3 \psi^a_{k'}(r, \omega) = +[\omega/(\omega^2)^{1/2}]\psi^a_{-k'}(r, \omega),$$

$$\rho_3 \psi^b_{k'}(r, \omega) = -[\omega/(\omega^2)^{1/2}]\psi^b_{-k'}(r, \omega),$$

and (B15)

$$\rho_3 \psi^{(\mp)}_{k'}(r, \omega) = \pm[\omega/(\omega^2)^{1/2}]\psi^{(\mp)}_{-k'}(r, \omega).$$

The states are now to be characterized by their helicity and total angular momentum,

$$\langle 0|\psi(x)| +, j, m', q', j', \epsilon, \text{out}\rangle = \frac{1 - i\gamma_5}{\sqrt{2}} \psi(x; j + \tfrac{1}{2}, m', q', j', \epsilon, \text{out}),$$

$$\langle 0|\psi(x)| -, j, m', q', j', \epsilon, \text{in}\rangle = \frac{1 + i\gamma_5}{\sqrt{2}} \psi(x; j + \tfrac{1}{2}, m', q', j', \epsilon, \text{in}). \tag{B16}$$

Also

$$(\alpha, \delta)^{k'}(\omega) = (\alpha, \delta)^{-k'}(\omega), \quad (\beta, \gamma)^{k'}(\omega) = -(\beta, \gamma)^{-k'}(\omega).$$

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