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Gravitational wave reception by a sphere

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The reception of gravitational waves by an elastic self-gravitating spherical detector is studied in detail. The equations of motion of a detector driven by a gravitational wave are presented in the intuitively convenient coordinate system of Fermi. An exact analytic solution is given for the homogeneous isotropic sphere. Nonlinear effects of a massive self-gravitating system are computed for a body of mass equal to that of the earth, and are shown to be numerically important.

INTRODUCTION

The prospect of the unambiguous detection of gravitational waves within the next decade appears not unreasonable.[1,2] One need not dwell upon the immense significance of such a discovery both as a check of the theory of gravitation and as a wholly new experimental technique to probe galactic and cosmic structure.

Most of the current experimental detection schemes utilize laboratory-size detectors which resonate at kilohertz frequencies which could be produced only by conjectured sources of gravitational waves. On the other hand, the strongest sources of gravitational radiation[3] according to standard theory are nearby double stars which emit waves with periods comparable to the fundamental period of the spheroidal free oscillations of the earth (54 min). Since the free spheroidal oscillations of a sphere have the correct symmetry to be excited by plane gravitational waves, it is of considerable interest to calculate the elastic response of the earth to monochromatic waves of frequencies in the range 10^{-4} Hz to 1 Hz. Previous calculations[4] have made simplifying assumptions in order to obtain an estimate of the response. The present investigation has a threefold purpose. First, it is useful to discuss the elastic response of an extended body to a gravitational wave in a coordinate system which appears as a local inertial frame at the detector's center. It is then fairly easy to relate the results to measurements carried out with conventional instruments such as gravimeters and seismometers. Second, we present in detail analytic results for the response of a uniform spherical detector having properties (mass, elastic moduli) comparable to those of the real earth. Third, we include the effects of self-stress generated by the body's own gravitational field, which are numerically important for bodies as massive as the earth or moon.

We shall not attempt to discuss the problems of noise encountered when one attempts to measure wave-induced vibrations of the earth. The ability to detect experimentally such vibrations does not appear to be out of the question, although it certainly entails improvements in present experimental techniques.

Section I is concerned with a discussion of the coordinate system in which the equations of motion and the gravitational field appear most readily interpretable in the Newtonian sense. A gauge transformation from comoving to Fermi coordinates is obtained explicitly and all subsequent calculations are carried out in this coordinate system. The advantage of using Fermi coordinates is that the gravitational wave field appears obviously as a classical driving force.

In Sec. II, the equations of motion of an elastic system are obtained by two independent methods—from the action principle and from local energy-momentum conservation laws. The equations of motion of the elastic displacement field describe the time dependence of those quantities conventionally measured in the laboratory.

The problem of the homogeneous and isotropic elastic sphere without self-stress but driven by a gravitational wave is solved analytically in Sec. III. The technique uses the appropriate vector spherical harmonics determined by symmetry considerations.

Sections IV and V are concerned with the more complicated problem of a self-stressed sphere. The general considerations of Sec. IV show why the self-stressing field of the earth must be taken into account if a realistic response of the earth to a gravitational wave is sought. Section V then presents the numerical results obtained from the model earth calculations. A discussion of the relation between the calculated displacements and observations carried out by gravimeters, seismometers, and strainmeters is given in Sec. VI.

The notations employed throughout the paper are summarized in the Appendix.

I. WEAK GRAVITATIONAL WAVES

The gravitational waves expected to bathe the earth are all weak in the precise sense that the metric tensor $g_{\mu\nu}$ is very nearly that of flat spacetime $\eta_{\mu\nu}$. If we write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, then the $h_{\mu\nu}$ are all small in magnitude compared to unity.

In the linearized theory of gravitational waves with the choice of a comoving coordinate system and the transverse traceless gauge, the metric assumes the form[5]

$$\begin{aligned} ds^2 &= g_{\mu\nu}\, dx^\mu\, dx^\nu \\ &= -(dx^0)^2 + (\eta_{mn} + h_{mn})dx^m dx^n , \\ &\partial^\lambda \partial_\lambda h_{mn} = 0 , \\ &h^m{}_m = 0, \quad h^n{}_{m,n} = 0, \quad h_{mn} = h_{nm} . \end{aligned} \tag{1.1}$$

Since all gravitational waves broadcast by sources of conceivably detectable intensity are far from the solar system, the gravitational waves impinging on the earth are plane waves to an extremely good approximation. The metric therefore may be assumed to have only the following nonvanishing h_{mn}:

$$\begin{aligned} &h_{rs} = h_{rs}(x^3 - x^0) , \\ &h^r{}_r = 0, \\ &(r, s) = (1, 2). \end{aligned} \tag{1.2}$$

The wave thus propagates along the three-direction. For convenience, the indices labeled by the letters r and s will from now on be understood to assume only the values 1 and 2 so that the metric tensor appears as

$$ds^2 = -(dx^0)^2 + (dx^3)^2 + (\eta_{rs} + h_{rs})dx^r dx^s . \tag{1.3}$$

To facilitate the physical interpretation of the response of a material medium to a gravitational wave, it proves useful to work in a Fermi coordinate system,[6] which will reduce to a local Lorentz coordinate system on a submanifold which will be specified below. Such a coordinate system is, in a sense, as "Newtonian" as possible. The transition to such a system is effected by the appropriate coordinate (gauge) transformation.

The transformation of coordinates will be carried out in two steps. First, consider the following transformation (here $x_\mu = \eta_{\mu\nu} x^\nu$):

$$\begin{aligned} &x_0 = \bar{x}_0 , \\ &x_3 = \bar{x}_3 , \\ &x_r = \bar{x}_r + \epsilon_r(\bar{x}) . \end{aligned} \tag{1.4}$$

Here $\epsilon_r(x)$ is to be determined but, in any event, is assumed small so that quadratic and higher terms in ϵ are discarded. (The ϵ are of the same order of smallness of h.) The metric appears in the transformed coordinate system as

$$\begin{aligned} ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu &+ 2\epsilon_{r,0} dx^r\, dx^0 + (\epsilon_{r,s} + \epsilon_{s,r})dx^r\, dx^s \\ &+ h_{rs}\, dx^r\, dx^s + 2\epsilon_{r,3} dx^r\, dx^3 , \\ h_{rs}(x) &\cong h_{rs}(\bar{x}) . \end{aligned} \tag{1.5}$$

The bars on the new coordinates have been omitted for notational convenience. Choose ϵ_r to satisfy

$$h_{rs} = -(\epsilon_{r,s} + \epsilon_{s,r}) . \tag{1.6}$$

The obvious solution involving a choice of arbitrary constants such that $\epsilon_r = 0$ at $x^r = 0$ is

$$\epsilon_r = -\tfrac{1}{2} h_{rs} x^s , \tag{1.7}$$

and the metric becomes

$$ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu + 2\epsilon_{r,0} dx^r\, dx^0 + 2\epsilon_{r,3} dx^r\, dx^3 . \tag{1.8}$$

The property to note is that in the new coordinate system $g_{\mu\nu} = \eta_{\mu\nu}$ on the submanifold $x^r = 0$.

A further coordinate transformation will serve to make the coordinate system a local Lorentz coordinate system on the submanifold $x^r = 0$. Coordinates with this property are called Fermi coordinates. Define new coordinates $\bar{x}_\mu$ by

$$\begin{aligned} &x_0 = \bar{x}_0 + \epsilon_0(\bar{x}) , \\ &x_3 = \bar{x}_3 + \epsilon_3(\bar{x}) , \\ &x_r = \bar{x}_r \end{aligned} \tag{1.9}$$

and, with neglect again of quadratic terms in ϵ, the new form of the metric is

$$\begin{aligned} ds^2 = -(dx^0)^2(1 - 2\epsilon_{0,0}) &+ 2\epsilon_{0,r}\, dx^0 dx^r + 2\epsilon_{0,3} dx^0 dx^3 \\ &+ (dx^3)^2(1 + 2\epsilon_{3,3}) + 2\epsilon_{3,0} dx^3 dx^0 + 2\epsilon_{3,r}\, dx^3 dx^r \\ &+ (dx^1)^2 + (dx^2)^2 + 2\epsilon_{r,0} dx^r\, dx^0 + 2\epsilon_{r,3} dx^r\, dx^3 . \end{aligned} \tag{1.10}$$

Note that $\epsilon_{0,r} \cong \partial\epsilon_0(\bar{x})/\partial\bar{x}^r$ in the approximation used and that again the bars on the coordinates have been dropped for notational convenience. Now choose ϵ_0 and ϵ_3 to satisfy

$$\epsilon_{0,r} = -\epsilon_{r,0}, \quad \epsilon_{3,r} = -\epsilon_{r,3} \tag{1.11}$$

so that

$$\epsilon_0 = \tfrac{1}{4} h_{rs,0} x^r\, x^s, \quad \epsilon_3 = \tfrac{1}{4} h_{rs,3} x^r\, x^s . \tag{1.12}$$

The resultant form of the metric for weak plane waves is

$$\begin{aligned} ds^2 = -(dx^0)^2(1 - \tfrac{1}{2} h_{rs,00} x^r\, x^s) &+ (dx^3)^2(1 + \tfrac{1}{2} h_{rs,33} x^r\, x^s) \\ &+ 2(\tfrac{1}{2} h_{rs,03} x^r\, x^s\, dx^0 dx^3) + (dx^1)^2 + (dx^2)^2 . \end{aligned} \tag{1.13}$$

This form of the metric tensor has the properties

$$g_{\mu\nu}=\eta_{\mu\nu}\,,\qquad \Gamma^{\mu}_{\rho\sigma}=0 \tag{1.14}$$

again on the submanifold $x^r=0$. Furthermore, along this line we find

$$\frac{\partial^a\Gamma^{\mu}_{\rho\sigma}}{(\partial x^3)^b(\partial x^0)^c}=0\,,\quad b+c=a\,. \tag{1.15}$$

These are the properties of a Fermi coordinate system. It shares many local properties with the Lorentz coordinate systems used in flat space-time physics.

The Riemann tensor assumes the particularly simple expression on the submanifold $x^r=0$:

$$R_{\mu r\lambda s}=\tfrac{1}{2}g_{\mu\lambda,rs}\,. \tag{1.16}$$

It is clear that the only nonvanishing components of $R_{\mu\rho\lambda\sigma}$ are the following and those which are obtained from these by using the symmetries of the Riemann tensor:

$$R_{0r0s}=\tfrac{1}{2}h_{rs,00}\,,\qquad R_{0r3s}=\tfrac{1}{2}h_{rs,03}\,,\qquad R_{3r3s}=\tfrac{1}{2}h_{rs,33}\,. \tag{1.17}$$

It is this Riemann tensor which a gravitational antenna detects.

The earth while irradiated with gravitational waves is also subject to its own gravitational field. The self-gravitational field of the earth is also a very weak field in the sense that

$$h\equiv\frac{V}{c^2}\ll 1\,. \tag{1.18}$$

Here V is the gravitational potential at a position $\vec{r}$ from the earth's center.

When in addition to the impinging gravitational wave there is a massive gravitating body present, the metric tensor can be written

$$ds^2=-(dx^0)^2[1-\tfrac{1}{2}h_{rs,00}x^rx^s+h(\vec{r})]+(dx^3)^2[1+\tfrac{1}{2}h_{rs,33}x^rx^s-h(\vec{r})]+h_{rs,03}x^rx^s\,dx^0dx^3+[(dx^1)^2+(dx^2)^2][1-h(\vec{r})]\,. \tag{1.19}$$

The center of mass of the earth has been chosen to have the coordinates $x^r=0$. The form of $h(\vec{r},t)$ in the presence of a gravitational wave will be computed in Secs. IV and V.

If the response of a system of particles to the gravitational wave is desired and the particles move with nonrelativistic speeds, only the Newtonian approximation to the metric is required. In this approximation, the *effective* metric is simply

$$ds^2=-(dx^0)^2\left[1-\tfrac{1}{2}h_{rs,00}x^rx^s+h(\vec{r})-\left(\frac{dx^1}{dx^0}\right)^2-\left(\frac{dx^2}{dx^0}\right)^2-\left(\frac{dx^3}{dx^0}\right)^2\right]. \tag{1.20}$$

We have written the metric in this form to emphasize that in the nonrelativistic Newtonian limit, only the coefficient of $(dx^0)^2$ need be modified from the flat space-time value since dx^m/dx^0 will always be of order v/c when the metric is used to calculate motion.

This effective metric cannot, of course, be used in relativistic motion problems such as the interaction of light with a gravitational wave. In this latter case, the full metric given by Eq. (1.19) must be retained.

II. RESPONSE OF MATTER TO A GRAVITATIONAL WAVE

The first step in detecting gravitational waves is to select a suitable antenna. Any physical object, since it will have energy, can in principle serve as a detector provided that it has an extended structure. The latter requirement follows from the fundamental property of observable gravitational waves appearing as "tidal" phenomena.

The basic response of a system of particles (each labeled by the index p) to a gravitational wave is given by the equations of geodesic deviation[7]:

$$m_pc^2\frac{D^2\delta x^{\mu}_p}{Ds^2}+m_pR^{\mu}{}_{\nu n\lambda}U^{\nu}\delta x^n_pU^{\lambda}=\delta F^{\mu}_p\,. \tag{2.1}$$

The coordinates of the particle labeled by p are designated by δx^{μ}_p to remind one that such an equation is valid only if the particle is located "close" to the fiducial geodesic traversed by the sequence of events whose four-velocity is $U^{\mu}=c\,dx^{\mu}/ds$. We shall assume that the fiducial world line is associated with the coordinates of the center of mass of the system in Newtonian approximation. The condition for closeness is that one requires $|\delta x^{\mu}_p|/R\ll 1$, where R is the radius of curvature of various space-time sections prescribed by the Riemann tensor. The force δF^{μ}_p is by definition the differential nongravitational force on the particle p relative to the force on the center-of-mass point. If the center of mass is in free fall, then δF^{μ}_p can be taken to be the nongravitational force F^{μ}_p which the particle p experiences from the rest of the particles, and δx^{μ}_p must then be taken as the distance x^{μ}_p from the center of mass.

In a normal Fermi coordinate system, in which

$U^{\mu} = (c, 0, 0, 0)$, the equation of geodesic deviation assumes the form

$$m_p \frac{d^2 x_p^m}{dt^2} = -m_p c^2 R^m{}_{0n0} x_p^n + F_p^m\,, \qquad m = 1, 2, 3\,. \tag{2.2}$$

Such an equation indicates that the presence of the gravitational wave can be taken into account by adding to the Newtonian equations of motion a gravitational tidal force term, viz. $-m_p c^2 R^m_{0n0} x_p^n$.

The particular system of concern here is the earth, which is assumed to be an elastic body of spherical symmetry, stressed by its own gravitational field. In this section, however, the self-stressing will be temporarily neglected for simplicity.

Two approaches are informative in deriving the equations of motion of the earth under the influence of a gravitational wave. Both lead to the equation of motion which perhaps is most easily surmised from Eq. (2.2).

In the first approach, the action principle[8] is used to derive the equations of motion. The action can be written in the form

$$A = \int L_0 dt + \int L_1 dt\,. \tag{2.3}$$

Here the L_0 term yields, with the use of the variational principle, the usual elastic-continuum equations of motion without a driving term. It is important to emphasize that the usual form of the equations results only in a Fermi coordinate system since only therein are the covariant derivatives replaceable by ordinary derivatives.[9] The interaction Lagrangian L_1 which describes the matter-gravity interaction assumes the following form in Newtonian gravitational approximation:

$$L_1 = \tfrac{1}{2} \int T^{\mu\nu} (g_{\mu\nu} - \eta_{\mu\nu}) d^3x\,, \tag{2.4}$$

where $T^{\mu\nu}$ is the stress-energy tensor and $h_{\mu\nu}$ is the gravitational wave potential; both are to be evaluated throughout the region of space occupied by the elastic body. Only T^{00} need be retained in Newtonian approximation and, in the same approximation,

$$T^{00} = \rho c^2\,, \tag{2.5}$$

where ρ is the equilibrium proper mass density of the elastic medium. The expression $(g_{00} - \eta_{00})$ is obtained from Eq. (1.20). An element of the continuum is labeled by x^m, the equilibrium coordinates, and the position of this element is prescribed by the positional field $y^m(x)$. The relation between the positional field y^r and the displacement field u^r is by definition

$$y^m(x^n, x^0) = x^m + u^m(x^n, x^0)\,. \tag{2.6}$$

The value of $(g_{00} - \eta_{00})$ is required at the position of the element of the elastic body labeled by x^m and is explicitly

$$g_{00} - \eta_{00} = \tfrac{1}{2} h_{rs,00}(y^3 - x^0) y^r y^s\,. \tag{2.7}$$

It will now be assumed that the wavelength of the incident gravitational radiation is large compared to the size of the detector. Consequently, the y^3 dependence of $h_{rs,00}$ is negligible. Variation of the action then yields in the standard way the gravitational force density

$$f_s(x) = \tfrac{1}{2} \rho c^2 h_{rs,00} x^r\,. \tag{2.8}$$

In this expression, the equilibrium position x^s of the element of the elastic continuum has been used since the amplitude of vibration of the elastic body will be very small and the gravitational potential h_{rs} is weak.

The second method of deriving the equations of motion follows by using the exact equations

$$T^{\mu}{}_{\nu;\mu} = 0 \tag{2.9}$$

which describe the movement of energy and momentum. The equation in more explicit form is

$$\frac{\partial T^{\mu\nu}}{\partial x^{\mu}} = -\Gamma^{\nu}_{\mu\rho} T^{\mu\rho} - \Gamma^{\mu}_{\mu\rho} T^{\rho\nu}\,. \tag{2.10}$$

The usual equations of motion are obtained from the momentum density equation:

$$\frac{\partial T^{0n}}{\partial x^0} + \frac{\partial T^{mn}}{\partial x^m} = -\Gamma^{n}_{\mu\rho} T^{\mu\rho} - \Gamma^{\mu}_{\mu\rho} T^{\rho n}\,. \tag{2.11}$$

In Newtonian approximation, the T^{0n} and T^{mn} on the left-hand side of the equation are the nonrelativistic momentum density and the negative of the stress tensor, respectively.

To the same approximation, only T^{00} terms need be retained on the right-hand side of the equation. The conventional definition of the elastic stress tensor is $T^{mn} = -\sigma^{mn}$ and thus

$$\frac{\partial T^{0n}}{\partial x^0} = \frac{\partial \sigma^{mn}}{\partial x^m} - \Gamma^{n}_{00} \rho c^2\,. \tag{2.12}$$

To determine Γ^n_{00} again only the Newtonian nonrelativistic approximation is retained,

$$\Gamma^n_{00} = -\tfrac{1}{2} g_{00,n}\,, \tag{2.13}$$

which becomes in the long-wavelength approximation:

$$\Gamma^n_{00} = -\tfrac{1}{2} h_{ns,00} x^s \tag{2.14}$$

The equations of motion of the elastic continuum which result from either of these methods are

$$\frac{\partial T^{0n}}{\partial x^0} = \frac{\partial \sigma^{mn}}{\partial x^m} + \tfrac{1}{2} \rho c^2 h_{ns,00} x^s\,. \tag{2.15}$$

The term involving h_{rs} has already been prescribed in form while the remaining quantities T^{0n} (momentum density) and σ^{mn} (stress tensor) are quantities familiar from the theory of elasticity.

III. RESPONSE OF AN ELASTIC SPHERE TO GRAVITATIONAL WAVES

Before turning to the problem of the self-stressed earth under the influence of a gravitational wave, it is worthwhile formulating and solving analytically the simpler problem of the forced vibrations of a homogenous elastic sphere. We shall assume the wavelength of the wave is very long compared to the radius of the sphere.

If it is assumed that the excursions of the elements of the elastic body from the equilibrium positions are small, the linear theory of elasticity is appropriate. Let the equilibrium positions of the elements of the elastic continuum be labeled by the coordinates x_i. (Since the Newtonian approximation has been made, it is no longer necessary to distinguish between covariant and contravariant components in a three-dimensional orthonormal-base-vector system.) The displacement from equilibrium of an element at x_i is prescribed by the time-dependent vector $u_n(x_i, t)$ which is assumed small. The momentum density is formed as the product $\partial(\rho u_n)/\partial t$ and, in keeping with the linearization, ρ can be considered to be a constant in a homogeneous medium.

The stress tensor σ_{mn} is related to the strain tensor[10] by a generalized Hooke's law, again under the assumption of small displacements. The strain of the elastic medium is defined by

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \qquad (3.1)$$

in Cartesian coordinates and is related to the stress (only in a Fermi coordinate system) by

$$\sigma_{ij} = \delta_{ij}\lambda u_{ll} + 2\mu u_{ij} , \qquad (3.2)$$

where λ and μ are the Lamé coefficients specifying the elastic properties of an isotropic medium.

Besides the stresses caused by strains in the medium, there are frictional forces which set up stresses resulting in dissipation of energy. The dissipative stress tensor is

$$\sigma'_{ij} = \delta_{ij}\zeta \dot{u}_{ll} + 2\eta(\dot{u}_{ij} - \tfrac{1}{3}\delta_{ij}\dot{u}_{ll}), \qquad (3.3)$$

where the dot signifies time differentiation. For now, the dissipative stress tensor will be neglected.

The equations of motion of the elastic medium in linearized version are thus

$$\rho\frac{\partial^2 u_n}{\partial t^2} = \frac{\partial}{\partial x^m}(\delta_{mn}\lambda u_{ll} + 2\mu u_{mn}) + \tfrac{1}{2}\rho c^2 h_{mn,00}x^m . \qquad (3.4)$$

The boundary condition is that the total force per unit area at the surface of the elastic medium vanishes in the direction normal to the surface of the elastic body. If the normal vector is η_n then the three conditions

$$\eta_n\sigma_{ns} = 0 \qquad (3.5)$$

must be satisfied everywhere on the surface.

While the equations of motion have the gravitational driving force as a volume force, a simple change of field variables leads to the effective consideration of the gravitational force as a surface force. Introduce the new variable Z_n by defining

$$Z_n = u_n - \tfrac{1}{2}h_{mn}x^m . \qquad (3.6)$$

We shall show later that the new field variable Z_n can in the absence of self-fields be directly measured by means of seismometers; at this stage it is introduced for mathematical convenience. Since h_{mn} depends only upon time when evaluated at the center of mass of the elastic body, it follows that for the long-wavelength approximation

$$Z_{mn} = u_{mn} - \tfrac{1}{2}h_{mn} . \qquad (3.7)$$

Consequently, the equations of motion assume the form

$$\rho\frac{\partial^2 Z_n}{\partial t^2} = \frac{\partial}{\partial x^m}[\delta_{mn}\lambda Z_{ll} + 2\mu(Z_{mn} + \tfrac{1}{2}h_{mn})] \qquad (3.8)$$

and since spatial derivatives of h_{mn} are neglected

$$\rho\frac{\partial^2 Z_n}{\partial t^2} = \frac{\partial}{\partial x^m}(\delta_{mn}\lambda Z_{ll} + 2\mu Z_{mn}) . \qquad (3.9)$$

The driving force has disappeared from the equations of motion only to appear in the boundary condition

$$\eta_m[\delta_{mn}\lambda Z_{ll} + 2\mu(Z_{mn} + \tfrac{1}{2}h_{mn})] = 0 . \qquad (3.10)$$

It is calculationally useful to solve the problem in this reformulated version since the solution of the equations of motion inside the elastic medium is that of free vibrations, as pointed out by Dyson.

For a spherical body, the coordinates naturally suited to the problem are not the Cartesian coordinates used in the above equations. A transformation to spherical coordinates is clearly needed.

The equations of motion[10] are easiest to survey in vector form:

$$\rho \frac{\partial^2 \vec{Z}}{\partial t^2} = \mu \nabla^2 \vec{Z} + (\lambda + \mu)\nabla(\nabla\cdot\vec{Z}) . \tag{3.11}$$

If the longitudinal and transverse vector fields are introduced by the decomposition

$$\vec{Z} = \vec{Z}^{(t)} + \vec{Z}^{(l)} ,\quad \nabla\cdot\vec{Z}^{(t)} = 0 ,\quad \nabla\times\vec{Z}^{(l)} = 0 , \tag{3.12}$$

the equation of motion breaks up into two wave equations:

$$\frac{\partial^2 \vec{Z}^{(t)}}{\partial t^2} = c_t^2 \nabla^2 \vec{Z}^{(t)}, \quad c_t = (\mu/\rho)^{1/2}$$
$$\frac{\partial^2 \vec{Z}^{(l)}}{\partial t^2} = c_l^2 \nabla^2 \vec{Z}^{(l)}, \quad c_l = [(\lambda + 2\mu)/\rho]^{1/2} . \tag{3.13}$$

Before turning to the solution of the wave equations, a look at the boundary conditions will point the way to selecting those solutions which satisfy the boundary conditions.

The plane-wave amplitude and polarization has been denoted previously in rectangular Cartesian coordinates by h_{mn}. In spherical coordinates, the corresponding tensor components are

$$h_{rr} = \sin^2\theta(h_{xx}\cos2\phi + h_{xy}\sin2\phi) ,$$
$$h_{r\theta} = \sin\theta\cos\theta(h_{xx}\cos2\phi + h_{xy}\sin2\phi) = \frac{1}{2}\frac{\partial h_{rr}}{\partial\theta} , \tag{3.14}$$
$$h_{r\phi} = \sin\theta(-h_{xx}\sin2\phi + h_{xy}\cos2\phi) = \frac{1}{2}\frac{1}{\sin\theta}\frac{\partial h_{rr}}{\partial\phi} .$$

We shall consider waves of arbitrary polarization and write

$$h_{rr} = \sin^2\theta f(a\cos2\phi + b\sin2\phi) \equiv 2fS_2(\theta, \phi) . \tag{3.15}$$

Since all equations are linear, a, b, and f can be chosen as complex quantities with the restriction

$$|a|^2 + |b|^2 = 1 ,\quad f = he^{-i\omega t}, \tag{3.16}$$

where ω is the frequency of the gravitational wave and h is a real number specifying the amplitude of the wave. Convention dictates that the real part of all subsequent solutions is the physical result.

The function $S_2(\theta, \phi)$ is given by

$$S_2 = \tfrac{1}{2}\sin^2\theta(a\cos2\phi + b\sin2\phi) \tag{3.17}$$

so that

$$h_{r\theta} = \frac{\partial S_2}{\partial\theta} he^{-i\omega t} ,\quad h_{r\phi} = \frac{1}{\sin\theta}\frac{\partial S_2}{\partial\phi} he^{-i\omega t} . \tag{3.18}$$

The stress components which occur in the boundary conditions assume the form

$$\sigma_{rr} = 2\mu Z_{rr} + \lambda\nabla\cdot\vec{Z} ,\quad \sigma_{r\theta} = 2\mu Z_{r\theta} ,\quad \sigma_{r\phi} = 2\mu Z_{r\phi} , \tag{3.19}$$

where

$$Z_{rr} = \frac{\partial Z_r}{\partial r} ,$$
$$Z_{r\theta} = \frac{1}{2}\left(\frac{\partial Z_\theta}{\partial r} - \frac{Z_\theta}{r} + \frac{1}{r}\frac{\partial Z_r}{\partial\theta}\right) , \tag{3.20}$$
$$Z_{r\phi} = \frac{1}{2}\left(\frac{1}{\sin\theta}\frac{\partial Z_r}{\partial\phi} + \frac{\partial Z_\phi}{\partial r} - \frac{Z_\phi}{r}\right).$$

The equations which must be satisfied on the surface of the sphere where $r = R$ are

$$2\mu(Z_{rr} + \tfrac{1}{2}h_{rr}) + \lambda\nabla\cdot\vec{Z} = 0 ,$$
$$2\mu(Z_{r\theta} + \tfrac{1}{2}h_{r\theta}) = 0 , \tag{3.21}$$
$$2\mu(Z_{r\phi} + \tfrac{1}{2}h_{r\phi}) = 0 .$$

To satisfy these equations, we construct three linearly independent vector-harmonic solutions of the equations of motion. Standard rotation-group techniques[11] indicate that fields which transform according to the $J = 2$ representation of the rotation group should be sought.

The longitudinal solution $\vec{Z}^{(l)}$ can be written as

$$\vec{Z}^{(l)} = \nabla\Phi , \tag{3.22}$$

where Φ obeys the wave equation of Eq. (3.13). The solution which will fit the boundary condition at the center of the sphere and has the proper symmetry is

$$\vec{Z}^{(l)} = \frac{1}{q^2}\nabla[j_2(qr)S_2]e^{-i\omega t} , \quad q = \omega/c_l . \tag{3.23}$$

Here $j_2(qr)$ is a spherical Bessel function and the factor q^{-2} has been inserted for dimensional reasons. Here and below we anticipate that the solution involves the same S_2 presented by the incoming gravitational wave [Eq. (3.17)].

The two linearly independent transverse fields $\vec{Z}^{(t)}$ and $\vec{Z}^{(t_1)}$ are most easily constructed by finding an appropriate solution of an auxiliary scalar equation

$$\nabla^2\chi = \frac{1}{c_t^2}\frac{\partial^2\chi}{\partial t^2} . \tag{3.24}$$

The solution with the right symmetry and obeying the boundary conditions at the origin is

$$\chi = j_2(kr)S_2 e^{-i\omega t} , \qquad k=\omega/c_t \quad . \tag{3.25}$$

One transverse field is obtained by applying the rotationally invariant operator $\vec{L}=\vec{r}\times\vec{\nabla}$ to both sides of Eq. (3.24) to generate a vector solution

$$\vec{Z}^{(t_1)} = \frac{1}{k^2} j_2(kr)\vec{L}S_2 . \tag{3.26}$$

It is easy to verify that $\vec{\nabla}\cdot\vec{Z}^{(t_1)}=0$.

Another transverse vector field obeying the equations of motion is obtained from $\vec{Z}^{(t_1)}$ as follows:

$$\vec{Z}^{(t)} = -\frac{1}{k^2} \nabla\times[j_2(kr)\vec{L}S_2]\, e^{-i\omega t} . \tag{3.27}$$

It is also clearly a transverse field.

The boundary condition on the surface of the sphere is satisfied by choosing a particular linear combination of these vector fields

$$\vec{Z} = C_1\vec{Z}^{(t)} + C_2\vec{Z}^{(l)} + C_3\vec{Z}^{(t_1)} . \tag{3.28}$$

The toroidal mode described by $\vec{Z}^{(t_1)}$ can be disposed of immediately. A short calculation indicates that the solution for $\vec{Z}$ cannot satisfy the boundary conditions unless $C_3=0$.

The desired solution must be a linear combination of the remaining two vector fields. The longitudinal vector field is explicitly

$$\vec{Z}^{(l)} = \frac{e^{-i\omega t}}{q^2}\left[\hat{r}\,\frac{dj_2(qr)}{dr}S_2 + \hat{\theta}\,\frac{j_2(qr)}{r}\,\frac{\partial S_2}{\partial\theta} + \hat{\phi}\,\frac{j_2(qr)}{r}\,\frac{1}{\sin\theta}\,\frac{\partial S_2}{\partial\phi}\right] , \tag{3.29}$$

while the excited transverse vector field is

$$\vec{Z}^{(t)} = \frac{e^{-i\omega t}}{k^2}\left[\hat{r}6\,\frac{j_2(kr)}{r}\,S_2 + \frac{1}{r}\,\frac{d[rj_2(kr)]}{dr}\left(\hat{\theta}\,\frac{\partial S_2}{\partial\theta} + \hat{\phi}\,\frac{1}{\sin\theta}\,\frac{\partial S_2}{\partial\phi}\right)\right] . \tag{3.30}$$

The tensor strain fields which these vector displacement fields generate can conveniently be written in terms of the auxiliary functions

$$f_0(x) = j_2(x)/x^2 ,$$

$$f_1(x) = \frac{d}{dx}\,[j_2(x)/x] ,$$

$$f_2(x) = \frac{d^2}{dx^2}\,[j_2(x)] .$$

The longitudinal strain field is then

$$\begin{aligned} Z_{rr}^{(l)} &= f_2(qr)S_2 e^{-i\omega t} ,\\ Z_{r\theta}^{(l)} &= f_1(qr)\,\frac{\partial S_2}{\partial\theta}\,e^{-i\omega t} ,\\ Z_{r\phi}^{(l)} &= f_1(qr)\,\frac{1}{\sin\theta}\,\frac{\partial S_2}{\partial\phi}\,e^{-i\omega t} . \end{aligned} \tag{3.31}$$

The transverse strain field is

$$\begin{aligned} Z_{rr}^{(t)} &= 6f_1(kr)S_2 e^{-i\omega t} ,\\ Z_{r\theta}^{(t)} &= \tfrac{1}{2}[f_2(kr)+4f_0(kr)]\,\frac{\partial S_2}{\partial\theta}\,e^{-i\omega t} ,\\ Z_{r\phi}^{(t)} &= \tfrac{1}{2}[f_2(kr)+4f_0(kr)]\,\frac{1}{\sin\theta}\,\frac{\partial S_2}{\partial\phi}\,e^{-i\omega t} . \end{aligned} \tag{3.32}$$

The only remaining quantity needed for the explicit solution is $\vec{\nabla}\cdot\vec{Z}$. Only the longitudinal field contributes to $\vec{\nabla}\cdot\vec{Z}$ and the quantity is readily evaluated by using the equations of motion,

$$\vec{\nabla}\cdot\vec{Z}^{(l)} = \frac{1}{q^2}\,\nabla^2[j_2(qr)S_2]\,e^{-i\omega t} = -j_2(qr)S_2 e^{-i\omega t} . \tag{3.33}$$

The equations determining the correct linear combination $\vec{Z}=C_1\vec{Z}^{(t)}+C_2\vec{Z}^{(l)}$ are

$$\{2\mu[6C_1f_1(kr)+C_2f_2(qr)] - C_2\lambda j_2(qr)\}S_2 = -2\mu hS_2 , \tag{3.34a}$$

$$\{\tfrac{1}{2}C_1[f_2(kr)+4f_0(kr)] + C_2f_1(qr)\}\,\frac{\partial S_2}{\partial\theta} = -\tfrac{1}{2}h\,\frac{\partial S_2}{\partial\theta} , \tag{3.34b}$$

$$\{\tfrac{1}{2}C_1[f_2(kr)+4f_0(kr)] + C_2f_1(qr)\}\,\frac{1}{\sin\theta}\,\frac{\partial S_2}{\partial\phi} = -\tfrac{1}{2}h\,\frac{1}{\sin\theta}\,\frac{\partial S_2}{\partial\phi} . \tag{3.34c}$$

All equations are to be evaluated at $r=R$. The form of the last two equations indicates that these are not independent.

The two coefficients C_1 and C_2 are now completely determined and specify uniquely the response of the elastic sphere to a gravitational wave. One need only solve the two simultaneous equations

$$\begin{aligned} C_1[12\mu f_1(kR)] + C_2[2\mu f_2(qR) - \lambda j_2(qR)] &= -2\mu h ,\\ C_1[\tfrac{1}{2}f_2(kR)+2f_0(kR)] + C_2[f_1(qR)] &= -\tfrac{1}{2}h . \end{aligned} \tag{3.35}$$

These coupled equations can be inverted to find C_1 and C_2 provided that the determinant of the coefficients of C_1 and C_2 does not vanish. The vanishing of the determinant is a signal that the frequency of the incident gravitational wave coincides with a natural frequency of the elastic system. For such a situation, it becomes important to treat the damping of the free vibrations of the sphere.

The more general version of this problem which includes a self-stress of the sphere is the subject of the subsequent section. Numerical evaluation of these results will be discussed there.

IV. EQUATIONS OF MOTION OF SELF-STRESSED SPHERE

Most astronomical bodies are sufficiently massive that they cannot be described as homogeneous spheres. In particular, the stress on the body due to its own gravitational field causes radial variations in density and elastic moduli in the equilibrium state. Further, alterations in the amplitude of the displacement field induced by a gravitational wave give rise to additional gravitational fields which can significantly affect the motion of the body. Such effects are nonlinear because they require calculations of corrections to the potential term $h(\vec{r},t)=V/c^2$ of Eq. (1.20), which are of first order in the amplitude of the incident gravitational wave. Similar effects have previously been considered in treating the free vibrations of the earth.[12]

Let $\rho(\vec{r},t)$ be the proper mass density which is displaced by the amount $\vec{u}(\vec{r},t)=\vec{y}-\vec{r}$ to a nearby location, and let $p(\vec{r},t)$ be the scalar pressure. The six variables to be determined are the displacement field $\vec{u}$, the density ρ, pressure p, and the potential V. Five equations are provided by Einstein's field equations; one of these is the field equation for V. Four more equations follow from the Bianchi identities: one equation representing energy conservation and three representing momentum conservation. The remaining equation is an equation of state relating stress to strain in terms of Lamé coefficients.

A double expansion in small quantities is to be made: All terms are expanded in powers of c^{-1} and only leading terms—corresponding to a Newtonian approximation—are retained; these terms are then expanded to first order in the amplitude of the incident wave.

The conservation equations are

$$T^{\mu\nu}{}_{;\nu}=0\,, \tag{4.1}$$

where the stress-energy tensor is

$$T^{\mu\nu}=\rho U^\mu U^\nu+p(g^{\mu\nu}+U^\mu U^\nu/c^2)-\sigma^{\mu\nu}_{\rm sh}\,. \tag{4.2}$$

Here, $U^\mu=c\,dx^\mu/ds$ is the four-velocity, and the projection operators $U^\mu U^\nu/c^2$ and $(g^{\mu\nu}+U^\mu U^\nu/c^2)$ project onto subspaces parallel and orthogonal to U^λ, respectively. For convenience, we have separated the stress into a contribution associated with dilation—the scalar pressure—and a contribution associated with shear, $\sigma^{\mu\nu}_{\rm sh}$.

The Eulerian equations of motion which follow from Eqs. (4.1) and (4.2) are

$$(\rho+p/c^2)U^\nu U^\mu{}_{;\nu}+(g^{\mu\nu}+U^\mu U^\nu/c^2)p_{;\nu} =(\delta^\mu_\lambda+U^\mu U_\lambda/c^2)\sigma^{\lambda\nu}_{\rm sh\,;\nu}\,, \tag{4.3}$$

with the momentum equations obtained by taking $\mu=1,2,3$. The remaining conservation equation follows from $U_\mu T^{\mu\nu}{}_{;\nu}=0$ and the requirement $U_\mu\sigma^{\mu\nu}_{\rm sh}=0$; it is

$$(\rho U^\nu)_{;\nu}+\frac{p}{c^2}U^\nu{}_{;\nu}=0\,. \tag{4.4}$$

To lowest order in c^{-1} the momentum equations assume the form

$$\rho c(U^k{}_{,0}+\Gamma^k_{00}c)+p^{,k}=\sigma^{km}{}_{,m}\,. \tag{4.5}$$

At this point, an expansion in powers of h_{kl} is made. Keeping only first-order corrections, we write

$$\begin{aligned}\rho&=\rho_0+\rho^{(1)}\,,\\ p&=p_0+p^{(1)}\,,\\ \Gamma^k_{00}&=\Gamma^{k(0)}_{00}+\Gamma^{k(1)}_{00}\,,\\ V&=V_0+V^{(1)}\,,\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}\Gamma^{k(0)}_{00}&=V_{0,k}/c^2\,,\\ \Gamma^{k(1)}_{00}&=V^{(1)}{}_{,k}/c^2-\tfrac{1}{2}h_{ks,00}x^s\,,\end{aligned} \tag{4.7}$$

Substitution of these expressions into (4.5) and collecting zeroth-order terms yields the expected equation

$$p_{0,k}=-\rho_0 V_{0,k} \tag{4.8}$$

which expresses the fact that the static radial pressure gradient balances the gravitational force. We are assuming that in the static equilibrium case, ρ_0, p_0, and V_0 are spherically symmetric. The first-order terms in (4.5) involve the displacement field u_k, since $cU^k{}_{,0}\cong\ddot{u}_k$. We have, on discarding all terms with factors c^{-1} or c^{-2}

$$\rho_0\ddot{u}_k+\rho_0V^{(1)}{}_{,k}+\rho^{(1)}V_{0,k}+p^{(1)}{}_{,k}-\sigma^{km}_{\rm sh,m}=\tfrac{1}{2}\rho_0\ddot{h}_{ks}x^s\,. \tag{4.9}$$

Similarly, expanding Eq. (4.4) yields the zeroth- and first-order expressions

$$\begin{aligned}\frac{\partial\rho_0}{\partial t}&=0\,,\\ \frac{\partial\rho^{(1)}}{\partial t}+\rho_0\nabla\cdot\vec{U}+\rho_{0,k}U^k&=0\,.\end{aligned} \tag{4.10}$$

Since $U_k \cong u_k$, the last equation implies

$$\rho^{(1)} = -\vec{u}\cdot\nabla\rho_0 - \rho_0\Delta\,, \tag{4.11}$$

where $\Delta = u_{l,l}$ is the fractional change in volume resulting from the displacement. In general a change δp of pressure can be related to a change $\delta\rho$ of density by means of the bulk modulus K, where

$$K = \lambda + \tfrac{2}{3}\mu\,. \tag{4.12}$$

The relation is

$$\delta p = K\delta\rho/\rho\,. \tag{4.13}$$

Combining Eqs. (4.11) and (4.13) yields the following expression for $p^{(1)}$:

$$p^{(1)} = -\vec{u}\cdot\nabla p_0 - K\Delta\,. \tag{4.14}$$

Equations (4.11) and (4.14) are equivalent to Rayleigh's method of accounting for the initial stress in a body. We also have for the shear stress

$$\sigma_{\rm sh}^{km} = 2\mu(u_{km} - \tfrac{1}{3}\delta_{km}\Delta)\,. \tag{4.15}$$

Using Eqs. (4.8), (4.11), (4.14), and (4.15) to eliminate p, $\rho^{(1)}$, and $\sigma_{\rm sh}^{km}$, we obtain the equation of motion

$$\rho_0\ddot{u}_k - (\lambda\Delta)_{,k} - (\mu u_{k,m})_{,m} - (\mu u_{m,k})_{,m} - (u_m\rho_{0,m} + \rho_0\Delta)V_{0,k} + \rho_0 V^{(1)}{}_{,k} + (\rho_0 u_m V_{0,m})_{,k} = \tfrac{1}{2}\rho_0\ddot{h}_{ks}x^s\,. \tag{4.16}$$

The equations determining V_0 and $V^{(1)}$ follow from Einstein's field equations which give in this approximation

$$\nabla^2 V = 4\pi G\rho\,,$$

which can be written

$$\nabla^2 V_0 = 4\pi G\rho_0\,,\qquad \nabla^2 V^{(1)} = -4\pi G(\rho_0\Delta + u_m\rho_{0,m})\,. \tag{4.17}$$

Given the zeroth-order solution for V_0, Eqs. (4.16) and (4.17) form a closed system of equations for the unknown quantities $\vec{u}$, $V^{(1)}$.

Viscous damping may be formally included by making the replacements

$$\lambda \to \lambda - i\omega(\zeta + \tfrac{2}{3}\eta),\quad \mu \to \mu - i\omega\eta\,, \tag{4.18}$$

where again the time dependence $e^{-i\omega t}$ has been assumed for all first-order quantities.

Boundary conditions are imposed by requiring that the strain vanish at the center of the sphere, and that the normal components of the stress vanish on the surface of the sphere at $r = R$:

$$\begin{aligned} &2\mu u_{r,r}(R) + \lambda\Delta(R) = 0\,,\\ &u_{\theta,r}(R) + [u_{r,\theta}(R) - u_\theta(R)]/R = 0\,,\\ &u_{\phi,r}(R) + [u_{r,\phi}(R)/\sin\theta - u_\phi(R)]/R = 0\,. \end{aligned} \tag{4.19}$$

The potentials V_0 and $V^{(1)}$ are continuous at the surface, and vanish as $r \to 0$ by gauge choice. For $r > R$ the potential $V^{(1)}$ satisfies Laplace's equation and can be expanded in spherical harmonics:

$$V^{(1)} = \sum_{l,m} C_{lm}P_l(r)Y_{lm}(\theta,\phi)\,. \tag{4.20}$$

However, only the terms for $l = 2$ will be excited by the gravitational wave; for these terms $P_l(r)$ is proportional to r^{-3}. Matching the interior solution of Eq. (4.17) with the exterior solution yields the following explicit condition on the interior solution:

$$\frac{\partial V^{(1)}(R)}{\partial R} + (l+1)\frac{V^{(1)}(R)}{R} = -4\pi G\rho_0 u_r(R)\,, \tag{4.21}$$

where only the $l = 2$ term has been assumed to contribute.

In the following section, the differential equations (4.16) and (4.17) are solved, subject to the boundary conditions (4.19) and (4.21).

V. SOLUTIONS FOR HOMOGENEOUS SELF-STRESSED SPHERE

If we now assume that a plane monochromatic wave is incident on the sphere, it is represented by h_{kl} or h_{rr}, $h_{r\theta}$, $h_{r\phi}$ as given by Eqs. (3.14). The differential equations to be solved are (4.11) and (4.12). Regarding the right-hand side of (4.11) as an inhomogenous driving term, the general solution of the system of equations consists of a particular solution of the inhomogeneous equations, to which is added the general solution of the homogeneous equations obtained by setting $h_{kl} = 0$. This method of solution is to be contrasted with that discussed in Sec. III, which involved a new field variable such that the effect of the gravitational driving terms only appeared in the boundary conditions. The particular solution is easily obtained for the case of uniform density, and uniform elastic constants; it is denoted by subscripts p:

$$u_p^k = \tfrac{1}{2}h_{kl}x^l,\quad V_p^{(1)} = -u_p^k V^{(0)}{}_{,k}\,,$$

where

$$V^{(0)} = \frac{2\pi G\rho_0}{3}r^2$$

is the static gravitational potential due to the spherical mass. Thus for the particular solution,

$$V_p^{(1)} = -\tfrac{1}{2}h_{kl}x^k x^l\frac{4\pi G\rho_0}{3} = -\tfrac{1}{2}Ar^2 h_{rr}\,, \tag{5.1}$$

where $A = 4\pi G\rho_0/3$, and h_{rr} has been defined in

Eq. (3.14).

In vector form the homogeneous elastic equations (4.11) are

$$\rho_0 \frac{\partial^2 \vec{u}}{\partial t^2} = \mu \nabla^2 \vec{u} + (\lambda + \mu)\nabla(\Delta) + A\rho_0 \Delta \vec{r} - \rho_0 \nabla V^{(1)} - A\rho_0 \nabla(r u_r) .$$

The terms arising from the self-gravitational field prevent the above equation from being easily separated into longitudinal and transverse parts as in Sec. IV.

Using the decomposition given in Eqs. (3.31) and (3.32), writing (no toroidal excitations)

$$u_r = U(r) S_2(\theta, \phi), \quad u_\theta = W(r) \frac{\partial S_2}{\partial \theta}, \quad u_\phi = W(r) \frac{1}{\sin\theta} \frac{\partial S_2}{\partial \phi} \tag{5.3}$$

and assuming

$$V^{(1)} = P(r) S_2 ,$$

we find after some calculation that the dilation is given by

$$\Delta = X(r) S_2 = (U_{,r} + 2U/r - 6W/r) S_2 . \tag{5.4}$$

The differential equation (4.12) for $V^{(1)}$ becomes

$$\nabla_1^2 P = -3AX . \tag{5.5}$$

The solution of the elastic equations may be obtained with the aid of the following definition:

$$H = \frac{\partial W}{\partial r} + (W - U)/r . \tag{5.6}$$

Then $\nabla \times \vec{u}$ may be expressed in terms of H as

$$\nabla \times \vec{u} = (\vec{r} \times \nabla)(H S_2) .$$

Taking the curl of the elastic equation then immediately yields the following equation for the transverse part of the elastic displacement:

$$\frac{\mu}{\rho_0} \nabla_1^2 H + \omega^2 H = AX . \tag{5.7}$$

With the help of the easily established identity

$$\nabla \cdot (\vec{r}\Delta) - \nabla^2(\vec{r} \cdot \vec{u}) = \Delta - 6 H S_2 ,$$

the divergence of the elastic equation yields the following equation for the longitudinal part of the elastic displacement:

$$\frac{\lambda + 2\mu}{\rho_0} \nabla_1^2 X + (\omega^2 + 4A) X = 6AH . \tag{5.8}$$

From the two equations [(5.7) and (5.8)] it is seen that the effect of the gravitational field of the mass is to couple the longitudinal and transverse displacements. When $A = 0$, the equations reduce to those of Sec. III.

Operating on the latter equation with $\nabla_1^2 + \omega^2 \rho_0/\mu$ and eliminating H yields a fourth-order differential equation for X which can be factorized:

$$(\nabla_1^2 + k^2)(\nabla_1^2 + q^2) X = 0 ,$$

where q^2 and where $k^2 > q^2$ are solutions of the following quadratic equation for x:

$$(x - \omega^2 \rho_0/\mu)[x - (\omega^2 + 4A)\rho_0/(\lambda + 2\mu)] - 6A^2 \rho_0^2/\mu(\lambda + 2\mu) = 0 .$$

The solution for X is then

$$X = -C_1 j_2(kr) - C_2 j_2(qr) , \tag{5.9}$$

where C_1 and C_2 are arbitrary constants and where minus signs have been introduced to facilitate comparison with the results given in Sec. III. For H we find

$$H = -\tfrac{1}{6}[S C_1 j_2(kr) + T C_2 j_2(qr)] , \tag{5.10}$$

where

$$S = [\omega^2 + 4A - k^2(\lambda + 2\mu)/\rho_0]/A ,$$

$$T = [\omega^2 + 4A - q^2(\lambda + 2\mu)/\rho_0]/A ,$$

and for P we find

$$P = -\frac{3AC_1}{k^2} j_2(kr) - \frac{3AC_2}{q^2} j_2(qr) + (\omega^2 - 2A) C_3 r^2 , \tag{5.11}$$

where for convenience the arbitrary constant which is the coefficient of r^2 has been written with an extra factor $\omega^2 - 2A$. The solutions for U and W may now be found in a straightforward way. They are

$$U(r) = C_1(2 + S) j_2(kr)/k^2 r - C_1 j_3(kr)/k + C_2(2 + T) j_2(qr)/q^2 r - C_2 j_3(qr)/q + 2C_3 r ,$$
$$W(r) = C_1(1 + \tfrac{1}{2}S) j_2(kr)/k^2 r - C_1 S j_3(kr)/6k + C_2(1 + \tfrac{1}{2}T) j_2(qr)/q^2 r - C_2 T j_3(qr)/q + C_3 r . \tag{5.12}$$

To the above general solution of the homogeneous differential equations must be added the particular solution which is given by

$$X_p = H_p = 0 , \quad P_p = -A r^2 h , \quad U_p = r h , \quad W_p = \tfrac{1}{2} r h , \tag{5.13}$$

where h is given in terms of h_{rr} by Eqs. (3.15) and (3.16).

The boundary conditions at $r=R$, the radius of the sphere, are given by Eqs. (4.14). We shall write these equations in the form

$$\sum_{j=1}^{3} a_{ij} C_j = \alpha_i h \quad (i=1, 2, 3), \tag{5.14}$$

where after considerable calculation we find the following expressions for a_{ij} and α_i:

$$\begin{aligned}
\alpha_1 &= -2\mu ,\\
\alpha_2 &= -1 ,\\
\alpha_3 &= 2A ;
\end{aligned}\tag{5.15}$$

$$\begin{aligned}
a_{11} &= 2\mu S f_1(kR) + 2\mu f_2(kR) - \lambda j_2(kR) ,\\
a_{12} &= 2\mu T f_1(qR) + 2\mu f_2(qR) - \lambda j_2(qR) ,\\
a_{13} &= 4\mu ,\\
a_{21} &= j_2(kR)[(2+S)/k^2R^2 - \tfrac{1}{6}S]\\
&\quad + j_3(kR)(-2+\tfrac{1}{3}S)/kR ,\\
a_{22} &= j_2(qR)[(2+T)/q^2R^2 - \tfrac{1}{6}T]\\
&\quad + j_3(qR)(-2+\tfrac{1}{3}T)/qR ,\\
a_{23} &= 2 ,\\
a_{31} &= Aj_2(kR)(3S-9)/k^2R^2 ,\\
a_{32} &= Aj_2(qR)(3T-9)/q^2R^2 ,\\
a_{33} &= (5\omega^2 - 4A) .
\end{aligned}\tag{5.16}$$

The constants C_i are then obtained by inverting the matrix a_{ij},

$$C_i = \sum_j (a^{-1})_{ij} \alpha_j h , \tag{5.17}$$

and from these values the functions U, W and the displacements u_r, u_θ, u_ϕ can be calculated. If damping is neglected, the resonant frequency of the freely oscillating sphere is obtained by setting $\det(a_{ij})=0$, and a^{-1} will not exist.

High-frequency limit. It is interesting to compare the results of the present calculations with those of Dyson, in the limit of high frequency (frequency high compared with the fundamental resonance frequencies of the sphere). In this limit we have

$$\begin{aligned}
&k^2 \simeq \omega^2\rho_0/\mu, \quad q^2 \simeq \omega^2\rho_0/(\lambda+2\mu), \quad \omega^2 >> A\\
&j_2(kR) \simeq -(\sin kR)/kR, \quad j_3(kR) = (\cos kR)/kR,\\
&S \simeq -\omega^2(\lambda+\mu)/A\mu, \quad T \simeq 6A(\lambda+2\mu)/\omega^2(\lambda+\mu) << 1.
\end{aligned}\tag{5.18}$$

Since $S >> 1$, it is convenient to obtain expressions for U and W by letting

$$C_1 \to 6C_1/S, \quad S \to \infty.$$

Then neglecting terms proportional to T, we have for the general solution, to which must be added the particular solution,

$$\begin{aligned}
U &= 6C_1 j_2(kr)/k^2 r + 2C_2 j_2(qr)/q^2 r\\
&\quad + C_2 j_3(qr)/q + 2C_3 r,\\
W &= 3C_1 j_2(kr)/k^2 r + C_1 j_3(kr)/k + C_2 j_2(qr)/q^2 r + C_3 r,\\
X &= -C_2 j_2(qr), \quad H = -C_1 j_2(kr),\\
P &= -3AC_2 j_2(qr)/qr + \omega^2 C_3 r^2.
\end{aligned}\tag{5.19}$$

For this case the matrix of coefficients becomes approximately

$$\begin{aligned}
&a_{11} = a_{22} = a_{31} = a_{32} \simeq 0,\\
&a_{12} = (\lambda+2\mu)(\sin qR)/qR,\\
&a_{13} = 4\mu, \quad a_{23} = 2,\\
&a_{21} = (\sin kR)/kR,\\
&a_{33} = 5\omega^2
\end{aligned}\tag{5.20}$$

The solutions are

$$\begin{aligned}
&C_3 = 2Ah/5\omega^2 \simeq 0,\\
&C_2 - 2\mu qRh/[(\lambda+2\mu)\sin qR],\\
&C_1 = -kRh/\sin kR.
\end{aligned}\tag{5.21}$$

The horizontal component of displacement at the surface in this limit, keeping only leading terms in order $1/\omega$, is given by

$$\begin{aligned}
W(R) &= W_p + C_1 j_3(kR)/k\\
&= \tfrac{1}{2}Rh - h\left(\frac{\mu}{\rho_0}\right)^{1/2}\frac{1}{\omega}\cot[\omega R(\rho_0/\mu)^{1/2}].
\end{aligned}\tag{5.22}$$

The horizontal component of displacement is then obtained by multiplying by an appropriate function of θ and ϕ; in this limit the self-stress does not affect the displacement. In the direction perpendicular to the propagation direction of the wave,

$$u_\phi = W(R)\frac{1}{\sin\theta}\frac{\partial S_2}{\partial\phi}.$$

The term $\frac{1}{2}Rh(1/\sin\theta)\partial S_2/\partial\phi$ corresponds to the horizontal component of the motion of a free test mass:

test-mass displacement

$$= \tfrac{1}{2}R\sin\theta(h_{xx}\sin 2\phi - h_{xy}\cos 2\phi).$$

The remaining term is the "seismic displacement" which should be compared with Dyson's result.[9] In the direction normal to the direction of propagation the horizontal seismic displacement is therefore

$$+\left(\frac{\mu}{\rho_0}\right)^{1/2}\frac{1}{\omega}\cot\left[\omega R\left(\frac{\rho_0}{\mu}\right)^{1/2}\right]\sin\theta(h_{xx}\sin 2\phi - h_{xy}\cos 2\phi). \tag{5.23}$$

At high frequencies the characteristic attenuation length of elastic waves is small compared to R. Then $\omega R/(\rho_0/\mu)^{1/2}$ effectively has a large imaginary part and the factor $\cot[\omega R(\rho_0/\mu)^{1/2}]$ becomes a phase factor, $e^{(-i\pi/2)}$.

Numerical calculations. Numerical calculations of $U(R)$ and $W(R)$ were carried out for a homogeneous spherical model of the earth with the following parameters: $\rho_0 = 5.52$ g/cm^3, $\mu = \lambda = 0.91 \times 10^{12}$ dyn/cm^2, $R = 6.37 \times 10^8$ cm, $\eta = 7.0 \times 10^{11}$ poise, $\zeta = \frac{1}{10}\eta$, corresponding to a fundamental spheroidal oscillation which occurs at a period of 3402 sec. The damping constants have been chosen to give a value of $Q \simeq 400$ for the fundamental.

The dimensionless vertical strain per unit gravitational wave amplitude $U(R)/Rh$ is plotted in Fig. 1 as a function of the period of the gravitational wave. The vertical displacement is seen to be significant only in the neighborhood of the fundamental. On the other hand, the dimensionless horizontal strain $W(R)/Rh$, which is plotted in Fig. 2, is significant in the neighborhood of the first harmonic, ${}_1S_2$.

The calculated value of the period of ${}_1S_2$ is 2355 sec whereas the observed period is only 1462.9 sec. This is because at higher frequencies the details of the earth's layered structure have a more significant effect on the resonance frequencies. It is nevertheless expected that for the real earth the principal horizontal response will occur at the ${}_1S_2$ mode frequency.

VI. MEASUREMENT OF WAVE-INDUCED DISPLACEMENTS

The displacement field u^k in general has both horizontal and vertical components; in this section we consider how these displacements may be observed in principle by means of various types of sensors—such as gravimeters and seismometers—attached to the surface of the self-gravitating sphere.

Gravimeter. A gravimeter measures acceleration relative to a local inertial frame. At low velocities, the motion of a gravimeter which is freely falling and which would measure zero acceleration will satisfy the equation

$$\ddot{x}^k + c^2\Gamma^k_{00} = 0. \tag{6.1}$$

A gravimeter of mass m acted on by a mechanical force F^k will move according to the equation

$$\ddot{x}^k + c^2\Gamma^k_{00} = F^k/m; \tag{6.2}$$

hence the measured value of the acceleration will be given by the left-hand side of Eq. (6.2). For a gravimeter attached to the surface of the earth, we may take

$$x^k = x^k_0 + u^k(R) \tag{6.3}$$

where x^k_0 is the equilibrium position on the surface. If the gravimeter measures only the time-dependent part Δg of the radial acceleration induced by an incident gravitational wave, we have

$$\Delta g = \ddot{u}_r(R) + c^2\Gamma^r_{00}(R + u_r(R)) - c^2\Gamma^{r(0)}_{00}(R)$$
$$= \left[\ddot{U}(R) - \ddot{h}R + AU + \frac{\partial P(R)}{\partial R}\right] S_2(\theta, \phi), \tag{6.4}$$

where the subscript or superscript r refers to the radial direction. The first term in Eq. (6.4) may be interpreted as the contribution to the observed variation in g due to radial acceleration of the gravimeter. The second term is due to the force exerted by the incoming wave. The third term is due to motion of the gravimeter through the body's

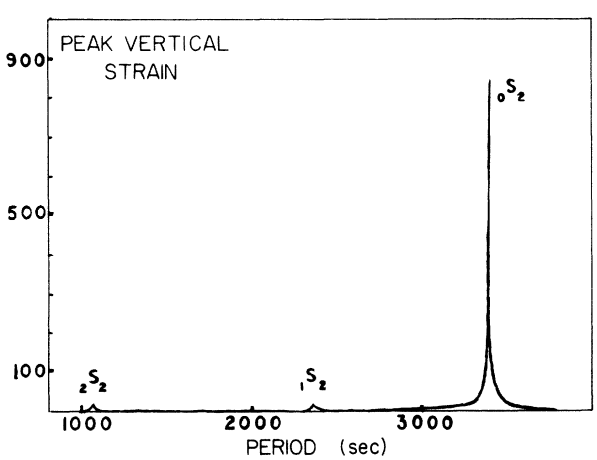

FIG. 1. Dimensionless vertical strain per unit gravitational wave amplitude $[U(R)/Rh]$ vs period of gravitational wave.

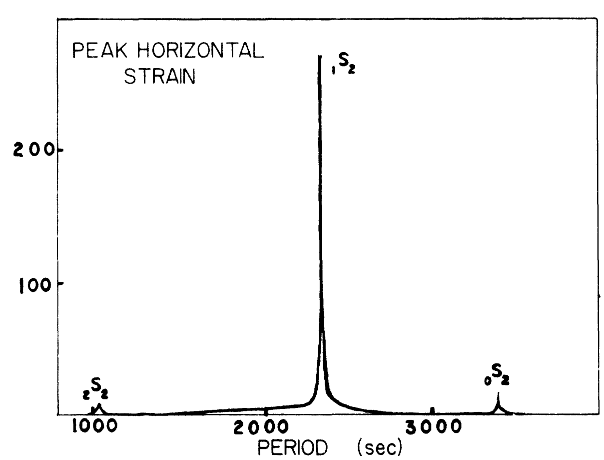

FIG. 2. Dimensionless horizontal strain per unit gravitational wave amplitude $[W(R)/Rh]$ vs period of gravitational wave.

gravitational field, and the fourth term arises from the changing mass distribution. The third and fourth terms in Eq (6.4) are written in a form appropriate for a gravimeter placed just below the surface of the body which has been modeled as a uniform sphere; if the instrument were placed just above the surface, the third term should be replaced by $-2AU$. For the real earth the correct form of this term should be determined by the static radial variation in g at the actual location of the instrument. For a spherical uniform model earth with a gravimeter placed just above the surface the fourth term should be replaced by

$$\left.\frac{\partial P}{\partial r}\right|_{r+\epsilon}=\left.\frac{\partial P}{\partial r}\right|_{r-\epsilon}+4\pi\rho_0 GU(R), \tag{6.5}$$

which follows from Eq (4 17).

Seismometer. A seismometer may be considered as a nearly free test mass supported by a force sufficient to balance out the static part of the earth's gravitational field. Thus the motion of the seismometer mass is described by the equation

$$\ddot{x}_S^k+c_2\Gamma_{00}^{k(1)}=0, \tag{6.6}$$

and hence for a monochromatic wave the position of the seismometer test mass will be given by

$$x_S^k=x_0^k+c^2\Gamma_{00}^{k(1)}/\omega^2. \tag{6.7}$$

The surface of the earth at this point will, however, move as

$$x_E^k=x_0^k+u^k(R) \tag{6.8}$$

and therefore the observed seismic displacement will be given by

$$x_E^k-x_S^k=u^k(R)-\tfrac{1}{2}h_l^k x_0^l+\frac{1}{2\omega^2}(PS_2)_{,k}\Big|_{r=R}\;. \tag{6.9}$$

The third term in this expression arises from the changing gravitational field of the earth. If the self-stress of the earth is omitted, then the seismic displacement is $u^k(R)-\frac{1}{2}h_l^k x_0^l$; this is the quantity Z^k defined in Eq. (3.6). We therefore conclude that Z^k is, in the absence of self-fields, the appropriate variable to call the displacement field of the elastic body measured relative to seismometer test masses. This result differs from that of Dyson[9] in the sign of the gravitational wave-driven term, $-\frac{1}{2}h_l^k x_0^l$.

Horizontal strainmeter. As a final example of a possible detection device we consider a strainmeter which measures the fractional change in length between two points on the earth's surface. Suppose that when in equilibrium the ends are at the positions x_l^k and x_r^k. Then when displaced due to motion of the surface, the two ends will be at $x_l^k+u^k(x_l)$ and $x_r^k+u^k(x_r)$, respectively. The change of length will be given in terms of the vector $\Delta l^k=u^k(x_r)-u^k(x_l)$. Since the difference $l^k=x_r^k-x_l^k$ may in practical cases be considered small compared to R, this length change can be expressed as

$$\begin{aligned}\Delta l^k&=\vec{l}\cdot\nabla u^k\\&=\frac{\partial u^k}{\partial l},\end{aligned} \tag{6.10}$$

where $\partial/\partial l$ denotes a derivative taken in the direction of the vector l^k. If only the horizontal components of strain are measured, then we may replace u^k by a combination of its horizontal components, u_θ or u_ϕ [see Eq (5.3)]. For example, for a north-south orientation of the strainmeter, the strain is

$$\begin{aligned}\frac{\Delta l_\theta}{l}&=\frac{1}{R}\frac{\partial u_\theta}{\partial\theta}\\&=\frac{W(R)}{R}\frac{\partial^2 S_2}{\partial\theta^2}.\end{aligned} \tag{6.11}$$

The function $\partial^2 S_2/\partial\theta^2$ is of order unity and depends on position on the surface, so that basic measure of horizontal strain at the surface is the ratio $W(R)/R$. A vertical strainmeter may be analyzed in a similar manner.

APPENDIX

The metric tensor uses the Landau-Lifshitz spacelike convention which for a Minkowski space is

$$ds^2=-(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2. \tag{A1}$$

Greek indices assume the values from 0 to 3.

Latin indices assume the values 1, 2, and 3 except when r and s are indices. In this latter case, r and s run over the indices 1 and 2.

In the Newtonian approximation used from Sec. III onwards, covariant and contravariant indices are not distinguished since an orthonormal basis is used.

The four-velocity U^μ is defined by

$$U^\mu=c\frac{dx^\mu}{ds}. \tag{A2}$$

The stress tensor σ_{mn} has the sign conventionally used in elasticity theory,[10] while the energy density assumes the form

$$T^{00}=\rho c^2 \tag{A3}$$

in Newtonian approximation.

[1]A recent survey of gravitational wave detection is given by Terrence J. Sejnowski, Phys. Today 27, 40 (1974).

[2]J. Weber, M. Lee, D. J. Getz, G. Rybeck, V. L. Trimble, and S. Steppel, Phys. Rev. Lett. 31, 779 (1973).

[3]W. H. Press and K. S. Thorne, Annu. Rev. Astron. Astrophys. 10, 335 (1972).

[4]J. Weber, Phys. Rev. Lett. 18, 498 (1967); F. J. Dyson, Astrophys. J. 156, 529 (1969).

[5]See, for example, C. W. Misner, K. S. Thorne, and J. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973), Chap. 35, or Steven Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), Chap. 10.

[6]Fermi coordinate systems are discussed by J. J. Stoker, *Differential Geometry* (Wiley-Interscience, New York, 1969), Chap. 9, p. 312. Note that because of the high symmetry of the plane gravitational wave metric, the coordinate system we use is locally Lorentzian everywhere in a plane. More general coordinate systems which are locally Lorentzian along a geodesic have been discussed by F. K. Manasse and C. W. Misner, J. Math. Phys. 4, 735 (1963).

[7]J. Weber, *General Relativity and Gravitational Waves* (Interscience, New York, 1961) Chap. 8, p. 124.

[8]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1971), Third English edition, Chap. 11, p. 266.

[9]F. J. Dyson, Ref. 4.

[10]The notation we use is that of L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon, Oxford, 1970).

[11]M. E. Rose, *Multipole Fields* (Wiley, New York, 1955).

[12]C. L. Pekeris and H. Jarosch, in *Contributions in Geophysics: In Honor of Beno Gutenberg*, edited by Hugo Benioff, Jr. *et al.* (Pergamon, Oxford, 1958), Chap. 13.