PHYSICAL REVIEW D VOLUME 12, NUMBER 10 15 NOVEMBER 1975

Effective Lagrangian for the Yang-Mills field

M. J. Duff*
Mathematics Department, King's College, London, England

M. Ramón-Medrano†
Physics Department, Imperial College, London, England
(Received 21 July 1975)

A recently proposed method for evaluating effective Lagrangians is applied to the Yang-Mills field.

I. INTRODUCTION

In a recent paper,[1] a simple method was presented for the evaluation in the quantum field theory of the effective Lagrangian induced by one-loop quantum effects. Using functional integrals and DeWitt's background-field method,[2] exact results could be obtained by imposing appropriate "quasilocal" conditions on the background fields.

In this paper we demonstrate that the methods described in Ref. 1 (henceforth referred to as paper I) may easily be generalized to accommodate closed loops of gauge quanta and the fictitious-particle contributions. Only the outline of the arguments is given here, and we refer the reader to paper I for specific details.

II. FORMALISM

We consider, as a typical example, the pure Yang-Mills theory

$$\mathcal{L}^{\rm YM}_{(0)} = -\tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}\,, \tag{2.1}$$

where

$$G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + gf^{abc}B^b_\mu B^c_\nu \tag{2.2}$$

with f^{abc} the completely antisymmetric structure constants of the group.

As in paper I, we make the background-field replacement

$$B^a_\mu \to B^a_\mu + b^a_\mu\,, \tag{2.3}$$

where b^a_μ is the quantum field variable and B^a_μ is now to be regarded as a classical background field. Now, however, we must also add a gauge-breaking term to $\mathcal{L}^{\rm YM}_{(0)}$ and take into account the corresponding fictitious-particle contribution. The gauge-breaking addition is chosen to be[3]

$$\mathcal{L}^{\rm gauge} = -\tfrac{1}{2}(\nabla_\mu b_\mu)^2\,, \tag{2.4}$$

where the covariant derivative

$$\nabla^{ab}_\mu = \partial_\mu \delta^{ab} + gf^{acb}B^c_\mu \tag{2.5}$$

depends only on the background field. As in paper I, the vecter single-loop effects will now be governed by that part of $\mathcal{L}^{\rm YM}_{(0)} + \mathcal{L}^{\rm gauge}$ which is bilinear in the quantum field b^a_μ. We denote the resulting Lagrangian by L_V. After a little algebra we find

$$L_V = -\tfrac{1}{2}(\nabla_\mu b_\nu)^2 - gf^{abc}G^a_{\mu\nu}b^b_\mu b^c_\nu\,. \tag{2.6}$$

Corresponding to the gauge choice (2.4), the fictitious-particle Lagrangian is

$$\mathcal{L}^{\rm fp} = -\nabla_\mu\omega^{*a}\nabla_\mu\omega^a + gf^{abc}\nabla_\mu\omega^{*a}\omega^b b^c_\mu \tag{2.7}$$

written in terms of the complex fermion fields ω and ω^*. These fields are purely quantum and have themselves no background part. Again keeping terms bilinear in the quantum variables, therefore, the fictitious single-loop effects will be governed by

$$L_{\rm fp} = -\tfrac{1}{2}(\nabla_\mu\eta_1)^2 - \tfrac{1}{2}(\nabla_\mu\eta_2)^2\,, \tag{2.8}$$

where, for convenience, we use the real fields η^a_1 and η^a_2 defined by

$$\omega^a = \frac{1}{\sqrt{2}}(\eta^a_1 + i\eta^a_2). \tag{2.9}$$

Following 't Hooft,[4] both L_V and $L_{\rm fp}$ may now be cast into the canonical form

$$L = -\tfrac{1}{2}\partial_\mu h^i\partial_\mu h^i + h^i N^{ij}_\mu\partial_\mu h^j - \tfrac{1}{2}h^i M^{ij}h^j \tag{2.10}$$

or defining

$$h_{;\mu} = \partial_\mu h + N_\mu h \tag{2.11}$$

and

$$X = M + N_\mu N_\mu\,, \tag{2.12}$$

$$L = -\tfrac{1}{2}h^i_{;\mu}h^i_{;\mu} - \tfrac{1}{2}h^i X^{ij}h^j\,, \tag{2.13}$$

which is manifestly invariant under the transformations

$$\delta h = \Lambda h,\quad \delta X = [\Lambda, X],\quad \delta N_\mu = -\partial_\mu\Lambda - [N_\mu, \Lambda] = -\Lambda_{;\mu}\,, \tag{2.14}$$

where $\Lambda(x)$ is an arbitrary, infinitesimal, antisymmetric matrix. The effective Lagrangian will now depend on the background field only via the tensor combinations X^{ij} and $Y^{ij}_{\mu\nu}$, where

$$Y_{\mu\nu} = \partial_\mu N_\nu - \partial_\nu N_\mu + [N_\mu, N_\nu]. \tag{2.15}$$

For the vector field, we make, from Eq. (2.6), the identifications

$$\begin{aligned} &h^i \to b^{\alpha a},\\ &X^{ij} \to X^{ab}_{\alpha\beta} = 2gf^{abc}G^c_{\alpha\beta},\\ &Y^{ij}_{\mu\nu} \to Y^{ab}_{\mu\nu\alpha\beta} = gf^{acb}G^c_{\mu\nu}\delta_{\alpha\beta} \end{aligned} \tag{2.16}$$

and for the fictitious field, from Eq. (2.8), we find

$$\begin{aligned} &h^i \to \eta^a_1, \quad X^{ij} \to \mathfrak{X}^{ab} = 0,\\ &Y^{ij}_{\mu\nu} \to \mathcal{Y}^{ab}_{\mu\nu} = gf^{acb}G^c_{\mu\nu}, \end{aligned} \tag{2.17}$$

and similarly for η^a_2.

III. THE EFFECTIVE LAGRANGIAN

The effective action induced by these quantum effects will, of course, be exceedingly complicated. Even in the one-loop approximation it will be a nonlocal functional of the background field, the one-loop Lagrangian $\mathfrak{L}_{(1)}$ depending on B^a_μ and all its derivatives. For arbitrarily varying fields, therefore, one must resort to some perturbative method of calculation. Instead, however, we wish to solve for $\mathfrak{L}_{(1)}$ *exactly* by placing what were called in paper I "quasilocal" conditions on the background field. Accordingly, we choose our background field B^a_μ to satisfy the condition

$$\nabla_\rho G^a_{\mu\nu} = 0. \tag{3.1}$$

(This is the non-Abelian analog of the condition

$$\partial_\rho F_{\mu\nu} = 0 \tag{3.2}$$

imposed by Schwinger[5] on the Maxwell field strength $F_{\mu\nu}$ in calculating one-loop effective Lagrangians for constant external electromagnetic fields.) From Eqs. (2.16) and (2.17), the condition (3.1) automatically insures that

$$X^{ij}_{;\mu} = 0 = Y^{ij}_{\mu\nu;\rho} \tag{3.3}$$

for both the vector and the fictitious fields. These conditions, in turn, mean that the matrices X and Y commute

$$[X, Y_{\mu\nu}] = 0 = [Y_{\mu\nu}, Y_{\rho\sigma}] \tag{3.4}$$

(see paper I for details).

Our task is now considerably simplified and the methods used in paper I to calculate the effective Lagrangian now carry over completely to the present situation. There it was shown that corresponding to the Lagrangian L of (2.13), and the conditions (3.3) and (3.4), the one-loop effective Lagrangian (in n space-time dimensions) was given by

$$\mathfrak{L}_{(1)} = \pm \frac{\hbar}{2(4\pi)^{n/2}}\,\mathrm{Tr}\int_0^\infty \frac{ds}{s^{1+n/2}}\,(e^{-X(B)s}e^{-F(Y;s)} - e^{-X(0)s}), \tag{3.5}$$

where

$$F(Y;s) = \tfrac{1}{2}\mathrm{tr}\ln[(\gamma s)^{-1}\sin(\gamma s)] \tag{3.6}$$

and

$$\gamma_{\mu\nu}{}^2 = Y_{\mu\alpha}(B)Y_{\nu\alpha}(B) \tag{3.7}$$

with B the background field. Here tr means trace over $\mu\nu$ indices and Tr means trace over ij indices. The over-all sign is plus for a boson loop and minus for a fermion loop. The one-loop Yang-Mills Lagrangian will now be given by

$$\mathfrak{L}^{\mathrm{YM}}_{(1)} = \mathfrak{L}^{V}_{(1)} + 2\mathfrak{L}^{\mathrm{fp}}_{(1)}, \tag{3.8}$$

the factor 2 accounting for the two fictitious fields η_1 and η_2. Remembering the change in sign for fermions, therefore, and that $\mathfrak{X} = 0$, we have

$$\mathfrak{L}^{\mathrm{YM}}_{(1)} = \frac{\hbar}{2(4\pi)^{n/2}}\,\mathrm{Tr}\int_0^\infty \frac{ds}{s^{1+n/2}}[(e^{-Xs}e^{-F(Y;s)} - 1) - 2(e^{-F(\mathcal{Y};s)} - 1)], \tag{3.9}$$

where $X^{ab}_{\mu\nu}$, $Y^{ab}_{\mu\nu\alpha\beta}$, and $\mathcal{Y}^{ab}_{\mu\nu}$ are given by Eqs. (2.16) and (2.17). Note that for both vector and fictitious fields $X(0) = 0$, which merely reflects the fact that we are dealing with a massless theory. This will prove significant when we consider the infrared problem.

IV. RENORMALIZATION

As it stands, the Lagrangian $\mathfrak{L}_{(1)}$ of Eq. (3.5) is ultraviolet divergent when evaluated at $n = 4$. These divergences may be removed, as in paper I, by the addition of counterterms which are equal to minus that part of $\mathfrak{L}_{(1)}$ which diverges for $n = 4$ at the lower limit of integration ($s = 0$). By expanding the exponentials and integrating over s, we find that the counterterms required are given by

$$\Delta\mathfrak{L} = \frac{\hbar}{32\pi^2(n-4)}\mathrm{Tr}(X^2 + \tfrac{1}{6}Y_{\mu\nu}Y_{\mu\nu}). \tag{4.1}$$

For the Yang-Mills case, therefore, we have from Eq. (3.9)

$$\Delta\mathfrak{L}^{\mathrm{YM}} = \frac{\hbar}{32\pi^2(n-4)}\mathrm{Tr}[X^2 + \tfrac{1}{6}Y_{\mu\nu}Y_{\mu\nu} + (-2)\tfrac{1}{6}\mathcal{Y}_{\mu\nu}\mathcal{Y}_{\mu\nu}]. \tag{4.2}$$

Defining C by

$$f_{abc}f_{abd} = C\delta_{cd}, \tag{4.3}$$

we find from (2.16) and (2.17) that

$$\Delta\mathfrak{L}^{\mathrm{YM}} = \frac{\hbar}{32\pi^2(n-4)} \times \tfrac{11}{3}Cg^2G^a_{\mu\nu}G^a_{\mu\nu}. \tag{4.4}$$

These counterterms were first obtained in this form by 't Hooft[4] using different methods. Alternatively (and, of course, equivalently), we may follow Schwinger and renormalize by subtracting off the leading terms of the exponentials, keeping $n=4$ throughout. The complete, ultraviolet-finite, Lagrangian then becomes

$$\mathfrak{L}^{\rm YM} = \mathfrak{L}^{\rm YM}_{(0)} + \mathfrak{L}^{\rm YM}_{(1)} \tag{4.5}$$

$$= -\tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \frac{\hbar}{2(4\pi)^2}\,\mathrm{Tr}\int_0^\infty \frac{ds}{s^3}\left[\left(e^{-Xs}e^{-F(Y;s)} - 1 - \frac{X^2}{2}s^2 - \tfrac{1}{12}Y_{\mu\nu}Y_{\mu\nu}s^2\right) - 2\left(e^{-F(\mathcal{Y};s)} - 1 - \tfrac{1}{12}\mathcal{Y}_{\mu\nu}\mathcal{Y}_{\mu\nu}s^2\right)\right], \tag{4.6}$$

where all quantities are now understood to be the renormalized quantities.

However, the final Lagrangian (4.6) still diverges at the upper limit of integration ($s=\infty$). Nor does it seem possible to deal with these infinities in the way described in paper I for massless scalar theories (e.g., massless $\lambda\phi^4$). This is, of course, just a manifestation of the well-known infrared catastrophe of massless Yang-Mills theory.

V. DISCUSSION

The formal expression (4.6) is not yet in a particularly simple form. Some simplification is achieved by noting that from (2.16) and (2.17)

$$Y^{ab}_{\mu\nu\alpha\beta} = \mathcal{Y}^{ab}_{\mu\nu}\,\delta_{\alpha\beta} = -\tfrac{1}{2}X^{ab}_{\mu\nu}\delta_{\alpha\beta}. \tag{5.1}$$

Furthermore, with knowledge of the eigenvalues of the above matrices, the trace operations could be performed explicitly. This should be reasonably straightforward provided the group in question is not too large, but we shall not pursue this here.

Although our effective action

$$\Gamma = \int dx\, \mathfrak{L}^{\rm YM} \tag{5.2}$$

is, from (4.6), manifestly invariant under gauge transformations of the background field, we must still discuss its dependence on the specific choice of gauge for the quantum field. For example, had we chosen, instead of (2.4), a gauge-breaking addition

$$\mathfrak{L}^{\rm gauge} = -\frac{1}{2\xi}(\nabla_\mu b_\mu)^2 \tag{5.3}$$

with ξ arbitrary, then in general the effective action Γ would depend on ξ.

In the present situation, however, we note that the condition (3.1) implies, in particular, that

$$\nabla_\mu G^a_{\mu\nu} = 0, \tag{5.4}$$

which are just the classical field equations. Moreover, since Γ depends only on $G^a_{\mu\nu}$, solutions of (5.4) will also be solutions of

$$\frac{\delta\Gamma}{\delta B^a_\mu} = 0. \tag{5.5}$$

One might speculate then that $\mathfrak{L}^{\rm YM}$ of (4.6) is, in fact, ξ independent in much the same way that effective potentials $V(\phi)$ are ξ independent[6] when evaluated at solutions of $dV/d\phi = 0$. Unfortunately, giving an arbitrary value to ξ renders the calculations much too cumbersome to achieve direct verification of this.

Finally, we refer the reader to a recent paper of Drummond and Fidler[7] who have stressed, within the context of Yang-Mills theories, the need for elucidating the structure of effective actions. Here, we simply present Eq. (4.6) as an interesting exact result of quantum field theory, showing that Schwinger's one-loop effective Lagrangian for Maxwell fields may be generalized to the non-Abelian case where the gauge fields are themselves quantized.

ACKNOWLEDGMENTS

We are grateful for discussions with M. R. Brown.

*Work supported by the Science Research Council.

†Work supported by G. I. F. T. (Grupo Interuniversitario de Fisica Teórica), Spain.

[1] M. R. Brown and M. J. Duff, Phys. Rev. D 11, 2124 (1975). See also M. R. Brown and M. J. Duff, Lett. Nuovo Cimento 11, 80 (1974); 11, 544 (1974).

[2] B. S. DeWitt, Phys. Rev. 162, 1195 (1967); 162, 1239 (1967).

[3] J. Honerkamp, Nucl. Phys. B48, 269 (1972).

[4] G. 't Hooft, Nucl. Phys. B62, 444 (1973).

[5] J. Schwinger, Phys. Rev. 82, 664 (1951).

[6] S. Weinberg, Phys. Rev. D 7, 2887 (1973).

[7] I. T. Drummond and R. W. Fidler, Nucl. Phys. B90, 77 (1975).