One-loop divergences of the nonlinear chiral theory

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An application of dimensional renormalization to chiral-invariant theories is presented. The naive Feynman rules may be used in this framework without introducing any noninvariant corrections to the on-mass-shell S matrix in the one-loop approximation. Moreover, the soft-pion theorem is fulfilled in all orders of perturbation theory.

Recently there have been several discussions on perturbation theory in the nonlinear chiral theory.[1-4] In spite of the nonrenormalizability of this theory it is expected that chiral invariance should constrain the counterterms in such a way that one may extract some useful results from the one-loop calculations, for instance. In doing any perturbative calculations in this theory we are faced with the following questions:

(1) Is the Adler condition (the soft-pion theorem) satisfied order by order in perturbation theory?

(2) Is the on-mass-shell S matrix computed in perturbation theory invariant under redefinitions of the pion fields?

If one worked without any caution (i.e., using naive perturbation theory) the answer to these questions would be negative. However, it was pointed out[2] that if one does the perturbation calculations more carefully, by adding an extra term multiplied by $\delta^4(0)$ to the Lagrangian, the Adler condition is automatically satisfied. The second question seems to be more complicated, and in order to give a proper answer one must use an invariant renormalization with respect to a kind of gauge group.[3,4] Using the background-field technique Ecker and Honerkamp[5] succeeded in calculating the one-loop counterterms of the nonlinear chiral-invariant pion Lagrangian. These counterterms are manifestly chiral-invariant and gauge-invariant, but they do not have the same structure as the initial Lagrangian, i.e., the theory remains invariant in the one-loop order but it is not renormalizable. Unfortunately, all these calculations contain a lot of meaningless things, and one has to introduce a regularization method to deal with them properly.

The purpose of this paper is to show that dimensional regularization[6] is very suitable for doing all calculations in this particular theory. We shall show that in this regularization method one-loop diagrams automatically satisfy the Adler condition, and the on-mass-shell S matrix is invariant under redefinitions of the pion field even though we use a naive perturbation theory.

The Lagrangian of our theory can be written as[3,5]

$$\mathcal{L} = \tfrac{1}{2} g_{ij}(\overline{\pi})\partial_\mu \overline{\pi}^i \partial^\mu \overline{\pi}^j \,, \tag{1}$$

$g_{ij}(\overline{\pi})$ being the metric in a curved isospace with constant curvature f_π^{-2}.

All one-loop diagrams can be obtained in a way explained by Coleman and Weinberg[7] if one starts with a prototype diagram, which is a simple circle in this case. Now it is easy to check that in the soft-pion limit (i.e., when all external momenta are zero) all these diagrams contain the integral

$$\int d^n k \,, \tag{2}$$

where n is the space-time dimension. But this term which is so meaningless in a usual regularization can be treated properly by a convenient definition of the n-dimensional integration.[8] After this redefinition it can be shown that this term vanishes in the framework of dimensional regularization.

Thus one can say that the Adler condition is indeed automatically satisfied in this regularization, at least in the one-loop approximation. We do not need to add any counterterm to satisfy this condition. This is easy to explain, and all the mystery of this cancellation consists in the occurrence of the $\delta^4(0)$ factor in the counterterm introduced by Gerstein, Jackiw, Lee, and Weinberg[2] which is now zero. It is natural to ask ourselves whether such a cancellation exists in higher orders of perturbation theory. The answer is affirmative,[9] but we do not give the proof here.

The second question can be solved by using the standard procedure of the background-field method and applying the technique developed by 't Hooft.[10] If we write

$$\overline{\pi}^i = \pi^i + \chi^i \,,$$

where π^i are background fields and χ^i quantum fields, one can expand the Lagrangian $\mathcal{L}$ in quantum fields about the background fields. The linear terms yield the equation of motion for the classical background fields

$$S_i = \frac{\delta S}{\delta \pi^i} = -g_{ij}(\pi)\Box\pi^j - \Gamma_{i,kj}(\pi)\partial_\mu \pi^j \partial^\mu \pi^k \ , \quad (3)$$

where $\Gamma_{k,ij} = \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k})$ are the Christoffel symbols for the metric g_{ij} ($g_{ik,j}$ means the derivative of g_{ik} with respect to the pion field π^j).

The quadratic part $\mathcal{L}_2$ of $\mathcal{L}$ has the form

$$\mathcal{L}_2 = \tfrac{1}{2} g_{ij}(\pi)\partial_\mu \chi^i \partial^\mu \chi^j + (g_{ij,k}\partial_\mu \pi^j)\partial^\mu \chi^i \chi^k + \tfrac{1}{2}(g_{ij,kl}\partial_\mu \pi^i \partial^\mu \pi^j)\chi^k \chi^l \ . \quad (4)$$

In order to apply the lemma of 't Hooft[10] we have to double the fields χ^k, make them complex, subtract a total derivative, and redefine $(\chi^i)^* \to (\chi^i)^* g_{ij}$ to arrive at the standard form

$$\mathcal{L} = -\chi_i^* \Box \chi^i + 2\chi_i^* N_j^{\mu,i} \partial_\mu \chi^j + \chi_i^* M_j^i \chi^j \ . \quad (5)$$

The values of $N_j^{\mu,i}$ and M_j^i can be read off from the complex version of $\mathcal{L}_2$ to be

$$N_j^{\mu,i} \equiv -\Gamma_{jk}^i \partial^\mu \pi^k \ ,$$
$$M_j^i = -g^{ik}(\partial_j \Gamma_{k,lm}\partial_\mu \pi^l \partial^\mu \pi^m + g_{lk,j}\Box \pi^l) \ . \quad (6)$$

The counterterm which should be added to $\mathcal{L}$ to render the theory finite in the one-loop approximation in the framework of dimensional regularization is given by 't Hooft's lemma[10] as

$$\Delta\mathcal{L} = \frac{1}{\epsilon}\mathrm{Tr}\left(\tfrac{1}{12} Y_{\mu\nu} Y^{\mu\nu} + \tfrac{1}{2} X^2\right) \ , \quad (7)$$

where the matrices $Y_{\mu\nu}$ and X are

$$Y_{\mu\nu} = \partial_\mu N_\nu - \partial_\nu N_\mu + [N_\mu, N_\nu] \ , \quad (8)$$

$$X = M - N_\mu N^\mu - \partial_\mu N^\mu \ , \quad (9)$$

and $\epsilon = 8\pi^2(n-4)$.

These matrices can be written explicitly as

$$Y_{\mu\nu,j}^i = R_{jkl}^i \partial_\mu \pi^k \partial_\nu \pi^l \ , \quad (10)$$

$$X_j^i = R_{kjl}^i \partial_\mu \pi^k \partial^\mu \pi^l - g^{ik}\Gamma_{kj}^m S_m \ , \quad (11)$$

where R_{jkl}^i is the Riemannian curvature tensor for the metric g_{ij}. The fact that isospace has a constant curvature f_π^{-2} implies

$$R_{ijkl} = f_\pi^{-2}(g_{ik}g_{jl} - g_{il}g_{jk}) \ . \quad (12)$$

Introducing these terms into (7) and not forgetting the factor 2 to undo the doubling we finally get

$$\Delta\mathcal{L} = \frac{1}{6\epsilon} f_\pi^{-4}[2(g_{ij}\partial_\mu \pi^i \partial_\nu \pi^j)^2 + (g_{ij}\partial_\mu \pi^i \partial^\mu \pi^j)^2] + \Delta\mathcal{L}' \ , \quad (13)$$

where $\Delta\mathcal{L}'$ contains S_i as a factor. We note that $\Delta\mathcal{L}'$ is not manifestly chiral-invariant but the first term in (13) is. On the other hand, when we calculate the on-mass-shell S matrix we can use the equations of motion (3), and $\Delta\mathcal{L}' = 0$ in this case. In other words, the on-mass-shell S matrix is invariant under redefinitions of the pion field, at least in the one-loop approximation.

Had we used a coordinate-independent perturbation expansion we would have obtained a manifestly chiral-invariant $\Delta\mathcal{L}$. In order to see this we simply notice that

$$\chi^i = \overline{\pi}^i - \pi^i = \sum_{n=1}^{\infty} \frac{1}{n!}\Gamma_{k_1,\ldots,k_n}^i \Gamma_0^{k_1} \ldots \Gamma_0^{k_n} \ , \quad (14)$$

where $\Gamma_0^i = d\xi^i/d\lambda|_{\lambda=1}$ is the derivative of the geodesic from π^i to $\overline{\pi}^i$, λ is the length for this curve and $\xi^i(0) = \pi^i$, $\xi^i(1) = \overline{\pi}^i$. $\Gamma_{k_1,\ldots,k_n}^i$ are the generalized Christoffel symbols. Using Γ_0^k instead of χ^i we can write $\mathcal{L}_2$ in the form (5) with N_j^i given by (6) but with M_j^i modified as follows:

$$\overline{M}_j^i = M_j^i - g^{ik}\Gamma_{kj}^m S_m \ .$$

In this case $\Delta\mathcal{L}' = 0$, and $\Delta\mathcal{L}$ is manifestly chiral-invariant as expected.

In conclusion we can say that the dimensional regularization and renormalization provide us with Green's functions satisfying the Adler condition and with on-mass-shell S-matrix elements which are independent of the parametrization of the pion field in the one-loop approximation. In particular the one-loop approximation π propagator is zero, and for the π-π scattering amplitude only one subtraction constant must be introduced. In order to satisfy not only the Adler condition but also the independence of the on-mass-shell S-matrix elements of the parametrization of the pion field, we do not have to do the canonical perturbation theory more carefully, but just apply it naively and use dimensional regularization.

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