Relativistic partial-wave analysis for three-meson systems. I*

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In previous Faddeev-type treatments of three-meson resonant systems, the Galilean two-body relative momentum has been employed in relativistic partial-wave analyses carried out in the three-body center-of-mass frame. However, this leads to difficulties in the interpretation of the Galilean form as the proper relativistic two-body relative momentum. We show that these difficulties are overcome if the two-body internal momentum is defined so as to have both its magnitude and direction independent of the Lorentz frame in which it is evaluated, analogous to the identical property of the Galilean relative momentum for various Galilean frames. Using such a relativistic version of the internal momentum, we carry out, as an example, a partial-wave analysis of the minimal-dynamics K-matrix equations using the Omnès coupling scheme. Finally we apply our results to the $I = 0$ channel of the three-pion system.

I. INTRODUCTION

In the past decade there have been a number of studies[1-4] whose purpose was to generate a Lorentz-covariant version of the Faddeev equations. There have also been a number of more or less unsuccessful attempts[5-7] to use such relativized Faddeev equations to examine the properties of three-meson resonant systems. As an alternative, other nonrelativistic three-particle theories, such as the boundary-condition model, have also been extended to relativistic situations with apparently a greater degree of success.[8,9] However, one inconsistency in the Faddeev-type calculations, independent of the particular model employed, has been the use of the nonrelativistic two-body relative momentum with relativistic one-body kinematics.

The purpose of the present study is to give a more complete treatment of the relativistic kinematics and to give a practical relativistic partial-wave analysis for a system of three spinless particles. For the sake of definiteness, we do this in the context of the minimal-dynamics K-matrix formalism for 3-to-3 scattering.[10] In this formalism, the connected part of the three-particle K matrix is ignored. This permits the three-body T matrix to be determined entirely by on-shell two-body t matrices. Despite the drastic approximations involved,[10] the formalism has the distinct advantages that we avoid the model dependences associated with specific off-shell extensions of the t matrices, and that below the inelastic threshold, the region in which we are most interested, the formalism is unitary.

The body of the present work will proceed as follows: In Sec. II, we will explain our notation, define the relevant two-body dynamical variables, and express the on-shell two-body t matrices in terms of these variables. These results are well known but have not been used to date in Faddeev-type calculations.[2,4-7] In Sec. III, we construct states of definite total angular momentum using the Omnès approach.[11] Although we follow the derivation of Ref. 11, an outline of the calculation is given in order to emphasize the necessity of using the relativistic relative momentum in the three-body center-of-mass frame in relativistic situations. In Sec. IV we construct eigenstates of parity which are compatible with the geometry considered in Sec. III. In Sec. V we perform a partial-wave analysis of the minimal K-matrix equations and apply the results to the $I=0$ channel of the three-pion system. The special complications arising from the symmetrization of the $I>0$ channel amplitudes will be treated in a separate paper. In Sec. VI we summarize our results.

II. RELATIVISTIC KINEMATICS

We begin with an arbitrary reference frame η and a set of fixed spatial coordinate axes in that frame, $(\hat{X}(\eta), \hat{Y}(\eta), \hat{Z}(\eta))$. We consider three spinless particles with masses $m_i > 0$, $i = 1, 2, 3$ whose momenta and energies in the η frame, $\vec{p}_i(\eta)$ and $e_i(\eta)$, respectively, satisfy the mass-shell constraint

$$e_i^2(\eta) = |\vec{p}_i(\eta)|^2 + m_i^2. \tag{2.1}$$

The two-body and three-body total energies and momenta in the η frame are defined by

$$(E_{ij}(\eta), \vec{P}_{ij}(\eta)) \equiv (e_i(\eta) + e_j(\eta), \vec{p}_i(\eta) + \vec{p}_j(\eta)) \tag{2.2}$$

and

$$(E(\eta), \vec{P}(\eta)) \equiv \left(\sum_{j=1}^{3} e_j(\eta), \sum_{j=1}^{3} \vec{p}_j(\eta)\right), \tag{2.3}$$

respectively. The subset of reference frames for which $\vec{P}(\eta^0)=0$ will be denoted by $\{\eta^0\}$. Each of these frames is called a three-body center-of-mass frame and any two elements of $\{\eta^0\}$ are related by a pure spatial rotation. Let us choose an arbitrary frame $\alpha\in\{\eta^0\}$ with respect to which we will define our three-particle states. We call

$$\mathfrak{M}\equiv E(\alpha)\equiv\sqrt{s} \tag{2.4}$$

the three-body invariant mass. The frame obtained by boosting from the α frame with the velocity $-\vec{P}_{ij}(\alpha)/E_{ij}(\alpha)$ will be called the γ_{ij} frame. In this frame we have $\vec{P}_{ij}(\gamma_{ij})=0$ so that the γ_{ij} frame is a two-body center-of-mass frame for the particles i and j. We take the coordinate axes of the γ_{ij} frame to be parallel to those of the α frame. Other i-j center-of-mass frames may be obtained by applying pure rotations to the γ_{ij} frame. We will not, however, be interested in these but will consider only the γ_{ij} frame obtained from our specific frame α by the preceding pure Lorentz boost. We call

$$M_{ij}\equiv E_{ij}(\gamma_{ij})\equiv\sqrt{\sigma_{ij}} \tag{2.5}$$

the two-body invariant mass. In an arbitrary frame η we have

$$E^2(\eta)-|\vec{P}(\eta)|^2=\mathfrak{M}^2 \tag{2.6}$$

and

$$E_{ij}^{\,2}(\eta)-|\vec{P}_{ij}(\eta)|^2=M_{ij}^{\,2}. \tag{2.7}$$

The connection between $\mathfrak{M}$ and M_{ij} is[12]

$$M_{ij}^{\,2}=\mathfrak{M}^2-2\mathfrak{M}e_k(\alpha)+m_k^{\,2}, \tag{2.8}$$

where i, j, and k are cyclic.

In order to elucidate the properties of the relativistic two-body relative momentum, we briefly recall the two-body Galilean kinematics. In transforming from the single-particle momenta $\vec{p}_i(\eta)$ and $\vec{p}_j(\eta)$ to the Galilean relative and center-of-mass momenta $\vec{k}_{ij}^G(\eta)$ and $\vec{P}_{ij}(\eta)$, respectively, we have

$$\vec{P}_{ij}(\eta)=\vec{p}_i(\eta)+\vec{p}_j(\eta)$$

and

$$\vec{k}_{ij}^G(\eta)=\frac{m_j\vec{p}_i(\eta)-m_i\vec{p}_j(\eta)}{m_i+m_j}. \tag{2.9}$$

The Jacobian of this transformation is

$$J\left(\frac{\vec{p}_i(\eta),\,\vec{p}_j(\eta)}{\vec{k}_{ij}^G(\eta),\,\vec{P}_{ij}(\eta)}\right)=1 \tag{2.10}$$

We recall two special properties of $\vec{k}_{ij}^G(\eta)$:

(i) When evaluated in the Galilean two-body center-of-mass frame γ_{ij}^G, the relative momentum evaluated in that frame is equal to the momentum of particle i in that frame,

$$\vec{k}_{ij}^G(\gamma_{ij}^G)=\vec{p}_i(\gamma_{ij}^G). \tag{2.11}$$

(ii) Both the magnitude and direction of $\vec{k}_{ij}^G(\eta)$ are independent of the Galilean frame η in which the relative momentum is evaluated, namely,

$$\vec{k}_{ij}^G(\eta)=\vec{p}_i(\gamma_{ij}^G). \tag{2.12}$$

Relativistically the form for the relative momentum which guarantees that properties (i) and (ii) are satisfied with respect to Lorentz frames is known[13] to be

$$\vec{k}_{ij}(\eta)=\frac{\vec{p}_i(\eta)[e_j(\eta)+e_j(\gamma_{ij})]-\vec{p}_j(\eta)[e_i(\eta)+e_i(\gamma_{ij})]}{E_{ij}(\eta)+M_{ij}}. \tag{2.13}$$

However, when transforming between single-particle momenta and relative and center-of-mass momenta, the Jacobian is frame dependent,[13] specifically,

$$J\left(\frac{\vec{p}_i(\eta),\,\vec{p}_j(\eta)}{\vec{k}_{ij}(\eta),\,\vec{P}_{ij}(\eta)}\right)=\frac{e_i(\eta)e_j(\eta)}{E_{ij}(\eta)}\,\frac{M_{ij}}{e_i(\gamma_{ij})e_j(\gamma_{ij})}. \tag{2.14}$$

In Sec. III it will be necessary to make use of the two quantities $|\vec{k}_{ij}(\eta)|^2$ and $\vec{k}_{ij}(\alpha)\cdot\vec{p}_k(\alpha)$. Using the Lorentz version of property (ii) for $\vec{k}_{ij}(\eta)$ and the relation

$$M_{ij}=[|\vec{k}_{ij}(\gamma_{ij})|^2+m_i^{\,2}]^{1/2}+[|\vec{k}_{ij}(\gamma_{ij})|^2+m_j^{\,2}]^{1/2}, \tag{2.15}$$

we obtain

$$|\vec{k}_{ij}(\eta)|^2=[M_{ij}^{\,2}-(m_i-m_j)^2][M_{ij}^{\,2}-(m_i+m_j)^2]/4M_{ij}^{\,2}. \tag{2.16}$$

Rewriting Eq. (2.13) in the form

$$\vec{k}_{ij}(\eta)=\frac{e_j(\eta)\vec{p}_i(\eta)-e_i(\eta)\vec{p}_j(\eta)}{M_{ij}}+\frac{\vec{P}_{ij}(\eta)\times[\vec{p}_j(\eta)\times\vec{p}_i(\eta)]}{M_{ij}[E_{ij}(\eta)+M_{ij}]} \tag{2.17}$$

and using the fact that $\vec{P}_{ij}(\alpha)=-\vec{p}_k(\alpha)$, we find that

$$\vec{k}_{ij}(\alpha)\cdot\vec{p}_k(\alpha)=\frac{M_{ij}}{2}[e_j(\alpha)-e_i(\alpha)]+\frac{1}{2M_{ij}}E_{ij}(\alpha)(m_i^{\,2}-m_j^{\,2}). \tag{2.18}$$

In previous relativistic three-body calculations,[2,4-9] Eq. (2.9) rather than Eq. (2.13) or Eq. (2.17) has been employed as a relativistic two-body internal momentum. Such a choice satisfies property (i) but violates property (ii) when *Lorentz* frames are considered.[14] As will be seen in Sec. III, the Lorentz frame independence of the magnitude and direction of $\vec{k}_{ij}(\eta)$ is crucial to the carrying out of a simple partial-wave analysis of the three-body equations in the three-body center-of-mass frame.

Henceforth we employ the notational shorthand

$$|\vec{p}_1\vec{p}_2\vec{p}_3\rangle_\eta \equiv |\vec{p}_1(\eta)\vec{p}_2(\eta)\vec{p}_3(\eta)\rangle$$

in describing three-body states in momentum space. We will find it convenient to expand our three-body equations in the three bases $|\vec{k}_{ij}\vec{P}_{ij}\vec{p}_k\rangle_\eta$ where (ijk) is a cyclic permutation of (123). This basis has the advantage that in considering the matrix elements of the two-body t matrix in the three-body space

$${}_\eta\langle \vec{k}'_{ij}\,\vec{P}'_{ij}\vec{p}'_k|t^k|\vec{k}_{ij}\vec{P}_{ij}\vec{p}_k\rangle_\eta,$$

the two-body portion of the matrix element is treated in the usual way, i.e., in terms of two-body relative and center-of-mass momentum-space states. Nonrelativistically we recall that with the normalizations

$${}_\eta\langle \vec{p}'_1\vec{p}'_2\vec{p}'_3|\vec{p}_1\vec{p}_2\vec{p}_3\rangle_\eta = (2\pi)^9\prod_j \delta(\vec{p}'_j(\eta)-\vec{p}_j(\eta)) \tag{2.19}$$

and

$${}_\eta\langle \vec{k}^{G\prime}_{ij}\vec{P}'_{ij}\vec{p}'_k|\vec{k}^G_{ij}\vec{P}_{ij}\vec{p}_k\rangle_\eta = (2\pi)^9\delta(\vec{k}^{G\prime}_{ij}(\eta)-\vec{k}^G_{ij}(\eta))\delta(\vec{P}'_{ij}(\eta)-\vec{P}_{ij}(\eta))\delta(\vec{p}'_k(\eta)-\vec{p}_k(\eta)) \tag{2.20}$$

the on-shell two-body t matrix in the three-body space takes form

$$\begin{aligned}{}_\eta\langle\vec{p}'_1\vec{p}'_2\vec{p}'_3|t^k|\vec{p}_1\vec{p}_2\vec{p}_3\rangle_\eta &= (2\pi)^3\delta(\vec{p}'_k(\eta)-\vec{p}_k(\eta))\,{}_\eta\langle\vec{p}'_i\vec{p}'_j|\hat{t}\,|\vec{p}_i\vec{p}_j\rangle_\eta\\ &= (2\pi)^6\delta(\vec{p}'_k(\eta)-\vec{p}_k(\eta))\delta(\vec{P}'_{ij}(\eta)-\vec{P}_{ij}(\eta))f(|\vec{k}^G_{ij}(\gamma^G_{ij})|^2,\theta_{ij}),\end{aligned} \tag{2.21}$$

where θ_{ij} is the angle between $\vec{p}_i(\gamma^G_{ij})$ and $\vec{p}'_i(\gamma^G_{ij})$, or equivalently [property (i)] the angle between $\vec{k}^G_{ij}(\gamma^G_{ij})$ and $\vec{k}^{G\prime}_{ij}(\gamma^G_{ij})$. On the other hand, in the relativistic case, choosing the normalizations

$${}_\eta\langle\vec{p}'_1\vec{p}'_2\vec{p}'_3|\vec{p}_1\vec{p}_2\vec{p}_3\rangle_\eta = (2\pi)^9 8e_1(\eta)e_2(\eta)e_3(\eta)\prod_j\delta(\vec{p}'_j(\eta)-\vec{p}_j(\eta)) \tag{2.22}$$

and

$${}_\eta\langle\vec{k}'_{ij}\vec{P}'_{ij}\vec{p}'_k|\vec{k}_{ij}\vec{P}_{ij}\vec{p}_k\rangle_\eta = (2\pi)^9 2e_k(\eta)\delta(\vec{k}'_{ij}(\eta)-\vec{k}_{ij}(\eta))\,\delta(\vec{P}'_{ij}(\eta)-\vec{P}_{ij}(\eta))\delta(\vec{p}'_k(\eta)-\vec{p}_k(\eta)), \tag{2.23}$$

we find, using Eq. (2.4) that the on-shell two-body t matrix in the three-body space becomes[13]

$${}_\eta\langle\vec{p}'_1\vec{p}'_2\vec{p}'_3|t^k|\vec{p}_1\vec{p}_2\vec{p}_3\rangle_\eta = 8(2\pi)^6 e_k(\eta)e_i(\gamma_{ij})e_j(\gamma_{ij})\delta(\vec{p}'_k(\eta)-\vec{p}_k(\eta))\delta(\vec{P}'_{ij}(\eta)-\vec{P}_{ij}(\eta))f(M_{ij},\theta_{ij}), \tag{2.24}$$

where θ_{ij} is again the angle between $\vec{p}_i(\gamma_{ij})$ and $\vec{p}'_i(\gamma_{ij})$, or equivalently [property (i)] the angle between $\vec{k}_{ij}(\gamma_{ij})$ and $\vec{k}'_{ij}(\gamma_{ij})$.

III STATES OF DEFINITE TOTAL ANGULAR MOMENTUM

With our intent being to obtain the matrix elements of the two-body t matrix in the three-body center-of-mass frame we first define three reference frames, α_k $(k=1,2,3)$ with coordinate axes $(\hat{x}_k(\alpha),\hat{y}_k(\alpha),\hat{z}_k(\alpha))$, which are obtained by applying the rotations $\mathcal{R}(A_k,B_k,C_k)$ to the α frame. The Euler angles A_k,B_k,C_k are defined such that the $\hat{x}_k(\alpha)\,\hat{z}_k(\alpha)$ plane is aligned with the $\vec{p}_1(\alpha)\vec{p}_2(\alpha)\vec{p}_3(\alpha)$ plane, with the $\hat{z}_k(\alpha)$ axis parallel to $\vec{p}_k(\alpha)$, and with the $\hat{y}_k(\alpha)$ axis parallel to $\vec{p}_k(\alpha)\times\vec{p}_i(\alpha)$. We may then construct the states of definite total angular momentum J, $|e_1e_2e_3JM\mu_k\rangle_\alpha$, from the single-particle states via the Wigner projection technique.[11,15] Here e_i are the single-particle energies in the α frame, M is the projection of J along $\hat{Z}(\alpha)$, and μ_k is the projection of J along $\hat{z}_k(\alpha)$. The normalizations of these states are chosen to be

$${}_\alpha\langle e'_1e'_2e'_3J'M'\mu'_k|\,e_1e_2e_3JM\mu_k\rangle_\alpha = \delta_{J'J}\,\delta_{M'M}\delta_{\mu'_k\mu_k}\prod_j\delta(e'_j(\alpha)-e_j(\alpha)). \tag{3.1}$$

The overlap between these states and the states $|\vec{p}_1\vec{p}_2\vec{p}_3\rangle_\alpha$ is then

$${}_\alpha\langle e_1e_2e_3JM\mu_k|\vec{p}_1\vec{p}_2\vec{p}_3\rangle_\alpha = \left(\frac{2J+1}{\pi^2}(2\pi)^9\right)^{\frac{1}{2}}\delta\Big(\sum_j\vec{p}_j(\alpha)\Big)\prod_j\,\delta(e_j(\alpha)-[p_j{}^2(\alpha)+m_j{}^2]^{\frac{1}{2}})\,\mathfrak{D}^{J*}_{M\mu_k}(A_kB_k\,C_k) \tag{3.2}$$

in agreement with Ref. 6.

Although the derivation of the expression for the on-shell amplitude

$$_{\alpha}\langle e_1' e_2' e_3' J' M' \mu_k' | t^k | e_1 e_2 e_3 J M \mu_k \rangle_{\alpha}$$

proceeds exactly in the manner of Ref. 11 except for our use of relativistic one-body and two-body kinematics, we will outline the derivation in order to emphasize a crucial point which occurs in the parametization of the scattering angle θ_{ij}. Introducing complete sets of single-particle momentum states and using Eq. (3.2), we have

$$_{\alpha}\langle e_1' e_2' e_3' J' M' \mu_k' | t^k | e_1 e_2 e_3 J M \mu_k \rangle_{\alpha}$$
$$= \int \frac{d\vec{p}_1'(\alpha) d\vec{p}_2'(\alpha) d\vec{p}_3'(\alpha) d\vec{p}_1(\alpha) d\vec{p}_2(\alpha) d\vec{p}_3(\alpha)}{(2\pi)^9 64 e_1'(\alpha) e_2'(\alpha) e_3'(\alpha) e_1(\alpha) e_2(\alpha) e_3(\alpha)}$$
$$\times \frac{(2J'+1)^{1/2}(2J+1)^{1/2}}{\pi^2} \delta\Big(\sum_j \vec{p}_j'(\alpha)\Big)\,\delta\Big(\sum_j \vec{p}_j(\alpha)\Big) \prod_j \delta(e_j'(\alpha) - [p_j'^2(\alpha)+m_j^2]^{1/2})\delta(e_j(\alpha) - [p_j^2(\alpha)+m_j^2]^{1/2})$$
$$\times \mathfrak{D}^{J'*}_{M'\mu_k'}(A_k' B_k' C_k') \mathfrak{D}^{J}_{M\mu_k}(A_k B_k C_k)\, {}_{\alpha}\langle \vec{p}_1' \vec{p}_2' \vec{p}_3' | t^k | \vec{p}_1 \vec{p}_2 \vec{p}_3 \rangle_{\alpha}. \qquad (3.3)$$

Evaluating one initial and one final momentum integral, transforming to energy-angle variables via

$$d\vec{p}_k(\alpha) d\vec{p}_i(\alpha) = e_1(\alpha) e_2(\alpha) e_3(\alpha) de_1(\alpha) de_2(\alpha) de_3(\alpha) dA_k d\cos B_k dC_k\,, \qquad (3.4)$$

using Eq. (2.24), transforming $\delta(\vec{p}_k'(\alpha) - \vec{p}_k(\alpha))$ to the energy-angle variables, and performing the energy integrations, we find that Eq. (3.3) simplifies to the expression

$$_{\alpha}\langle e_1' e_2' e_3' J' M' \mu_k' | t^k | e_1 e_2 e_3 J M \mu_k \rangle_{\alpha} = \int dA_k' d\cos B_k' dC_k' dA_k d\cos B_k dC_k$$
$$\times \frac{(2J'+1)^{1/2}(2J+1)^{1/2}}{8\pi^2 (2\pi)^3} \frac{\delta(e_k'(\alpha) - e_k(\alpha))}{p_k(\alpha)} e_i(\gamma_{ij}) e_j(\gamma_{ij}) \delta(\cos\theta_\alpha'^k - \cos\theta_\alpha^k)$$
$$\times \delta(\phi_\alpha'^k - \phi_\alpha^k) \mathfrak{D}^{J'*}_{M'\mu_k'}(A_k' B_k' C_k') \mathfrak{D}^{J}_{M\mu_k}(A_k B_k C_k) f(M_{ij}, \theta_{ij}), \qquad (3.5)$$

where $(\theta_\alpha^k, \phi_\alpha^k)$ are the polar angles of $\vec{p}_k(\alpha)$ in the α frame. The angles $\theta_\alpha'^k$ and $\phi_\alpha'^k$ are defined in a similar way in the primed coordinate system. We notice that in Eq. (3.5), if θ_α^k, $\theta_\alpha'^k$, ϕ_α^k, $\varphi_\alpha'^k$, and θ_{ij} depend in a complicated way on the Euler angles $(A_k B_k C_k)$ and $(A_k' B_k' C_k')$, the Euler angle integrations become in general quite complicated.

We now take note of two crucial points. First of all, our particular choice of Euler angles allows us to make the identification $\theta_\alpha^k = B_k$, $\theta_\alpha'^k = B_k'$, $\phi_\alpha^k = A_k$, and $\phi_\alpha'^k = A_k'$. Secondly, we stated in the previous section that θ_{ij} is defined in the γ_{ij} frame, it being the angle between $\vec{k}_{ij}(\gamma_{ij})$ and $\vec{k}_{ij}'(\gamma_{ij})$. If we choose to work with this definition we encounter problems in relating θ_{ij} to the two sets of Euler angles since the Euler angles are defined with respect to the directions of the momenta in the α frame while θ_{ij} is known in terms of γ_{ij} frame quantities. But if we recall that the direction of our relative momentum given by Eq. (2.13) is Lorentz frame independent, we see that the angle between $\vec{k}_{ij}(\gamma_{ij})$ and $\vec{k}_{ij}'(\gamma_{ij})$ is the same as the angle between $\vec{k}_{ij}(\alpha)$ and $\vec{k}_{ij}'(\alpha)$. In this way we are able to write θ_{ij} in terms of α frame variables and connect it to the Euler angles which are also written in terms of α frame variables. Referring to Fig. 1, in the α frame, with our particular choice of Euler angles, the law of cosines for spherical triangles in this frame gives

$$\cos\theta_{ij} = \cos\xi_k \cos\xi_k' + \sin\xi_k \sin\xi_k' \cos U_{ij}\,, \qquad (3.6)$$

where

$$U_{ij} \equiv C_k - C_k' \qquad (3.7)$$

and

$$\cos\xi_k \equiv \frac{\vec{k}_{ij}(\alpha)\cdot\vec{p}_k(\alpha)}{|\vec{k}_{ij}(\alpha)||\vec{p}_k(\alpha)|}\,, \qquad (3.8)$$

where again ξ_k' is defined in a similar way in the primed coordinate system. Then we can easily perform the $dA_k' d\cos B_k'$ integrations in Eq. (3.5), use the change of variables of Eq. (3.7), and carry out the $dA_k d\cos B_k dC_k$ integrals using the conventions of Edmonds[16] for the rotation matrices, which then yield

$$_\alpha\langle e_1'e_2'e_3'J'M'\mu_k'|t^k|e_1e_2e_3JM\mu_k\rangle_\alpha=\delta_{J'J}\delta_{M'M}\delta_{\mu_k'\mu_k}\frac{\delta(e_k'(\alpha)-e_k(\alpha))}{(2\pi)^3p_k(\alpha)}e_i(\gamma_{ij})e_j(\gamma_{ij})\int_0^{2\pi}dU_{ij}e^{i\mu_kU_{ij}}f(M_{ij},\theta_{ij}(\xi_k,\xi_k',U_{ij})) \tag{3.9}$$

in analogy to the Galilean result of Omnès.[11]

Finally, we define a "standard body-fixed" set of axes $(\hat{x}(\alpha),\hat{y}(\alpha),\hat{z}(\alpha))$ with respect to the α frame. These axes are oriented so that the $\hat{x}(\alpha)\hat{z}(\alpha)$ plane is coincident with the $\vec{p}_1(\alpha)\vec{p}_2(\alpha)\vec{p}_3(\alpha)$ plane with $\hat{y}(\alpha)$ parallel to $\hat{y}_k(\alpha)$ and with D_k denoting the angle between $\hat{z}(\alpha)$ and $\hat{z}_k(\alpha)$. We then define the states $|e_1e_2e_3JM\mu\rangle_\alpha$, where μ is the projection of J along $\hat{z}(\alpha)$, by

$$|e_1e_2e_3JM\mu\rangle_\alpha\equiv\sum_{\mu_k}d^J_{\mu_k\mu}(D_k)|e_1e_2e_3JM\mu_k\rangle_\alpha. \tag{3.10}$$

In this new basis the on-shell two-body t matrix becomes

$$_\alpha\langle e_1'e_2'e_3'J'M'\mu'|t^k|e_1e_2e_3JM\mu\rangle_\alpha=\delta_{J'J}\delta_{M'M}\frac{\delta(e_k'(\alpha)-e_k(\alpha))}{(2\pi)^3p_k(\alpha)}e_i(\gamma_{ij})e_j(\gamma_{ij})$$
$$\times\sum_{\mu_k}d^J_{\mu'\mu_k}(-D_k')d^J_{\mu_k\mu}(D_k)\int_0^{2\pi}dU_{ij}e^{i\mu_kU_{ij}}f(M_{ij},\theta_{ij}(\xi_k,\xi_k',U_{ij})). \tag{3.11}$$

For a simple specification of our standard body-fixed axis, we choose the $(\hat{x}(\alpha),\hat{y}(\alpha),\hat{z}(\alpha))$ axes to be coincident with the $(\hat{x}_1(\alpha),\hat{y}_1(\alpha),\hat{z}_1(\alpha))$ axes. For this choice, the angle D_k is the angle between $\vec{p}_1(\alpha)$ and $\vec{p}_k(\alpha)$. The advantages of this choice will become evident in Sec. V.

It is useful to comment upon the previous treatments of this sort. In the nonrelativistic case, it was this same Galilean frame independence of the direction of $\vec{k}^G_{ij}(\eta)$ which allowed Omnès[11] to write θ_{ij} in terms of the α frame variables rather than in terms of the γ^G_{ij} frame variables and thus connect θ_{ij} to the α frame Euler angles via the law of cosines for spherical triangles in the Galilean three-body center-of-mass frame. As to the relativistic treatments, Mennessier *et al.*[7] do not specify which form they use for the two-body relative momentum. However, in their Omnès-type formalism, they choose to apply the law of cosines for spherical triangles in the γ_{ij} frame rather than in the α frame. This causes problems in that their U_{ij} is written in terms of γ_{ij} frame variables and must still be reexpressed in terms of the α frame variables before the Euler angle integrations may be performed. Mishima *et al.*[4] and Basdevant and Kreps[6] define their α frame Euler angles so as to make the $\hat{x}_k(\alpha)\hat{y}_k(\alpha)$ plane coincident with the $\vec{p}_1(\alpha)\vec{p}_2(\alpha)\vec{p}_3(\alpha)$ plane with $\hat{x}_k(\alpha)$ parallel to $\vec{p}_k(\alpha)\times\vec{p}_i(\alpha)$. For this choice of Euler angles, they still arrive at an expression for the two-body t matrix analogous to Eq. (3.5) but they cannot make the simple identifications $\theta^k_\alpha=B_k$ and $\phi^k_\alpha=A_k$. This causes problems which are compounded by the fact that they use Eq. (2.9) rather than Eq. (2.13) for the relative momentum. Since

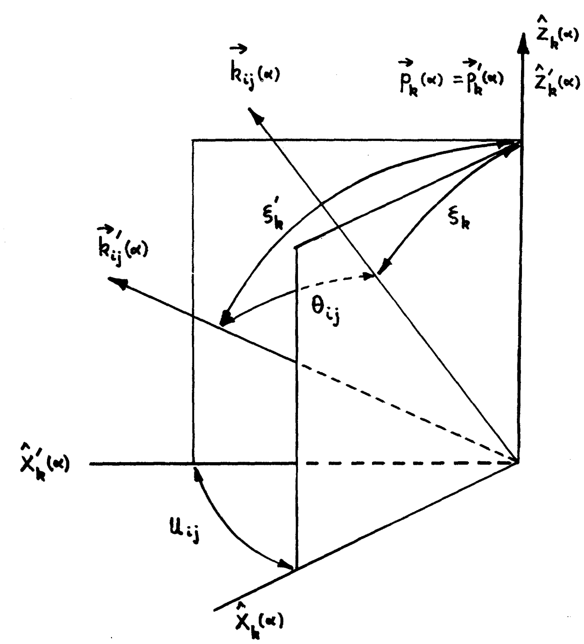

FIG. 1. The scattering of relativistic particles i and j with relativistic spectator k as seen in the three-body center-of-mass frame, α. The unprimed (initial) momenta $\vec{p}_1(\alpha)$, $\vec{p}_2(\alpha)$, $\vec{p}_3(\alpha)$ lie in the $\hat{x}_k(\alpha)\hat{z}_k(\alpha)$ plane and primed (final) momenta $\vec{p}_1'(\alpha)$, $\vec{p}_2'(\alpha)$, $\vec{p}_3'(\alpha)$ lie in the $\hat{x}_k'(\alpha)\hat{z}_k'(\alpha)$ plane. The initial and final relative momenta $\vec{k}_{ij}(\alpha)$ and $\vec{k}_{ij}'(\alpha)$ are defined in Eq. (2.13). The angle between the initial and final momentum planes is C_k-C_k' and the scattering angle is θ_{ij} as explained in the text.

both the magnitudes and directions of $\vec{k}^G_{ij}$ and $\vec{k}'^G_{ij}$ vary by differing amounts between the γ_{ij} and α frames, the angle between $\vec{k}^G_{ij}(\alpha)$ and $\vec{k}'^G_{ij}(\alpha)$ which is used in their analyses is not the scattering angle θ_{ij}.

An alternative method for constructing states of definite total angular momentum has been employed in some previous calculations.[2,5,8,9] This consists of coupling the angular momentum of the relative motion of particles i and j to the angular momentum of the relative motion of the ij center of mass and particle k. We will denote such states in an abbreviated way by $|(ij)k\rangle$. The connection between the nonrelativistic version of this coupling scheme[5] and the nonrelativistic Omnès method has been made by Balian and Brèzin.[17] Relativistically, this successive coupling scheme has been studied by Wick[18] using the helicity formalism and by MacFarlane[19] using the canonical formalism. The connection between these two successive coupling schemes has been made by McKerrell.[20] For relativistic situations, in the evaluation of the recoupling coefficients $\langle (ij)k|(jk)i\rangle$ as in Eq. (35) of Ref. 18 or Eq. (5.4) of Ref. 20, one of the Wick angles is the angle between $\vec{k}_{ij}(\gamma_{ij})$ and $\vec{p}_k(\alpha)$. Again two reference frames are involved and we must either write $\vec{p}_k(\alpha)$ in terms of γ_{ij} frame variables or write $\vec{k}_{ij}(\gamma_{ij})$ in terms of α frame variables. Using Eq. (2.13) simply accomplishes the second program while the use of Eq. (2.9) yields a vector $\vec{k}^G_{ij}(\alpha)$ whose magnitude and direction are different from that of $\vec{k}_{ij}(\gamma_{ij})$ $=\vec{p}_i(\gamma_{ij})$.

In Sec. V we will see that the partial-wave-analyzed two-body t matrix using the Omnès approach is quite similar in structure to the two-body t matrix analyzed using the successive coupling approach as in Eq. (7.4) of Ref. 2.

IV. STATES OF DEFINITE PARITY

We assume that parity is conserved as is expected in purely hadronic interactions. Using the Omnès approach of the previous section with the Euler angle convention of Basdevant and Kreps[6] and of Berman and Jacob,[15] parity eigenstates are easily obtained. However, as seen in the previous section, this choice of Euler angles leads to difficulties in performing the Euler angle integrations and is thus less practical for our purposes. However, with our specific shoice of Euler angles delineated in the previous section, the parity eigenstates can be constructed in the manner of Werle[21] and Mennessier *et al.*[7] Following these authors, if $\mathcal{P}$ is the parity operator and π_i is the intrinsic parity of particle i, we have

$$\mathcal{P}|e_1e_2e_3JM\mu\rangle_\alpha = \pi_1\pi_2\pi_3(-1)^{J-\mu}|e_1e_2e_3JM-\mu\rangle_\alpha. \quad (4.1)$$

Then we construct the parity eigenstates

$$|e_1e_2e_3JM\mu\lambda\rangle_\alpha \equiv \tfrac{1}{2}|e_1e_2e_3JM\mu\rangle_\alpha + \frac{\lambda}{2}(-1)^{J-\mu}|e_1e_2e_3JM-\mu\rangle_\alpha, \quad (4.2)$$

where $\lambda=\pm 1$. For these states we have

$$\mathcal{P}|e_1e_2e_3JM\mu\lambda\rangle_\alpha = \pi_1\pi_2\pi_3\lambda|e_1e_2e_3JM\mu\lambda\rangle_\alpha \quad (4.3)$$

and

$${}_\alpha\langle e_1'e_2'e_3'J'M'\mu'\lambda'|e_1e_2e_3JM\mu\lambda\rangle_\alpha = \tfrac{1}{2}\delta_{J'J}\delta_{M'M}\delta_{\lambda'\lambda}\prod_j \delta(e_j'(\alpha)-e_j(\alpha))[\delta_{\mu'\mu}+\lambda(-1)^{J-\mu}\delta_{\mu',-\mu}]. \quad (4.4)$$

From Eq. (4.2), we infer that $|e_1e_2e_3JM0\lambda\rangle_\alpha$ is identically zero for $\lambda(-1)^J=-1$. In this new parity basis, Eq. (3.11) assumes the form

$${}_\alpha\langle e_1'e_2'e_3'J'M'\mu'\lambda'|t^k|e_1e_2e_3JM\mu\lambda\rangle_\alpha = \delta_{J'J}\delta_{M'M}\delta_{\lambda'\lambda}\frac{\delta(e_k'(\alpha)-e_k(\alpha))}{2(2\pi)^3p_k(\alpha)}e_i(\gamma_{ij})e_j(\gamma_{ij}) \times\sum_{\mu_k}[d^J_{\mu'\mu_k}(-D_k')+\lambda d^J_{\mu'\mu_k}(\pi-D_k')]d^J_{\mu_k\mu}(D_k)\int_0^{2\pi}dU_{ij}\cos(\mu_kU_{ij})f(M_{ij},\theta_{ij}(\xi_k,\xi_k',U_{ij})). \quad (4.5)$$

V FINAL EQUATIONS

We are now in a position to make an expansion of the minimum-dynamics K-matrix equations in the $|e_1e_2e_3JM\mu\lambda\rangle_\alpha$ basis. Introducing the notation

$$T^i_{JM\lambda}(e_1'e_2'e_3';e_1e_2e_3;\mu'\mu)\equiv{}_\alpha\langle e_1'e_2'e_3'JM\mu'\lambda|T^i|e_1e_2e_3JM\mu\lambda\rangle_\alpha,$$

we find that Eq. (3.2) of Ref. 10 has the explicit realization

$$T^i_{JM\lambda}(e_1'e_2'e_3';e_1e_2e_3;\,\mu'\mu) = {}_\alpha\langle e_1'e_2'e_3'JM\mu'\lambda|t^i|e_1e_2e_3JM\mu\lambda\rangle_\alpha - \int de_1''(\alpha)de_2''(\alpha)de_3''(\alpha)\sum_{\mu''}{}_\alpha\langle e_1'e_2'e_3'JM\mu'\lambda|t^i|e_1''e_2''e_3''JM\mu''\lambda\rangle_\alpha \times i\pi\delta(\mathfrak{M}''-\mathfrak{M})[T^j_{JM\lambda}(e_1''e_2''e_3'';e_1e_2e_3;\,\mu''\mu) + T^k_{JM\lambda}(e_1''e_2''e_3'';e_1e_2e_3;\,\mu''\mu)], \tag{5.1}$$

where $\mathfrak{M}'=\mathfrak{M}$ and where the two-body t-matrix elements are given in Eq. (4.5). Applying these results to the $I=0$ channel of the three-pion system and noting that the only two-body isospin system which enters in this channel is $i_{ij}=1$, we may expand the symmetrized scattering amplitude $f(M_{ij},\theta_{ij})$ as[22,23]

$$f(M_{ij},\theta_{ij}) = \frac{16\pi}{M_{ij}k_{ij}(\gamma_{ij})}\sum_{l=\text{odd}}(2l+1)e^{i\delta_l^1}\sin\delta_l^1 P_l(\cos\theta_{ij}), \tag{5.2}$$

where δ_l^1 is the π-π phase shift in the $(l, i_{ij}=1)$ channel. With this partial-wave expansion, the two-body t matrix of Eq. (4.5) simplifies to

$${}_\alpha\langle e_1'e_2'e_3'J'M'\mu'\lambda'|t^k|e_1e_2e_3JM\mu\lambda\rangle_\alpha = \delta_{J'J}\delta_{M'M}\delta_{\lambda'\lambda}\delta(e_k'(\alpha)-e_k(\alpha))\frac{8e_i(\gamma_{ij})e_j(\gamma_{ij})}{p_k(\alpha)k_{ij}(\gamma_{ij})M_{ij}} \times \sum_{l=\text{odd}} e^{i\delta_l^1}\sin\delta_l^1\sum_{\mu_k}[d^J_{\mu'\mu_k}(-D_k') + \lambda d^J_{\mu'\mu_k}(\pi-D_k')]d^J_{\mu_k\mu}(D_k)Y_l^{\mu_k}(\xi_k,0)Y_l^{\mu_k}(\xi_k',0). \tag{5.3}$$

As a result of the Bose symmetry of the three-pion scattering amplitude, the three coupled integral equations of Eq. (5.1) may be rewritten as a single integral equation using the transformation employed in Ref. 6,

$$X_{JM\lambda}(e_1'e_2'e_3';e_1e_2e_3;\,\mu'\mu) \equiv T^1_{JM\lambda}(e_1'e_2'e_3';e_1e_2e_3;\,\mu'\mu) + T^2_{JM\lambda}(e_3'e_1'e_2';e_1e_2e_3;\,\mu'\mu) + T^3_{JM\lambda}(e_2'e_3'e_1';e_1e_2e_3;\,\mu'\mu). \tag{5.4}$$

Then permuting the final-state and intermediate-state indices in Eq. (5.1) and adding the resulting expressions, we obtain

$$X_{JM\lambda}(e_1'e_2'e_3';e_1e_2e_3;\,\mu'\mu) = L_{JM\lambda}(e_1'e_2'e_3';e_1e_2e_3;\,\mu'\mu) - \int de_1''(\alpha)de_2''(\alpha)de_3''(\alpha) \times\sum_{\mu''} i\pi\delta(\mathfrak{M}''-\mathfrak{M})[{}_\alpha\langle e_1'e_2'e_3'JM\mu'\lambda|t^1|e_3''e_1''e_2''\,JM\mu''\lambda\rangle_\alpha + {}_\alpha\langle e_1'e_2'e_3'JM\mu'\lambda|t^1|e_2''e_3''e_1''JM\mu''\lambda\rangle_\alpha] \times X_{JM\lambda}(e_1''e_2''e_3'';e_1e_2e_3;\,\mu''\mu), \tag{5.5}$$

where $L_{JM\lambda}(e_1'e_2'e_3';e_1e_2e_3;\,\mu'\mu)$ is the inhomogeneous term defined as in Eq. (5.4) with t^i replacing T^i. The fully symmetrized three-pion scattering amplitude is given by

$$X_{JM\lambda}(\mathcal{A}(e_1'e_2'e_3');e_1e_2e_3;\,\mu'\mu),$$

where $\mathcal{A}$ denotes the antisymmetrization operator. Owing to the relative unimportance of the $i_{ij}=1$, $l\geq 3$ π-π phase shifts in comparison to the $i_{ij}=1$, $l=1$ phase shift for $M_{ij}\lesssim 1.5$ GeV,[24] which is the region where we will use these equations and expect them to be most reliable[10] we truncate the partial-wave expansion of Eq. (5.2) at the p wave. Finally, substituting Eq. (5.3) into Eq. (5.5), using our convention for the angles D_k and defining

$$\chi_{JM\lambda}(e_1'e_2'e_3';e_1e_2e_3;\,\mu'\mu) = X_{JM\lambda}(e_1'e_2'e_3';e_1e_2e_3;\,\mu'\mu) - X_{JM\lambda}(e_1'e_3'e_2';e_1e_2e_3;\,\mu'\,\mu), \tag{5.6}$$

we obtain for the $I=0$ three-pion channel the finite domain one-dimensional integral equation

$$\chi_{JM\lambda}(e_1'e_2'e_3';e_1e_2e_3;\,\mu'\mu) = \mathfrak{L}_{JM\lambda}(e_1'e_2'e_3';e_1e_2e_3;\,\mu'\mu) - \frac{4i\pi M_{23}'}{(M_{23}'^2-4m_\pi^2)^{1/2}[e_1'(\alpha)^2-m_\pi^2]^{1/2}} \times e^{i\delta_1^1(M_{23}')}\sin\delta_1^1(M_{23}')Y_1^{\mu'}(\xi_1',0)\sum_{\mu''}\int de_1''(\alpha)[d^J_{\mu'\mu''}(D'')+\lambda d^J_{\mu'\mu''}(\pi+D'')] \times Y_1^{\mu'}(\xi'',0)\chi_{JM\lambda}(e_1'',e_2'+e_3'-e_1'',e_1';e_1e_2e_3;\,\mu''\mu), \tag{5.7}$$

where $\mathcal{L}_{JM\lambda}$ is defined as in Eq. (5.6) with $L_{JM\lambda}$ replacing $X_{JM\lambda}$, where

$$\cos D'' = \frac{M_{23}'^2 - 2e_1''(\alpha)E_{23}'(\alpha)}{2[e_1''(\alpha)^2 - m_\pi^2]^{1/2}[e_1'(\alpha)^2 - m_\pi^2]^{1/2}} \qquad (5.8)$$

and

$$\cos\xi'' = \frac{M_{23}'[E_{23}'(\alpha) - 2e_1''(\alpha)]}{(M_{23}'^2 - 4m_\pi^2)^{1/2}[e_1'(\alpha)^2 - m_\pi^2]^{1/2}}, \qquad (5.9)$$

and where the limits of integration are

$$\frac{E_{23}'(\alpha)}{2} \pm \frac{[e_1'(\alpha)^2 - m_\pi^2]^{1/2}(M_{23}'^2 - 4m_\pi^2)^{1/2}}{2M_{23}'}.$$

Calculations using Eq. (5.7) to examine the resonant behavior of the $I=0$ three-pion system are now in progress.

VI. SUMMARY

In summary then, we have pointed out that in order to perform a straightforward partial-wave decomposition in the three-body center-of-mass frame for relativistic three-particle systems, the relative two-body momentum must be defined in a relativistic way as in Eq. (2.13). As an explicit example of the use of this relativistic internal momentum our formalism was applied to the Omnès coupling scheme with the minimal-dynamics K-matrix equations in Sec. V and specifically to the three-pion system in the $I=0$ channel yielding Eq. (5.7). This relativistic treatment of the internal momentum may also be applied to the particular Faddeev-type off-shell models considered in Refs. 1–7. The application of the present analysis in symmetrizing the more complicated $I>0$ channel amplitudes for the three-pion system will be treated in a forthcoming paper.

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