Simple interpretation of the Melosh transformation*

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We discuss a simple and physically transparent derivation of the Melosh transformation. This enables us to investigate several difficulties in the field.

The transformation connecting current quarks and constituent quarks has been the subject of many recent discussions. From the free-quark study of Melosh[1] one can abstract algebraic properties which have recently been applied to hadron decays. Whereas the algebraic properties have been remarkably successful,[2] the free-quark study itself has been under considerable debate.[3-6] The most difficult point in the free-quark study is the so-called *angular condition.*

We discuss the angular condition by demanding that the nucleon state at rest have well-defined spin. This gives a concise and physically simple derivation of the Melosh transformation. Using this simple derivation we can understand several points which are not transparent in the original derivation. We also investigate a transformation for the case where SU(3) is broken by quark masses. In addition we discuss the problem of magnetic moments.

The major problem in defining the Melosh transformation is how to classify single quark states having transverse momentum $\vec{p}_t \neq 0$ under both constituent and current $SU(6)_W$. For states with $\vec{p}_t = 0$ the classification is simple and self-evident because the third component of W spin coincides with the ordinary spin projection for both algebras. Quark states with $\vec{p}_t \neq 0$ are easily defined for the $SU(6)_W$ of currents by using the Lorentz generators

$$E^1 = K^1 + J^2, \quad E^2 = K^2 - J^1, \quad K^3, \tag{1}$$

where K^i are boosts and J^i rotation generators. The null plane ($x_+ = x^0 + x^3 = 0$) on which the lightlike charges are defined is left invariant by the operators in Eq. (1). These operators also commute with the generators of $SU(6)_W$ of currents. We thus define the lightlike helicity basis (LLHB) which transforms simply under $SU(6)_W$ currents:

$$|p_+, \vec{p}_t, \lambda\rangle_L = e^{i\vec{\beta}\cdot\vec{E}} e^{-i\omega K_3} |m/\sqrt{2}, \vec{p}_t = 0, \lambda\rangle, \tag{2}$$

where

$$p_+ = (E + p_3)/\sqrt{2},$$
$$\omega = \ln(\sqrt{2}\, p_+/m),$$
$$\vec{\beta} = \vec{p}_t/\sqrt{2}\, p_+,$$

and λ represents the helicity of the quark. We have omitted for simplicity all SU(3) and color indices.

The $SU(6)_W$ strong symmetry of constituents is a collinear symmetry. If we discuss only the *free-quark model,* then quarks with $\vec{p}_t \neq 0$ need not have simple transformation properties under this collinear symmetry. On the other hand, if we build a baryon state out of three constituent quarks, the quark wave function will involve $\vec{p}_t \neq 0$ even when the baryon is restricted to $\vec{p}_t = 0$. Thus a classification of quarks with $\vec{p}_t \neq 0$ is needed in order to classify baryons under $SU(6)_W$ of constituents. Our major clue for such a classification is the angular condition which we define as follows: *A baryon state built out of three quarks, as is done in the constituent-quark model, must have well-defined J and spin.*

In the free-quark model a baryon state at rest is built by superposing state vectors of the three-quark sector of Fock space

$$|M, O, S_W, S_3\rangle = \int \prod_{i=1}^{3} \frac{d^3 p_i}{p_i^0} f(\vec{p}_1, \vec{p}_2, \vec{p}_3,)\delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) |\vec{p}_1, \vec{p}_2, \vec{p}_3, S_W, S_3\rangle. \tag{3}$$

From the W-spin subgroup of $SU(6)_W$ we have

$$|\vec{p}_1, \vec{p}_2, \vec{p}_3; S_W, S_3\rangle = \sum_{\lambda_1,\lambda_2,\lambda_3} C^{S_W, S_3}_{\lambda_1,\lambda_2,\lambda_3} |\vec{p}_1, \lambda_1\rangle |\vec{p}_2, \lambda_2\rangle |\vec{p}_3, \lambda_3\rangle, \tag{4}$$

where $C^{S_W, S_3}_{\lambda_1,\lambda_2,\lambda_3}$ couple the three quark W spins into a total W spin S_W with the projection along the third projection along the third axis S_3. For baryons at rest made purely out of quarks, W spin coincides with ordinary spin.

The angular momentum condition for the baryons at rest tells us that a rotation does not change the spin and therefore the W spin remains unchanged. Consider now an arbitrary rotation

$$R=e^{i\vec{\omega}\cdot\vec{J}},\quad \vec{p}_i \xrightarrow{R} R\vec{p}_i,\quad R|\vec{p}_i,\lambda_i\rangle=\sum_{\lambda_i'}|R\vec{p}_i,\lambda_i'\rangle D^{1/2}_{\lambda_i',\lambda_i}(\vec{p}_i,\vec{\omega}). \tag{5}$$

For an arbitrary spin basis the representation matrices $D^{1/2}_{\lambda',\lambda}$ depend not only on $\vec{\omega}$ but also on $\vec{p}_i$. Consequently, we have

$$R|\vec{p}_1,\vec{p}_2,\vec{p}_3;S_W,S_3\rangle=\sum_{\lambda_i,\lambda_i'}C^{S_W,S_3}_{\lambda_1,\lambda_2,\lambda_3}|R\vec{p}_1,\lambda_1'\rangle|R\vec{p}_2,\lambda_2'\rangle|R\vec{p}_3,\lambda_3'\rangle D^{1/2}_{\lambda_1',\lambda_1}(\vec{p}_1,\vec{\omega})D^{1/2}_{\lambda_2',\lambda_2}(\vec{p}_2,\vec{\omega})D^{1/2}_{\lambda_3',\lambda_3}(\vec{p}_3,\vec{\omega}). \tag{6}$$

Only if the matrices $D^{1/2}_{\lambda_i,\lambda_i'}(\vec{p}_i,\vec{\omega})$ do not depend on $\vec{p}_i$ does S_W remain unchanged; then

$$\sum_{\lambda}D^{1/2}_{\lambda_1',\lambda_1}(\vec{\omega})D^{1/2}_{\lambda_2',\lambda_2}(\vec{\omega})D^{1/2}_{\lambda_3',\lambda_3}(\vec{\omega})=\sum_{\lambda'}D^{S_W}_{S_3',S_3}(\vec{\omega})C^{S_W,S_3'}_{\lambda_1',\lambda_2',\lambda_3'}, \tag{7}$$

where $S_3'=\lambda_1'+\lambda_2'+\lambda_3'$ and the angular condition is satisfied, i.e., the baryon transforms correctly under rotation.

The only spin basis which fulfills this condition is the canonical spin basis, which should therefore be identified with the *constituent spin basis*. A state of momentum p and spin projection λ in this basis is obtained by a boost in the direction of the momentum from a state at rest with spin projection λ along the third axis:

$$|\vec{p},\lambda\rangle=e^{i\chi\vec{p}\cdot\vec{K}/|p|}|\vec{0},\lambda\rangle, \tag{8}$$

where

$$\chi=\operatorname{arctanh}(|\vec{p}|/p^0).$$

The canonical spin basis and the LLHB are connected by the following Wigner rotation:

$$|\vec{p},\lambda\rangle=\sum_{\lambda'}\mathcal{V}_{\lambda',\lambda}(p)|p_+,\vec{p}_t,\lambda'\rangle_L,$$

$$\mathcal{V}_{\lambda',\lambda}(p)=\chi^\dagger_{\lambda'}\frac{\sqrt{2}\,p_+ +m+i\,\vec{p}_t\times\vec{\sigma}}{[2\sqrt{2}\,p_+(p^0+m)]^{1/2}}\chi_\lambda, \tag{9}$$

where χ_λ are two-component Pauli spinors. The fact that the constituents' spin basis is identical to the canonical spin basis is due to the fact that in this spin basis the spin and the orbital angular momentum are decoupled[7] and so a $SU(6)_W\times O(3)$ classification becomes possible.

The formula (9) defines the unitary transformation between current- and constituent-quark states. This unitary transformation performs the same Wigner rotation as the transformation found by Melosh[1] in the frame of the free-field theory for baryons at rest and consequently is equivalent to it.

Let us now consider the classification of the baryon states with three-momentum different from zero. Such states can be obtained from (3) by a Lorentz transformation. Any Lorentz transformation will change the original canonical quark spin basis so that the new baryon states will lose their transformation properties under the $SU(6)_W$ strong algebra for baryons at rest. In order to classify baryons with three-momentum different from zero a momentum-dependent set of $SU(6)_W$ operators has to be used. They are obtained from the operators of the $SU(6)_W$ strong algebra for the baryons at rest by the same Lorentz transformation used to boost the baryon states. In particular, one can choose this Lorentz transformation such that the $SU(6)_W$ current algebra remains invariant.

The spin basis for the moving baryons will be then the lightlike helicity basis:

$$|P^0,\mathrm{P},S_W,S_3\rangle_L = e^{i\vec{P}_t\cdot\vec{E}/(\sqrt{2}P_+)}e^{i\ln(\sqrt{2}P_+K_3/M)}|M,\vec{0},S_W,S_3\rangle. \tag{10}$$

The Lorentz transformation used in (10) does not introduce any new Wigner rotation between the current and constituent quarks, so that the spin rotation which brings the current into the constituent quarks may be obtained from (9) by replacing the actual quark momenta p by their momenta in the baryon rest frame $[q_+=p_+M/2P_+,\ \vec{q}_t=(1-p_+/P_+)\vec{p}_t$ when P is the baryon four-momentum, p the actual quark momentum, and q the quark momentum in the baryon rest frame]:

$$\mathcal{V}_{\lambda,\lambda'}(p,P)=\chi^\dagger_{\lambda'}\frac{\sqrt{2}\,q_+ +m+i\,\vec{q}_t\times\vec{\sigma}}{[2\sqrt{2}\,q_+(q^0+m)]^{1/2}}\chi_\lambda. \tag{11}$$

A similar formula was found by Osborn[5] in terms of the quark positions, momenta, and spin operators in the frame of a first quantization theory and which leads to our formula (11) by sandwiching it between the baryon states (10).

The present interpretation of the Melosh transformation may shed some light on the discussion of the matrix elements of the electromagnetic dipole operator between nucleon states at rest. Melosh used the formula [Eq. (41) of Ref. 1]

$$\frac{\mu}{2M} = i\left\langle M, \vec{0}, -\tfrac{1}{2}\right|\left(\int d^4x\,\delta(x_+)x_1F^+_{\rm em}(x)+\frac{1}{M}QE_1\right)\left|M, \vec{0}, \tfrac{1}{2}\right\rangle \tag{12}$$

and argued that the second term is proportional to the Dirac magnetic moment. However, it was pointed out that the matrix elements of E_1 between the plane waves are not well defined, and so the matrix elements of the dipole operator cannot be identified with the total magnetic moment. The present model allows us now to easily evaluate the matrix elements of E_1 between baryon states at rest. A crucial point is the fact that the spin basis of the constituent quarks is the canonical spin basis. In this basis E_1 has the following representation:

$$E_1 = \sum_k \left(ip_k^0\frac{\partial}{\partial p_k^1} - \frac{p_k^2\sigma_k^3 - p_k^3\sigma_k^2}{p_k^0+M} + p_k^1\frac{\partial}{\partial p_k^3} - p_k^3\frac{\partial}{\partial p_k^1} + \sigma_k{}^2\right), \tag{13}$$

where p_k and σ_k are the momentum and the spin of the kth quark. Also essential is the fact that the baryon states are classified not only according to O(2) but according to the full O(3) algebra of the orbital angular momentum. Between states with $l=0$, the use of spherical wave packets becomes mandatory in our approach. All the terms in the expression of E_1 depending on the momentum disappear, and we are left with the result

$$\langle M, \vec{0}, -\tfrac{1}{2}|E_1|M, \vec{0}, \tfrac{1}{2}\rangle = \langle M, \vec{0}, -\tfrac{1}{2}|S_2|M, \vec{0}, \tfrac{1}{2}\rangle, \tag{14}$$

where S is the total-spin operator. This allows the identification of the matrix element of the dipole operator with the total magnetic moment.

On the other hand, it has been shown that the $l_3=0$ part of the dipole operator behaves as a $(1, 8) + (8, 1)$ tensor which leads to the $-\frac{3}{2}$ ratio of the total magnetic moments of the nucleons. In this approach a non-$\underline{35}$ part of the dipole operator found by Osborn, and which in his formalism appears from the derivatives of the mass operator of the many-quark system with respect to the relative quark momenta, does not appear because the mass is kept constant and equal to the nucleon mass. Such a term may, however, appear when the dipole operator is sandwiched between resonant states.

Another problem which receives a simple interpretation in the present approach in the problem of the SU(3)-symmetry breaking in the free-quark model with nondegenerate masses: $m_{\mathcal{P}} = m_{\mathfrak{N}} \neq m_\lambda$. It was recently pointed out[8] that in this case the operators of the SU(3) current algebra do not fulfill the angular momentum condition, and no solution of the angular momentum condition has been found.

Let us consider the lightlike SU(3) charge which carries a $\mathcal{P}$ quark into a λ quark with a different mass:

$$F^{4+i5} = \int d^4x\,\delta(x_+) :q_+^\dagger(x)\lambda^{4+i5}q_+(x):$$
$$= \tfrac{1}{2}\int dp_+ d^2p_t \sum_{\alpha,\beta} \lambda^{4+i5}_{\alpha,\beta} \sum_\lambda [a^\dagger(p_+, \vec{p}_t, \lambda, \alpha)a(p_+, \vec{p}_t, \lambda, \beta) - b^\dagger(p_+, \vec{p}_t, \lambda, \beta)b(p_+, \vec{p}_t, \lambda, \alpha)], \tag{15}$$

where $a^\dagger(p_+, \vec{p}_t, \lambda, \alpha)$ and $b^\dagger(p_+, \vec{p}_t, \lambda, \beta)$ are quark creation operators in the LLHB.

The action of F^{4+i5} on the wave function (3) consists in changing a $\mathcal{P}$ quark into a λ quark with the same p_+, p_t, and lightlike helicity. However, if $m_{\mathcal{P}} \neq m_\lambda$, p_0 and p_3 will change. This leads to a change of the canonical spin as well as to a change of the rotation symmetry of the $f(\vec{p}_1, \vec{p}_2, \vec{p}_3)$ wave function. The new state obtained will then be a mixture of states with different S_W spins and different orbital angular momenta. One may want a unitary transformation V which transforms the F^a charges into SU(3)-strong operators W^a which do not change the spin-orbital momentum structure of the wave function. In terms of creation and annihilation operators the W^a's for hadrons at rest are expected to be

$$W^a = \tfrac{1}{2}\int \frac{d^3p}{p^0}\sum_{\alpha,\beta}\lambda^\alpha_{\alpha\beta}\sum_\lambda [a^\dagger(\vec{p}, \lambda, \alpha)a(\vec{p}, \lambda, \beta) - b^\dagger(\vec{p}, \lambda, \beta)b(\vec{p}, \lambda, \alpha)], \tag{16}$$

where $a^\dagger(\vec{p}, \lambda, \alpha)$ and $b^\dagger(\vec{p}, \lambda, \beta)$ are creation operators for quarks with three-momentum $\vec{p}$ and canonical spin λ and SU(3) indices α and β.

The observations made here are just a reflection of the statement that if SU(3) is not conserved, the charges as defined on null planes are not Lorentz scalars. They change momenta and carry angular momentum. One could modify the transformation by adding a Lorentz boost acting on λ quarks only. This will change the created λ quark momentum to that of the initial $\mathcal{P}$ quark. This procedure is not a complete solution to the problem since SU(3) generators will remain frame-dependent objects.

We have seen how the constraint that three quark wave packets have well-defined spin generates the Melosh transformation. Such a demand for physical hadrons is obviously reasonable. In the free-quark model there are no hadronic bound states, and imposing an angular condition on nonstable wave packets may be somewhat arbitrary. This arbitrariness becomes even more obvious when we note that in order to impose the angular condition we had to make use of the simple form of the wave function in (3). In interacting theory the wave function may be very different. Thus the details of the angular condition are interaction dependent.[4] It is therefore up to us to find an interacting model and try to justify the empirical successes in a more realistic approach.

After completing this work we have learned that similar physical arguments were given by Buccella, Savoy, and Sorba.[9]

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