Time graph of the unstable particle and the nonunitary representations of the Poincaré group

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The second-quantization scheme for neutral scalar fields describing an unstable particle is formally developed within the framework of the nonunitary representations of the Poincaré group, $\bar{\mathcal{P}}_+^\dagger$. The fields satisfy the postulates of conventional field theories with a modified spectral condition. The exponential decay law for the time graph of the unstable particle is derived from a study of the asymptotic behavior of the causal Green's function.

I. INTRODUCTION

In the traditional developments of quantum field theory, the unstable particle continues to enjoy a special status and is normally attacked in the framework of the Lehmann spectral representation defining the propagator. It is suggested that there is a pole in the lower half-plane of the second Riemann sheet of the propagator and the real and imaginary parts of the pole provide the mass and lifetime of the unstable particle. Lévy examined the method of the propagator in the context of the Lee model. The time graph in all such models contains the characteristic exponential decay terms and the nonexponential decay terms as well. The latter were attributed to the time distribution of production and detection events. Further, in the presence of elastic channels, one opens up the Riemann surfaces and there seems to be no unique prescription for the analytic continuation of the propagator function.[1]

At the group-theoretical level the classic work of Wigner[2] on unitary irreducible representations of the Poincaré group provides a basis *only* for classifying the *stable* particle states. The rediscovery of Majorana wave equations made no contribution to this problem, since the stable particles and resonances with various spin contents are treated on the same footing.[3] Fleming's analysis of the unstable particle on the hyperplane, however, has shed some light on the dynamical features of this problem.[4] We wish to present here a systematic analysis of the field-theoretic aspect of this problem based on the work of Kawai and Goto.[5] The relevance of nonunitary representations of the Poincaré group associated with complex four-momentum was first noted by Zwanziger,[6] and later Schulman analyzed the quantum-mechanical aspects of this problem through the Poincaré semigroup.

In a previous communication, we studied the lifetime and mass spectra of unstable particles quantitatively from the generalized wave equations involving complex four-momentum, and the agreement seems to be fairly good with the observed data.[8] Here, we confine ourselves to a systematic quantization scheme for the neutral scalar field describing an unstable particle and deduce the exponential decay law for the time graph. Our material is arranged as follows.

In Sec. II we briefly survey the classification of nonunitary representations of the Poincaré group with complex four-momentum, namely (i) the degenerate class and (ii) the nondegenerate class. In Sec. III we study the mathematical properties of the generalized functions relevant to our analysis and introduce the field $\phi(x)$ for the unstable particle as a functional on the space $\mathfrak{D}$ of infinitely differentiable functions of compact support.[9] The Fourier transform of $\phi(x)$ for complex four-momentum λ is a functional in the space dual to $\hat{\mathfrak{D}}$. It is understood that in the conventional sense, the Fourier transform of $\phi(x)$ is undefined for complex λ; *only* in the sense of a distribution is it a well-behaved function. The properties of the field $\phi(x)$ are studied in the light of the *Wightman axioms.* In Sec. IV we introduce the "smoothed out" creation and annihilation operators for the field $\phi(x)$. We carry on then second quantization of the neutral scalar field and compute explicitly the Pauli-Jordan operators $D^{(\pm)}(x)$ and the causal Green's function $D^c(x)$. The exponential decay law for the time graph of the unstable particle is obtained from the asymptotic behavior of $D^c(x)$. Finally, we conclude our discussions in Sec. V spotlighting some of the salient features of our analysis.

II. NONUNITARY IRREDUCIBLE REPRESENTATIONS

To each element $(a,A)\in\bar{\mathcal{P}}_+^\dagger$, the universal covering group of the connected Poincaré group $\mathcal{P}_+^\dagger$ where $a\in T_4$ (the translation group) and $A\in SL(2,C)$, we associate an operator $U(a,A)$ such that

$$U(a,A)\phi(\lambda,s)=e^{i\lambda\cdot a}Q(\lambda,A)\phi(\Lambda(A^{-1})\lambda,s)\,. \tag{2.1}$$

In (2.1), the square-integrable functions $\{\phi(\lambda, s)\}$ form a basis in a linear space $\mathfrak{L}$ for an arbitrary complex λ $(=p+iq)$ and

$$U(a,A)=U(a,I)U(0,A)=T(a)Q(A)P(A)\,, \qquad (2.2a)$$

$$T(a)\phi(\lambda,s)=e^{i\lambda\cdot a}\phi(\lambda,s)\,, \qquad (2.2b)$$

$$P(A)\phi(\lambda,s)=\phi(\Lambda(A^{-1})\lambda,s)\,, \qquad (2.2c)$$

$$Q(A)\phi(\lambda,s)=Q(\lambda,A)\phi(\lambda,s)\,. \qquad (2.2d)$$

The operators $P(A)$, $Q(A)$, and $T(a)$ satisfy the following properties:

$$T(a_1)T(a_2)=T(a_1+a_2),\quad T(0)=I\,, \qquad (2.3a)$$

$$P(A)T(a)=T(\Lambda(A)a)P(A)\,, \qquad (2.3b)$$

$$Q(A)T(a)=T(a)Q(A)\,, \qquad (2.3c)$$

$$Q(\lambda,A_1)Q(\Lambda^{-1}(A_1)\lambda,A_2)=Q(\lambda,A_1A_2)\,. \qquad (2.3d)$$

Equation (2.3d) denotes the multiplication law for the operators $Q(\lambda,A)$ which can be derived from the group properties of $U(0,A)$. It also implies that the subgroup $G_\lambda\subset A$ satisfying the condition $\Lambda(A)\subset G'=G_p'\cap G_q'$ defines a matrix representation $Q(\lambda,A)$ $[=Q(p,q,A)]$ of G_λ. Without any loss of generality, we can consider the group $G_{\hat\lambda}\approx G_\lambda$, $\hat\lambda$ being any four-vector on the orbit of λ. Thus, for given λ on the orbit of $\hat\lambda$, we can choose a definite $A_{\lambda\leftarrow\hat\lambda}$ such that $\Lambda(A_{\lambda\leftarrow\hat\lambda})\lambda=\hat\lambda$; and for any $A\in\bar{\mathcal{P}}_+^\uparrow$, the matrix $\bar{A}$ $[A_{\lambda\leftarrow\hat\lambda}{}^{-1}A\,A_{\lambda'\leftarrow\hat\lambda}$, where $\lambda'=\Lambda(A^{-1})\lambda]$ belongs to $G_{\hat\lambda}$. So, we define

$$\begin{aligned}Q(\lambda,A)&=Q(p,q,A)\\&=D(A_{\lambda\leftarrow\hat\lambda}{}^{-1}A\,A_{\lambda'\leftarrow\hat\lambda})\,.\end{aligned} \qquad (2.4)$$

It can be easily checked that (2.4) satisfies (2.3d).

A. Classification of the little groups

(a) Degenerate class. In this case, p and q are proportional. Using $q_\mu=(-\Gamma/2m)p_\mu$, we have

$$\lambda_\mu=p_\mu+iq_\mu=(1-i\Gamma/2m)p_\mu\,.$$

Thus,

$$G_p\cap G_q=G_\lambda=G_p=G_q\,, \qquad (2.5)$$

i.e., apart from a complex factor in λ_μ, the structure of G_λ is the same as in the classical case; namely for $p^2>0$, $=0$, <0, the little groups are SU(2), E(2), and SU(1,1), respectively. For $p^2>0$, $p_0>0$, the irreducible representations of $\bar{\mathcal{P}}_+^\uparrow$ are characterized by (M,s), where s is the intrinsic spin and $M=m-i\Gamma/2$.

(b) Nondegenerate class. We obtain the nondegenerate class of representations when p and q are linearly independent. The little group $G_\lambda=G_p\cap G_q$ is a one-parameter group and the intersection of the groups can be taken only along a common one-parameter subgroup of both the groups. The irreducible representations of such groups are one dimensional and are given by

$$Q(p,q,A)\phi(\lambda,s)=e^{i\alpha s}\phi(\lambda,s)\,. \qquad (2.6)$$

Note that s can be integer, half-integer, or complex. For a detailed discussion of these representations, the reader is referred to the work of Schulman.[7]

III. CONSTRUCTION OF THE FIELDS $\phi(x)$

A. Mathematical preliminaries

Let $\{f(x)\}\in\mathfrak{D}$ be a set of C^∞ functions with compact support[10] in the domain $G_\alpha\{=|x|\le\alpha\}$ in space-time. Let $\mathfrak{D}'$ be conjugate to $\mathfrak{D}$ (the dual); i.e., the elements of $\mathfrak{D}'$ are distributions. We define the Fourier transform $\mathfrak{F}f$ or $\hat f$ by

$$(\mathfrak{F}f)(x)\equiv\hat f(\lambda)=\int_{G_\alpha}f(x)e^{i\lambda\cdot x}d^4x\,, \qquad (3.1)$$

where $\lambda\cdot x=\lambda_0x_0-\vec\lambda\cdot\vec{x}$ and λ is a Minkowski four-vector. Let $\lambda=p+iq$; then,

$$\hat f(\lambda)=\int_{G_\alpha}f(x)e^{ip\cdot x}e^{-q\cdot x}d^4x\,. \qquad (3.2)$$

$\{f(x)\}$ in general could decrease to zero more rapidly than any power of $1/|x|$ as $x\to\infty$.[11] The space of functions $\{f(x)\}$ satisfies the condition

$$\begin{aligned}\|f(x)\|_m&=\sup_x|x^kD^lf(x)|\quad(|k|,|l|\le m)\\&=\sup_xM_m(x)|D^lf(x)|\quad(|l|\le m)\\&=C_{kl},\quad m=0,1,2,\ldots\end{aligned} \qquad (3.3)$$

where

$$\begin{aligned}M_m(x)&=\sup_x|x^k|\quad(|k|\le m)\\&=\sup_x|x_0^{k_0}\cdots x_3^{k_3}|\quad(|k|\le m)\,.\end{aligned}$$

For all $f(x)\in\mathfrak{D}$, $M_m(x)D^lf(x)$ are continuous and bounded for $|l|\le m$. In particular $\|f(x)\|_m$ are finite and that where $M_m(x)=\infty$, there necessarily $D^lf(x)=0$. Thus,

$$\begin{aligned}M_m(x)&=1,\quad x\notin G_\alpha\\&=0,\quad x\notin G_\alpha\,.\end{aligned} \qquad (3.4)$$

Thus, the Fourier transforms of $\{f(x)\}$ are entire and analytic functions of λ and have growth ≤1 and of type $\le\alpha$ (see Ref. 12), i.e.,

$$\begin{aligned}|\lambda^k\hat f(\lambda)|&=\left|\int_{G_\alpha}D^kf(x)e^{i\lambda\cdot x}d^4x\right|\\&\le e^{\alpha|q|}\int_{G_\alpha}|D^kf(x)|d^4x\\&=e^{\alpha|q|}C_k(\hat f(x))\,.\end{aligned} \qquad (3.5)$$

$|D^k f(x)|$ are bounded, so also is $C_k(\hat{f}(x))$. We prescribe the topology in this space by using the countable set of norms,

$$\|\hat{f}(\lambda)\|_l = \sup_\lambda M_l(\lambda)|\hat{f}(\lambda)|,$$

where

$$M_l(\lambda) = e^{-\alpha|q|}\,|\lambda^k|_{\max|k|\le l}$$
$$= e^{-(\alpha_0|q_0|+\cdots+\alpha_3|q_3|)}|\lambda_0^{k_0}\cdots\lambda_3^{k_3}|_{\max|k|\le l},$$
$$|k| = k_0+k_1+k_2+k_3 = 0,1,2,\ldots, \quad (3.6)$$
$$|\alpha| = \alpha_0+\alpha_1+\alpha_2+\alpha_3 = 0,1,2,\ldots .$$

B. Fourier transform of the generalized functions $\phi(x)$

Let $\{\hat{f}_L(\lambda)\}\in\hat{\mathfrak{D}}_L$ be a set of functions restricted on the ray L $(=L_0\times L_1\times L_2\times L_3)$ in the λ plane. For degenerate class representations,

$$q_\mu = (-\Gamma/2m)p_\mu,$$

and the ray L passes through the origin to infinity at an angle $\theta=\arctan(-\Gamma/2m)$ with the real λ axis.

Definition 1. We define the functional $\hat{\phi}(\lambda)\in\hat{\mathfrak{D}}_L'$ as the Fourier transform of $\phi(x)$ according to

$$(\hat{\phi}(\lambda),\hat{f}(\lambda))_L = \int_L \hat{\phi}^*(\lambda)\hat{f}(\lambda)d^4\lambda$$
$$= (2\pi)^4(\phi,f)$$
$$= (2\pi)^4\phi(f).$$

We assume that along L, $\hat{\phi}(\lambda)e^{\alpha|q|}$ is absolutely integrable for any real α. Thus,

$$(\hat{\phi}(\lambda),\hat{f}(\lambda))_L = \int_L[\hat{\phi}(\lambda)]^*\hat{f}(\lambda)d^4\lambda$$
$$= \int_L \hat{\phi}^*(\lambda)\left(\int e^{i\lambda\cdot x}f(x)d^4x\right)d^4\lambda$$
$$= \int f(x)\left(\int_L \hat{\phi}(\lambda)e^{-i\lambda\cdot x}d^4\lambda\right)^* d^4x$$

(interchanging the λ and x integration by virtue of the the absolute convergence of the double integral)

$$= (2\pi)^4\int f(x)\phi^*(x)d^4x$$
$$= (2\pi)^4(\phi(x),f(x))$$
$$= (2\pi)^4\phi(f),$$

where

$$\phi(x) = \frac{1}{(2\pi)^4}\int_L \hat{\phi}(\lambda)e^{-i\lambda\cdot x}d^4\lambda \quad (3.7)$$

defines the Fourier transformation along L for complex λ.

Definition 2. We define the complex conjugate and Hermitian adjoint of $\hat{\phi}(\lambda)$ as[5]

$$(\hat{\phi}^*(\lambda),\hat{f}(\lambda))_L = e^{-2i\theta}(\hat{\phi}(\lambda),\hat{f}(\lambda))_L^*, \quad (3.8a)$$
$$(\hat{\phi}^\dagger(\lambda),\hat{f}(\lambda))_L = e^{-2i\theta}(\hat{\phi}(\lambda),\hat{f}(\lambda))_L^\dagger, \quad (3.8b)$$
$$\theta = \arctan(-\Gamma/2m).$$

C. Properties of the fields $\phi(x)$

One may now analyze the properties of the fields $\phi(x)$ in the framework of the Wightman axioms.[13]

(i) Hilbert space of states. The space of states is a separable Hilbert space $\mathcal{H} = (\{\psi_j\}: \|\psi_j\|^2<\infty)$ and the field $\phi(x)$ acts as an operator on it. Further, there exists a strongly continuous linear representation of $\bar{\mathcal{P}}_+^\dagger$ on $\mathcal{H}$.

(ii) $\phi(x)$ as an operator-valued distribution. The test function space $\mathfrak{D}$ and the set of fields $\{\phi(x)\}$ are mapped into linear operators $\phi(f)$; $\{f\}\in\mathfrak{D}$ over $\mathcal{H}$. The operators $\{\phi(f)\}$ are defined on a common invariant dense domain[14] $D\in\mathcal{H}$ such that $(\psi_1, \phi(f)\psi_2)$ is a distribution for $\psi_1,\psi_2\in D$ and

$$\phi(f)D\subset D, \quad \psi_0\in D$$
$$\phi^\dagger(f)D\subset D, \quad U(a,A)D\subset D.$$

(iii) Covariance of the fields. To each element $(a,A)\in\bar{\mathcal{P}}_+^\dagger$ there exists a continuous linear operator $U(a,A)$ such that

$$U(a,A)\phi(f)U^{-1}(a,A) = \phi(f_{(a,A)}), \quad f\in\mathfrak{D}$$

where

$$f_{(a,A)}(x) = f(A^{-1}(x-a)).$$

The action of $U(a,A)$ on a vector $\phi(f)\psi_0\in D$ is thus given by

$$U(a,A)\phi(f)\psi_0 = \phi(f_{(a,A)})\psi_0.$$

Since $f_{(a,A)}\in\mathfrak{D}$, we have

$$U(a,A)D=D.$$

(iv) Spectral condition. For $a\in T_4$, we have the following spectral decomposition for the nonunitary continuous linear operator $T(a)$ (using Stone's theorem):

$$T(a) = U(a,I) = \exp(ia_\mu\lambda^\mu),$$

where $\lambda_\mu = (1-i\Gamma/2m)p_\mu$ is interpreted as the complex energy-momentum operator. As usual, the spectral support of p_μ lies in the closure of the positive light cone; however, for complex λ_μ, $\bar{V}_+$ is modulated by a complex factor.

(v) Local commutativity. If the supports of $f(x)$ and $g(y)\in\mathfrak{D}$ are spacelike separated, i.e., if $f(x)g(y)=0$ for $(x-y)^2<0$, then

$$[\phi(f),\phi(g)]\psi=0, \quad \forall\psi\in D.$$

(vi) Particle interpretation. The S-matrix formulation for the fields $\phi(x)$ will be discussed elsewhere. In the present communication, we will confine ourselves to compute the Green's function for $\phi(x)$ and study its asymptotic behavior.

IV. PROPAGATOR FUNCTIONS

Our first task is now to compute the Pauli-Jordan operators $D^{(\pm)}(x)$ for the fields $\phi(x)$. The causal Green's function can then be expressed in terms of $D^{(+)}(x)$ and $D^{(-)}(x)$ by defining the vacuum expectation values of the time-ordered product, namely

$$\langle 0 | T(\phi(x)\phi(y)) | 0 \rangle .$$

To furnish this, we first introduce the Fock representation for the creation and annihilation operators of the free field $\phi(x)$:

$$\phi(f) = \frac{1}{(2\pi)^{3/2}} \int_L \frac{d^3\lambda}{(2\omega_\lambda)^{1/2}} [a(\vec{\lambda})\hat{f}(-\vec{\lambda}, -\omega_\lambda) + a^\dagger(\vec{\lambda})\hat{f}(\vec{\lambda}, \omega_\lambda)] . \tag{4.1}$$

Symbolically, we write

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int_L \frac{d^3\lambda}{(2\omega_\lambda)^{1/2}} [a(\vec{\lambda})e^{-i\lambda\cdot x} + a^\dagger(\vec{\lambda})e^{i\lambda\cdot x}] .$$

The creation and annihilation operators satisfy the following commutation relations, namely

$$[a(\vec{\lambda}), a^\dagger(\vec{\lambda}')]_L = e^{-3i\theta}\delta_L^3(\vec{\lambda} - \vec{\lambda}') , \tag{4.2}$$

where $\theta = \arctan(-\Gamma/2m)$ and $\delta_L^3(\vec{\lambda} - \vec{\lambda}')$ is Dirac's δ functional taken along L. We have introduced the extra phase factor on the right-hand side of (4.2) to make the commutator real.[15] We now define the Hermitian number operator N as

$$N = \int_L |d^3\lambda| a^\dagger(\vec{\lambda})a(\vec{\lambda}) . \tag{4.3}$$

We note that the creation and annihilation operators have the following commutation relations with N:

$$[N, a^\dagger(\vec{\lambda})]_L = a^\dagger(\vec{\lambda}) , \quad [N, a(\vec{\lambda})]_L = -a(\vec{\lambda}) . \tag{4.4}$$

We also require that $a(\vec{\lambda})|0\rangle = 0$. The transformation properties of $a(\vec{\lambda})$ and $a^\dagger(\vec{\lambda})$ are given by

$$U(b,A)a^\dagger(\vec{\lambda})U^{-1}(b,A) = e^{i\Lambda\lambda\cdot b}a^\dagger(\Lambda(A)\vec{\lambda}) , \quad U(b,A)a(\vec{\lambda})U^{-1}(b,A) = e^{-i\Lambda\lambda\cdot b}a(\Lambda(A)\vec{\lambda}) . \tag{4.5}$$

Similarly, under a pure translation,

$$U(b,I)|\lambda\rangle = T(b)|\lambda\rangle = e^{i\lambda\cdot b}|\lambda\rangle . \tag{4.6}$$

The complex energy-momentum four-vector is given by

$$\int_L |d^3\lambda| \lambda_\mu a^\dagger(\vec{\lambda})a(\vec{\lambda}) . \tag{4.7}$$

A. Evaluation of $[\phi(f), \phi(g)]$

Let

$$\phi(f) = \frac{1}{(2\pi)^{3/2}} \int_L \frac{d^3\lambda}{(2\omega_\lambda)^{1/2}} [a(\vec{\lambda})\hat{f}(-\vec{\lambda}, -\omega_\lambda) + a^\dagger(\vec{\lambda})\hat{f}(\vec{\lambda}, \omega_\lambda)] = \phi^{(-)}(f) + \phi^{(+)}(f) . \tag{4.8}$$

Then

$$[\phi^{(-)}(f), \phi^{(+)}(g)] = \frac{1}{(2\pi)^3} \int_L \frac{d^3\lambda}{(2\omega_\lambda)^{1/2}} \int_L \frac{d^3\lambda'}{(2\omega_{\lambda'})^{1/2}} [a(\vec{\lambda}), a^\dagger(\vec{\lambda}')]_L \times \hat{f}(-\vec{\lambda}, -\omega_\lambda)\hat{g}(\vec{\lambda}', \omega_{\lambda'}) .$$

Using (4.2) we have

$$[\phi^{(-)}(f), \phi^{(+)}(g)] = \frac{e^{-3i\theta}}{(2\pi)^3} \int_L \frac{d^3\lambda}{2\omega_\lambda} \hat{f}(-\vec{\lambda}, -\omega_\lambda)\hat{g}(\vec{\lambda}, \omega_\lambda) . \tag{4.9a}$$

Similarly,

$$[\phi^{(+)}(f), \phi^{(-)}(g)] = -\frac{e^{-3i\theta}}{(2\pi)^3} \int_L \frac{d^3\lambda}{2\omega_\lambda} \hat{f}(\vec{\lambda}, \omega_\lambda) \times \hat{g}(-\vec{\lambda}, -\omega_\lambda) . \tag{4.9b}$$

Using the property of the contribution for the δ_L functional (with complex argument),

$$\begin{aligned}(\delta * f, g)_L &= (\delta(z) \times f(z'), g(z + z'))_L \\ &= (f(z'), \delta(z), g(z + z'))_L \\ &= (f(z'), g(z'))_L \\ &= (f, g)_L \end{aligned} \tag{4.10}$$

and

$$\delta_L(\lambda^2 - M^2) = \frac{-i\pi}{\omega_\lambda} [\delta_L(\lambda_0 - \omega_\lambda) - \delta_L(\lambda_0 + \omega_\lambda)] , \tag{4.11}$$

we obtain

$$[\phi^{(-)}(f), \phi^{(+)}(g)] = \frac{i}{(2\pi)^4} e^{-3i\theta} \int_L d^4\lambda\, \theta(-\lambda_0)\delta_L(\lambda^2 - M^2) \times \hat{f}(-\lambda)\hat{g}(\lambda) \tag{4.12}$$

and

$$[\phi^{(+)}(f), \phi^{(-)}(g)] = \frac{-i}{(2\pi)^4} e^{-3i\theta} \int_L d^4\lambda\, \theta(\lambda_0)\delta_L(\lambda^2 - M^2) \times \hat{f}(-\lambda)\hat{g}(\lambda) . \tag{4.13}$$

Note that

$$\theta(\lambda_0)=1 \text{ for } p_0>0\,,$$
$$\theta(-\lambda_0)=1 \text{ for } p_0<0\,.$$

Combining (4.12) and (4.13) and using $\epsilon(\lambda_0)=\theta(\lambda_0)-\theta(-\lambda_0)$,

$$[\phi(f),\phi(g)]=\frac{-i}{(2\pi)^4}e^{-3i\theta}\int_L d^4\lambda\,\delta_L(\lambda^2-M^2)\epsilon(\lambda_0)\times\hat{f}(-\lambda)\hat{g}(\lambda)=-ie^{-3i\theta}(\hat{\Delta}*\hat{g},\hat{f})_L\,, \qquad (4.14)$$

where

$$(\hat{\Delta},\hat{f})_L=\int_L\hat{\Delta}^*(\lambda)\hat{f}(\lambda)d^4\lambda=\int_L\hat{\Delta}^*(\lambda)\left(\int f(x)e^{i\lambda\cdot x}d^4x\right)d^4\lambda=\int f(x)\left(\int_L\hat{\Delta}(\lambda)e^{-i\lambda\cdot x}d^4\lambda\right)^* d^4x=(2\pi)^4\int f(x)\hat{\Delta}^*(x,M)d^4x=(2\pi)^4(\Delta,f) \qquad (4.15)$$

Thus (see Ref. 16)

$$[\phi(f),\phi(g)]=-ie^{-3i\theta}(\Delta(x,M)*g,f)\,. \qquad (4.16)$$

From the property of the convolution, it follows that if the supports of $f(x), g(y)$ are spacelike separated, then $[\phi(f),\phi(g)]=0$.[17]

B. The causal Green's function $D^c(x)$ and the time graph

To estimate the asymptotic behavior of the time graph of an unstable particle, we analyze the property of the causal Green's function. We define the causal Green's function as usual by

$$D^c(x-y)=i\langle T(\phi(x)\phi(y))\rangle_0=\theta(x_0-y_0)D^{(-)}(x-y)-\theta(y_0-x_0)D^{(+)}(x-y)\,, \qquad (4.17)$$

where $D^{(\pm)}(x-v)$ are the free-particle propagator functions defined by (4.12) and (4.13), i.e.,

$$D^{(+)}(x)=\frac{e^{-3i\theta}}{(2\pi)^4}\int_L d^4\lambda\,\theta(\lambda_0)\delta_L(\lambda^2-M^2)e^{-i\lambda\cdot x}\,,$$
$$D^{(-)}(x)=\frac{e^{-3i\theta}}{(2\pi)^4}\int_L d^4\lambda\,\theta(-\lambda_0)\delta_L(\lambda^2-M^2)e^{-i\lambda\cdot x}\,. \qquad (4.18)$$

We now carry out the λ_0 integration in (4.18) and obtain

$$D^{(+)}(x)=\frac{1}{4\pi r}\frac{\partial}{\partial r}f(x)\,,\qquad D^{(-)}(x)=\frac{1}{4\pi r}\frac{\partial}{\partial r}f^*(x)\,, \qquad (4.19)$$

where

$$f(x)=\frac{i}{2\pi}e^{-3i\theta}\int_L\frac{d\lambda}{\lambda_0}e^{i(\lambda r+\lambda_0x_0)}\,,$$
$$f^*(x)=\frac{-i}{2\pi}e^{-3i\theta}\int_L\frac{d\lambda}{\lambda_0}e^{-i(\lambda r+\lambda_0x_0)}\,,$$

and

$$\lambda=(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{1/2},\quad r=\left(\sum_i(x_i^2)\right)^{1/2}. \qquad (4.20)$$

Let

$$\lambda=M\sinh\varphi=(m-i\Gamma/2)\sinh\varphi\,,\qquad \lambda_0=M\cosh\varphi=(m-i\Gamma/2)\cosh\varphi\,. \qquad (4.21)$$

Then,

$$f(x)=\frac{i}{2\pi}e^{-3i\theta}\int_{L'}d\varphi\,e^{iM(x_0\cosh\varphi+r\sinh\varphi)}\,. \qquad (4.22)$$

The φ integration is taken along L' as shown in Fig. 1. Here, we have to distinguish the four possibilities:

(1) $x_0>0,\quad x_0>r$;

(2) $x_0>0,\quad x_0<r$;

(3) $x_0<0,\quad |x_0|>r$;

(4) $x_0<0,\quad |x_0|<r$; (4.23)

Substituting (4.23) in (4.22) and using the integral representations of the cylindrical functions, we have

$$(1)\ \int_{L'}d\varphi\exp[iM\sqrt{u}\cosh(\varphi+\varphi_0)]=i\pi H_0^{(1)}(M\sqrt{u})=i\pi[J_0(M\sqrt{u})+iN_0(M\sqrt{u})]\,,$$

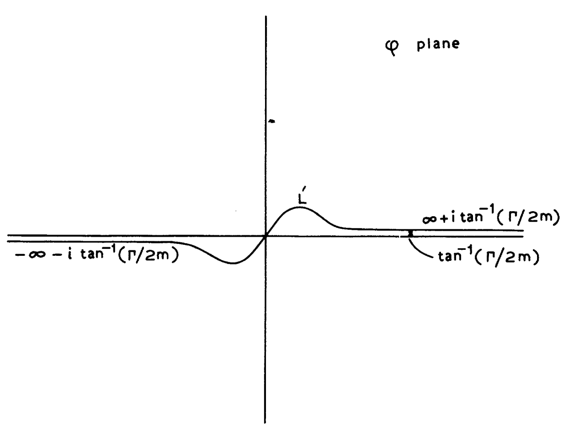

FIG. 1. The contour L' in the φ plane.

(2) $\int_{L'} d\varphi \exp[iM\sqrt{-u}\sinh(\varphi+\varphi_0)] = 2K_0(M\sqrt{-u})$,

(3) $\int_{L'} d\varphi \exp[-iM\sqrt{u}\cosh(\varphi-\varphi_0)]$

$$= -i\pi H_0^{(2)}(M\sqrt{u}) = -i\pi[J_0(M\sqrt{u}) - iN_0(M\sqrt{u})] ,$$

(4) $\int_{L'} d\varphi \exp[-iM\sqrt{-u}\sinh(\varphi-\varphi_0)] = 2K_0(M\sqrt{-u})$,

$$u = x^2 = (x_0)^2 - (\vec{x})^2 . \tag{4.24}$$

From (4.22) and (4.24), we finally obtain

$$f(x) = (1/2i)N_0(M\sqrt{u}) - \tfrac{1}{2}\epsilon(x_0)J_0(M\sqrt{u}), \quad u>0$$
$$= \frac{i}{\pi}K_0(M\sqrt{-u}), \quad u<0 . \tag{4.25}$$

Substituting (4.25) in (4.19), we have

$$D^{(\pm)}(x) = e^{-3i\theta}\left\{\frac{1}{4\pi}\epsilon(x_0)\delta(u) \mp \frac{iM}{8\pi\sqrt{u}}\theta(u)[N_1(M\sqrt{u}) \mp i\epsilon(x_0)J_1(M\sqrt{u})] \pm \theta(-u)\frac{iM}{4\pi^2\sqrt{-u}}K_1(M\sqrt{-u})\right\} . \tag{4.26}$$

The Pauli-Jordan function is now given by

$$D(x) = D^{(+)}(x) + D^{(-)}(x) = e^{-3i\theta}\left[\frac{1}{2\pi}\epsilon(x_0)\delta(u) - \frac{M}{4\pi\sqrt{u}}\theta(u)\epsilon(x_0)J_1(M\sqrt{u})\right] . \tag{4.27}$$

We note that $D(x)$ vanishes outside the light cones (for $u<0$). Now, the causal Green's function $D^c(x)$ is given by

$$D^c(x) = \theta(x_0)D^{(-)}(x) - \theta(-x_0)D^{(+)}(x)$$
$$= \frac{e^{-3i\theta}}{4\pi}\left\{\delta(u) - \frac{M}{2\sqrt{u}}\theta(u)[J_1(M\sqrt{u}) - iN_1(M\sqrt{u})] + \frac{iM}{\pi\sqrt{-u}}\theta(-u)K_1(M\sqrt{-u})\right\} ,$$

or

$$D^c(x) = \frac{e^{-3i\theta}}{4\pi}\left[\delta(u) - \frac{M}{2\sqrt{u}}\theta(u)H_1^{(2)}(M\sqrt{u}) + \frac{iM}{\pi\sqrt{-u}}\theta(-u)K_1(M\sqrt{-u}\right] . \tag{4.28}$$

It is understood that the Fourier transformation of (4.28) exists in the sense of the generalized functions of Gelfond and Shilov[12] and is given by

$$\hat{D}_F(\lambda) = \frac{i}{\lambda^2 - M^2} . \tag{4.29}$$

We note that the relation (4.29) was derived by Simonius[18] from an entirely different and heuristic way. The time graph was never shown explicitly.

To obtain the time graph of the unstable particle, we consider the asymptotic behavior of (4.28), i.e., for large values of u, we have

$$D^c(x) \sim -\theta(u)\left(\frac{iM}{32\pi^3 u^{3/2}}\right)^{1/2}\exp(-iM\sqrt{u}) + \theta(-u)\left[\frac{M}{32\pi^3(-u)^{3/2}}\right]^{1/2}\exp(-M\sqrt{-u}) . \tag{4.30}$$

Equation (4.30) displays the remarkable exponential decay law for the unstable particle.

V. CONCLUSION

The field theory of the unstable particle has been analyzed by resorting to the covariant property of the fields under the degenerate class representations of $\bar{\mathcal{P}}_+^\dagger$. The field describing the unstable particle is an operator-valued distribution on the space $\mathfrak{D}$ of infinitely differentiable functions of compact support. In fact, the choice of $\mathfrak{D}$ could be relaxed and the analysis could be analogously carried out in the more general space of type S_α or S_α^β [correspondingly, the Fourier transform of $\phi(x)$ is described in $\tilde{S}_\alpha = S^\alpha$, or $\tilde{S}_\alpha^\beta = S_\beta^\alpha$] of Gelfond and Shilov.[12] The Fourier transform of $\phi(x)$ for complex four-momentum λ is a functional in $\hat{\mathfrak{D}}'$ and such that $(\hat{\phi}(\lambda), \hat{f}(\lambda))_L$ is convergent and regular along the contour L in the complex λ plane. In our opinion, this is rather the customary practice in quantum field theory to extract the finite part from a divergent series or integral by suitably choosing the test function space $\mathfrak{D}$. Only in this sense $\hat{\phi}(\lambda)$ is well behaved or else it is undefined conventionally.

The fields satisfy the axioms of conventional field theories with a modified spectral condition. From the properties of the generalized functions, we show that the Pauli-Jordan operator $\Delta(x,M)$ smeared with the test function $f(x)$, i.e., $(\Delta(x,M), f(x))$ is finite[16] and identically vanishes for spacelike x.[17]

It is worthwhile to make some remarks on the work of Ref. 18 in the light of our analysis. In Ref. 18, the splitting of the field into positive- and negative-frequency parts is nonlocal since the two frequency parts are defined separately on two different Hilbert spaces. The definition of the Fourier transform of the field is also not clear when the momentum is complex. So the introduction of the Fourier transform of the field as an operator-valued distribution as in our present analysis is rather imperative. The regularization introduced for computation of the Pauli-Jordan operators seems to be far from explicit. Further, unlike our present analysis, the time graph for the unstable particle has not been computed in Ref. 18. In Sec. IV we have rather made some adaptations of the work of Bogoliubov and Shirkov[19] and by suitably defining the contours have computed the Green's function $D^c(x)$.

The physical implication of the nondegenerate series representations of $\overline{\mathcal{P}}_+^\dagger$ for exchange scattering and tachyons will be reported separately.

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[9]The space D is, in fact, dense in S, the space of infinitely differentiable functions.

[10]Suppose $\mathfrak{D}$ consists of infinitely differentiable functions $\{f(x)\}$ satisfying the inequalities

$$|x^k f^{(n)}(x)| \le C_n \alpha^k ,$$

where the constants C_n and α depend upon $f(x)$. Dividing both sides by $|x^k|$ and taking the least upper bound (lub) of k on the right-hand side, we have

$$|f^{(n)}(x)| \le C_n \inf_k |\alpha|x|^k = C_n \xi(x/\alpha),$$

where,

$$\xi(x/\alpha) = \inf_k 1/|x/\alpha|^k = \begin{cases} 1 \text{ for } |x| \le \alpha \\ 0 \text{ for } |x| > \alpha \end{cases},$$

i.e., if $f(x)$ is infinitely differentiable which vanishes for $|x| > \alpha$, then

$$|x^k D^n f(x)| \le C_n \alpha^k.$$

Since α is arbitrary, $\{f(x)\}$ are infinitely differentiable with compact support. If, however, $\{f(x)\} \in S_\alpha$ (see, e.g., Gelfond and Shilov, Ref. 12, p. 171), then $\{f(x)\}$ together with all their derivatives decrease exponentially at infinity, with an order $\le 1/\alpha$ and a type $\ge a$ dependent on the function $f(x)$, i.e.,

$$|f^{(n)}(x)| \le C_n \exp(-a|x|^{1/\alpha}).$$

[11]We have from (3.2) $\hat{f}(\lambda) = \int f(x) e^{ip\cdot x} e^{-q\cdot x} d^4x$. Suppose $x_0 \ge 0$, $\vec{x}^2 \le x_0^2$. Then

$$q\cdot x = q_0 x_0 - \vec{q}\cdot\vec{x} > (q_0 - |\vec{q}|)x_0 > \tfrac{1}{2}(q_0 - |\vec{q}|)(|x_0| + |\vec{x}|).$$

For $q^2 > 0$, $q_0 > 0$, we have

$$e^{-q\cdot x} \le \exp[-\tfrac{1}{2}(q_0 - |\vec{q}|)(|x_0| + |\vec{x}|)] = \exp[-\alpha(|x_0| + |\vec{x}|)], \quad \alpha > 0.$$

Thus, from a quick and rough estimation, we find that $e^{-q\cdot x}$ will play the role of cutoff or regularization. By a similar argument, we can also construct entire and analytic functions of $f(p)$ in the lower half-plane with $\lambda = p - iq$, $q^2 > 0$, $q_0 > 0$ (for $x_0 < 0$). Since in the limit $q \to 0$ both these functions coincide, we can in general speak of a single analytic function (by the edge of the wedge theorem) whose convergence properties are defined with respect to the norm (3.6).

[12]I. M. Gelfond and G. E. Shilov, *Generalized Functions* (Academic, New York, 1966), Vol. 2, p. 130.

[13]R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and all that* (Benjamin, New York, 1963).

[14]The dense domain D is generated by the polynomials over fields smeared with test functions $\{f(x)\}$ applied to the vacuum state $\psi_0 \in \mathcal{H}$.

[15]This follows from the definition of the complex conjugate of the functional. Substituting $\delta_L(z - z')$ for $\hat{\phi}(\lambda)$ in (3.8), we have

$$(\delta_L(z - z'), \hat{f}(z))_L = e^{-2i\theta}[(\delta_L(z - z'), \hat{f}^*(z))]_L^* ,$$

$\forall z, z' \in C^1$. Thus follows (4.2).

[16]The convergence properties of (4.15) can be proved as follows. We have

$$(\Delta(x, M), f(x)) = \int d^4x f(x) \Delta^*(x, M) = \frac{-i}{(2\pi)^4} e^{-3i\theta} \int_L d^4\lambda \hat{f}(\lambda) \delta_L(\lambda^2 - M^2) \epsilon(\lambda_0).$$

Using

$$\delta_L(\lambda^2-M^2)=\frac{-i\pi}{(\lambda_k^2+M^2)^{1/2}}[\delta_L(\lambda_0-(\lambda_k^2+M^2)^{1/2})-\delta_L(\lambda_0+(\lambda_k^2+M^2)^{1/2})],\quad k=1,2,3$$

we find

$$(\Delta(x,M),f(x))=\frac{e^{-3i\theta}}{(2\pi)^3}\int_{L_k}\frac{d^3\lambda_k}{2(\lambda_k^2+M^2)^{1/2}}\times[\hat f(\lambda_k,(\lambda_k^2+M^2)^{1/2})-\hat f(\lambda_k,-(\lambda_k^2+M^2)^{1/2})].$$

Note that L_k passes through the origin and is the same for all the components λ_k. Since $\hat f(\lambda)\in\hat{\mathfrak{D}}_L$, we have $|\hat f(\lambda)|\le\text{const}\times e^{\alpha|q|}$; the constant might depend upon $\hat f(\lambda)$. Now,

$$|(\Delta(x,M),f(x))|\le\text{const}\times\sup_\lambda\left|\int_{L_k}\frac{d^3\lambda_k}{2(\lambda_k^2+M^2)^{1/2}}[\hat f(\lambda_k,(\lambda_k^2+M^2)^{1/2})-\hat f(\lambda_k,-(\lambda_k^2+M^2)^{1/2})]\right|$$
$$\le\text{const}\times\sup_\lambda\left|\int_{L_k'}\frac{d^3p_k}{2(p_k^2+M^2)^{1/2}}[\hat f(p_k,(p_k^2+M^2)^{1/2})-\hat f(p_k,-(p_k^2+M^2)^{1/2})]\right|.$$

In this equation we have made use of the fact that when $L_k\to\infty$, the contribution to the integral

$$\int_{L_k}\frac{d^3\lambda_k}{2(\lambda_k^2+M^2)^{1/2}}[\hat f(\lambda_k,(\lambda_k^2+M^2)^{1/2})-\hat f(\lambda_k,-(\lambda_k^2+M^2)^{1/2})]$$

in the second and fourth quadrants identically vanishes by using the asymptotic properties of $\hat f(\lambda)$. So, by a counterclockwise rotation, we can take the integral along L_k' on the real axis in the complex λ_k plane. Thus,

$$|(\Delta(x,M),f(x))|\le\text{const}\times\lim_{L_k'\to\infty}\|\hat f(p_k,(p_k^2+M^2)^{1/2})\|\to\text{const}'\quad\text{(QED)}.$$

[17]The vanishing of (4.16) for spacelike supports of $f(x)$, $g(y)$ can be demonstrated as follows. Symbolically, we have

$$\Delta(x,M)=\frac{1}{(2\pi)^3}e^{-3i\theta}\int_{L_k}\frac{d^3\lambda_k}{2(\lambda_k^2+M^2)^{1/2}}\times\{\exp[-i(\lambda_k x_k-(\lambda_k^2+M^2)^{1/2}x_0)]-\exp[i(\lambda_k x_k-(\lambda_k^2+M^2)^{1/2}x_0)]\}.$$

Now, going to the Lorentz frame where $x_0=0$, and replacing $\lambda_k\to-\lambda_k$ in the second term of the above integrand, we find

$$\Delta(x_k,M)=\frac{1}{(2\pi)^3}e^{-3i\theta}\int_{L_k}\frac{d^3\lambda_k}{2(\lambda_k^2+M^2)^{1/2}}(e^{-i\lambda_k x_k}-e^{-i\lambda_k x_k})=0.$$

[18]M. Simonius, Helv. Phys. Acta 43, 223 (1970).
[19]N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959), p. 147.