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Gravitational two-body problem with arbitrary masses, spins, and quadrupole moments

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We find the precession of the spin and the precession of the orbit for the two-body problem in general relativity with arbitrary masses, spins, and quadrupole moments. One notable result which emerges is that, in the case of arbitrary masses m_1 and m_2, the spin-orbit contribution to the *spin* precession of body 1 is a factor $(m_2+\mu/3)/(m_1+m_2)$ times what it would be for a test body moving in the field of a fixed central mass (m_1+m_2). Here μ denotes the reduced mass $m_1 m_2/(m_1+m_2)$. This contrasts with the result of Robertson for the *periastron* precession where the corresponding factor is unity. These results may be of interest for binary neutron stars and, in particular, for binary pulsars such as PSR1913+16.

I. INTRODUCTION

The gravitational two-body equations of motion for arbitrary masses without spin were first derived by Einstein, Infeld, and Hoffmann[1] (EIH). Corinaldesi,[2] using the quantum theory of gravitation, seemed to have derived the EIH equations of motion from the one-graviton exchange interaction. But later Iwasaki[3] showed that the two-graviton exchange interaction was also needed to obtain the G^2 term in the Hamiltonian from which the EIH equations of motion could be derived.

Papapetrou[4,5] and Corinaldesi[5] derived equations of motion of a spinning test body in a given gravitational field. Then Schiff[6] used these results to find the precession of a gyroscope in orbit about the earth.

In a recent paper[7] we derived the precession of the spin and the precession of the orbit of a spinning test body in the gravitational field of a much larger spinning body. In our procedure we used the one-graviton exchange interaction of two spin-$\frac{1}{2}$ particles[8] derived from Gupta's[9] quantum theory of gravitation. This potential energy was first converted to a classical potential energy and then (as we were interested in the gyroscope problem) the large-mass approximation was made.

Because of the recent interest in binary neutron stars and, in particular, binary pulsars such as[10] PSR1913+16 we now think it is appropriate to give the results for the precession of the spin and the precession of the orbit for two bodies with arbitrary masses, spins, and quadrupole moments. We need only proceed as before[7] but without making the large-mass approximation.

Because of the fact that the spin-independent part of the Hamiltonian which we use[7,8] appears to be different from that of the EIH Hamiltonian, we present in Sec. II a general Hamiltonian for the two-body problem for arbitrary masses without spin and show that the results of Barker, Gupta, and Haracz[8] are related to the EIH Hamiltonian by a coordinate transformation. The essential point here is that all the different forms of the Hamiltonian lead to the same *observable* result for the precession of the orbit.

In Sec. III we shall write down the Hamiltonian and Lagrangian for the two-body problem for arbitrary masses, spins, and quadrupole moments, while in Sec. IV we shall give the precession of the spin and in Sec. V the precession of the orbit. Here we also point out that the spin-dependent parts of the equations of motion may be written in several different ways (depending on the choice of the spin supplementary condition) but they all lead to the same *observable* results for the precession of the orbit.

We shall present our conclusions in Sec. VI.

II. HAMILTONIAN WITHOUT SPIN

The spin-independent part of the Hamiltonian which we use[7,8] [Eqs. (2), (10), and (11) below] is apparently different from the EIH Hamiltonian[1]. For instance, it does not contain a $(\vec{P}\cdot\vec{r})^2$ term, whereas the latter does. It is our basic purpose in this section (a) to show that our Hamiltonian[7,8] is related to that of EIH by a particular coordinate transformation, and (b) to write down a general Hamiltonian [Eqs. (1)–(4) below], which is obtained from that of EIH by a general coordinate transformation [Eqs. (6)–(7) below].

On reflection, the fact that the Hamiltonian can be written in a variety of ways should not be surprising. It is simply related to the fact that the metric tensor may be written in a variety of ways, depending on the choice of coordinate conditions. For example, the Schwarzschild exterior solution of Einstein's field equations, for a spherically symmetric mass distribution, may be written in standard, isotropic, or harmonic coordinates.

These are common choices, but of course, an infinity of possible choices exists.

Our procedure is to start with the EIH Hamiltonian and then make a general transformation [see Eqs. (6)–(7) below, where α is an arbitrary parameter]. This gives us a very general Hamiltonian, containing the parameter α [Eqs. (1)–(4) below]. We then point out that, for a particular choice of α, we obtain the Hamiltonian which agrees with the one-gravitation exchange interaction of Refs. 7 and 8. Other interesting choices of α are then discussed.

Let m_1, $\vec{r}_1$, $\vec{P}_1$ and m_2, $\vec{r}_2$, $\vec{P}_2$ be the mass, position and momentum of the first and second particles, respectively. We shall be interested in the Hamiltonian in the center-of-mass system which is given by Hiida and Okamura[11] as

$$\mathcal{H}(\alpha) = \mathcal{H}_0 + V_1(\alpha) + V_2(\alpha), \tag{1}$$

where

$$\mathcal{H}_0 = \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{P}^2 - \frac{1}{8}\left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right)\frac{\vec{P}^4}{c^2}, \tag{2}$$

$$V_1(\alpha) = \frac{-Gm_1m_2}{r}\left\{1 + \left[\tfrac{1}{2} + (\tfrac{3}{2} - \alpha)\frac{M}{\mu}\right]\frac{\vec{P}^2}{m_1m_2c^2} + \left[\tfrac{1}{2} + \alpha\frac{M}{\mu}\right]\frac{(\vec{P}\cdot\vec{r})^2}{m_1m_2c^2r^2}\right\}, \tag{3}$$

$$V_2(\alpha) = (1 - 2\alpha)\frac{G^2\mu M^2}{2c^2r^2}, \tag{4}$$

and α is an arbitrary dimensionless parameter. The reduced mass and total mass are given by

$$\mu = \frac{m_1m_2}{m_1 + m_2}, \quad M = m_1 + m_2. \tag{5}$$

We also note that $\vec{r} = \vec{r}_1 - \vec{r}_2$, $\vec{P} = \vec{P}_1 = -\vec{P}_2$, and c and G are the speed of light and the gravitational constant, respectively. The Hamiltonian of Eq. (1) can be obtained from the EIH Hamiltonian [Eq. (1) with $\alpha = 0$ and $\vec{r}$ and $\vec{P}$ replaced by $\vec{r}_{\mathrm{EIH}}$ and $\vec{P}_{\mathrm{EIH}}$, respectively] by the coordinate transformation

$$\vec{r}_{\mathrm{EIH}} = \vec{r}\left(1 - \alpha\frac{GM}{c^2r}\right), \tag{6}$$

which implies that

$$\vec{P}_{\mathrm{EIH}} = \vec{P} + \alpha\frac{GM}{c^2r}\left[\vec{P} - \frac{(\vec{P}\cdot\vec{r})\vec{r}}{r^2}\right]. \tag{7}$$

There are two values of α that Hiida and Okamura[11] singled out for special attention. They are $\alpha = 0$, which corresponds to the EIH Hamiltonian, and $\alpha = \frac{1}{2}$, which gives a Hamiltonian without a G^2 term. However, the restriction to mass-independent values of α is not necessary. In fact, it is also possible to eliminate the $\vec{P}^2$ term in $V_1(\alpha)$ by choosing $\alpha = \frac{3}{2} + \frac{1}{2}\mu/M$ *or* to eliminate the $(\vec{P}\cdot\vec{r})^2$ term in in $V_1(\alpha)$ by choosing $\alpha = -\frac{1}{2}\mu/M$. The results are

$$V_1(\alpha = \tfrac{3}{2} + \tfrac{1}{2}\mu/M) = -\frac{Gm_1m_2}{r}\left[1 + \left(4 + \frac{3m_1}{2m_2} + \frac{3m_2}{2m_1}\right)\frac{(\vec{P}\cdot\vec{r})^2}{m_1m_2c^2r^2}\right], \tag{8}$$

$$V_2(\alpha = \tfrac{3}{2} + \tfrac{1}{2}\mu/M) = -\frac{G^2\mu M(\mu + 2M)}{2c^2r^2} \tag{9}$$

and

$$V_1(\alpha = -\tfrac{1}{2}\mu/M) = -\frac{Gm_1m_2}{r}\left[1 + \left(4 + \frac{3m_1}{2m_2} + \frac{3m_2}{2m_1}\right)\frac{\vec{P}^2}{m_1m_2c^2}\right], \tag{10}$$

$$V_2(\alpha = -\tfrac{1}{2}\mu/M) = \frac{G^2\mu M(\mu + M)}{2c^2r^2}. \tag{11}$$

Note that the potential-energy term $V_1(\alpha = -\frac{1}{2}\mu/M)$ of Eq. (10) agrees with the one-graviton exchange interaction of Barker, Gupta, and Haracz.[8] If the large-mass approximation ($m_2 \gg m_1$) is made, the EIH Hamiltonian ($\alpha = 0$) becomes identical to the Hamiltonian with $\alpha = -\frac{1}{2}\mu/M$.

Another Hamiltonian that is of interest is one where $\alpha = 1 + \lambda\mu/M$ and λ is a constant, independent of the masses, for when the large-mass approximation is made the coordinate system of this Hamiltonian will be a Schwarzschild coordinate system.

Let us now look at an important aspect in the derivation of the one-graviton exchange interaction from quantum field theory. Consider the term[3,8,11]

$$\frac{1}{k^2} = \frac{1}{\vec{k}^2 - k_0^2}, \tag{12}$$

where

$$k = p' - p = q - q'. \tag{13}$$

The initial and final propagation four-vectors for the particle of mass m_1 are p and p', while those for the particle of mass m_2 are q and q', respectively. The propagation four-vector p has a momentum of $\hbar\vec{p}$ and an energy of $c\hbar p_0$. Equation (12) can be written as[11]

$$\frac{1}{k^2}=\frac{1}{\vec{k}^2+x(p_0'-p_0)(q_0'-q_0)-\frac{1}{2}(1-x)[(p_0'-p_0)^2+(q_0'-q_0)^2]}\ . \tag{14}$$

Then using

$$p_0'-p_0=\frac{\vec{k}\cdot(\vec{p}\,'+\vec{p})}{p_0'+p_0}\ , \tag{15}$$

$$q_0-q_0'=\frac{\vec{k}\cdot(\vec{q}\,'+\vec{q})}{q_0'+q_0}\ , \tag{16}$$

in the denominator of Eq. (14) we get

$$k^2=\vec{k}^2-x\,\frac{\vec{k}\cdot(\vec{p}\,'+\vec{p})\vec{k}\cdot(\vec{q}\,'+\vec{q})}{(p_0'+p_0)(q_0'+q_0)}-\tfrac{1}{2}(1-x)\left[\frac{\vec{k}\cdot(\vec{p}\,'+\vec{p})\vec{k}\cdot(\vec{p}\,'+\vec{p})}{(p_0'+p_0)^2}+\frac{\vec{k}\cdot(\vec{q}\,'+\vec{q})\vec{k}\cdot(\vec{q}\,'+\vec{q})}{(q_0'+q_0)^2}\right]. \tag{17}$$

Hiida and Okamura[11] have shown that the relation between x of Eqs. (14) and (17) and α of Eq. (1) is

$$\alpha=-\tfrac{1}{4}(1-x),\quad x=4\alpha+1\ . \tag{18}$$

In the center-of-mass system we have $\vec{p}\,'=-\vec{q}\,'$, $\vec{p}=-\vec{q}$ and $p_0=p_0'$, $q_0=q_0'$, so that in Eq. (17) we may set[8]

$$k^2=\vec{k}^2 \tag{19}$$

if we choose

$$\frac{x}{p_0q_0}=\frac{1-x}{2}\left(\frac{1}{p_0^{\;2}}+\frac{1}{q_0^{\;2}}\right), \tag{20}$$

so that the $k_0^{\;2}$ term of Eq. (17) will be zero. We then have

$$x=\frac{p_0^{\;2}+q_0^{\;2}}{(p_0+q_0)^2}\approx\frac{m_1^{\;2}+m_2^{\;2}}{(m_1+m_2)^2}=1-2\,\frac{\mu}{M}\ . \tag{21}$$

Using $x=1-2\mu/M$ in Eq. (18) we find $\alpha=-\frac{1}{2}\mu/M$. We conclude that the particular form of the potential energy that appears in Ref. 8 results from taking $k_0^{\;2}=0$ in the denominator of Eq. (12).

It is also of interest[8,12] to consider the particular case of equal masses, i.e., $m_1=m_2=m$. Hence $\mu=m/2$ and $M=2m$. It follows that

$$\mathcal{H}_0=\frac{\vec{P}^2}{m}-\frac{\vec{P}^4}{4m^3c^2}\ , \tag{22}$$

$$V_1(\alpha)=-\frac{Gm^2}{r}\left[1+(\tfrac{13}{2}-4\alpha)\frac{\vec{P}^2}{m^2c^2}+(\tfrac{1}{2}+4\alpha)\frac{(\vec{P}\cdot\vec{r})^2}{m^2c^2r^2}\right], \tag{23}$$

$$V_2(\alpha)=(1-2\alpha)\frac{G^2m^3}{c^2r^2}\ . \tag{24}$$

Hence the corresponding potential-energy terms for the $\alpha=\frac{3}{2}+\frac{1}{2}\mu/M=\frac{13}{8}$ case are

$$V_1(\alpha=\tfrac{13}{8})=-\frac{Gm^2}{r}\left[1+7\frac{(\vec{P}\cdot\vec{r})^2}{m^2c^2r^2}\right], \tag{25}$$

$$V_2(\alpha=\tfrac{13}{8})=-\frac{9}{4}\,\frac{G^2m^3}{c^2r^2}\ , \tag{26}$$

while the corresponding potential-energy terms[8] for the $\alpha=-\frac{1}{2}\mu/M=-\frac{1}{8}$ case are

$$V_1(\alpha=-\tfrac{1}{8})=-\frac{Gm^2}{r}\left(1+7\frac{\vec{P}^2}{m^2c^2}\right), \tag{27}$$

$$V_2(\alpha=-\tfrac{1}{8})=\frac{5}{4}\,\frac{G^2m^3}{c^2r^2}\ . \tag{28}$$

III. TOTAL HAMILTONIAN AND LAGRANGIAN

The total Hamiltonian for arbitrary masses, spins, and quadrupole moments is given by [see Eqs. (1) and (62) of Ref. 7 and Eqs. (46) and (47) of Ref. 13]

$$\mathcal{H}_t(\alpha)=Mc^2+\mathcal{H}(\alpha)+V_{S1}+V_{S2}+V_{S1,S2}+V_{Q1}+V_{Q2}\ , \tag{29}$$

where $\mathcal{H}(\alpha)$ is given by Eq. (1) and

$$V_{S1}=\frac{G}{c^2r^3}\left(2+\frac{3m_2}{2m_1}\right)\vec{S}^{(1)}\cdot(\vec{r}\times\vec{P}), \tag{30}$$

$$V_{S2}=\frac{G}{c^2r^3}\left(2+\frac{3m_1}{2m_2}\right)\vec{S}^{(2)}\cdot(\vec{r}\times\vec{P}), \tag{31}$$

$$V_{S1,S2}=\frac{G}{c^2r^3}\left(\frac{3(\vec{S}^{(1)}\cdot\vec{r})(\vec{S}^{(2)}\cdot\vec{r})}{r^2}-\vec{S}^{(1)}\cdot\vec{S}^{(2)}\right), \tag{32}$$

$$V_{Q1}=\frac{GJ_2^{(1)}m_1m_2}{2r^3}\left(\frac{3(\vec{n}^{(1)}\cdot\vec{r})^2}{r^2}-1\right), \tag{33}$$

$$V_{Q2}=\frac{GJ_2^{(2)}m_1m_2}{2r^3}\left(\frac{3(\vec{n}^{(2)}\cdot\vec{r})^2}{r^2}-1\right), \tag{34}$$

where $\vec{S}^{(1)}$, $I^{(1)}$, $\vec{\omega}^{(1)}$, $\vec{v}_1$ and $\vec{S}^{(2)}$, $I^{(2)}$, $\vec{\omega}^{(2)}$, $\vec{v}_2$ are the spin, moment of inertia, angular velocity, and velocity of bodies 1 and 2, respectively. To first order, $\vec{S}^{(1)}=I^{(1)}\vec{\omega}^{(1)}$, $\vec{S}^{(2)}=I^{(2)}\vec{\omega}^{(2)}$, and $\vec{P}=\mu\vec{v}$, where $\vec{v}=\vec{v}_1-\vec{v}_2$. Also, $\vec{n}^{(1)}$ and $\vec{n}^{(2)}$ are unit vectors in the $\vec{S}^{(1)}$ and $\vec{S}^{(2)}$ directions, respectively. The quantities $J_2^{(1)}$ and $J_2^{(2)}$ for bodies 1 and 2, respectively, are given by [see Eqs. (38) and (40) of Ref. 7 and Eqs. (6) and (48) of Ref. 13]

$$m_1 J_2^{(1)} = \Delta I^{(1)} = \tfrac{1}{2}\int dV'[r'^2 - 3(\vec{n}^{(1)}\cdot\vec{r}')^2]\rho_1(\vec{r}'), \quad (35)$$

$$m_2 J_2^{(2)} = \Delta I^{(2)} = \tfrac{1}{2}\int dV'[r'^2 - 3(\vec{n}^{(2)}\cdot\vec{r}')^2]\rho_2(\vec{r}'), \quad (36)$$

and $\rho_1(\vec{r}')$ and $\rho_2(\vec{r}')$ are the mass densities of body 1 and body 2, respectively.

We note that m_1c^2 and m_2c^2 contain the rotational energy of body 1 and body 2 as well as the rest energy. We thus have[7]

$$Mc^2 = (m_{01}c^2 + m_{02}c^2) + (\tfrac{1}{2}I^{(1)}\omega^{(1)^2} + \tfrac{1}{2}I^{(2)}\omega^{(2)^2} + \cdots). \quad (37)$$

The total Lagrangian corresponding to Eq. (29) is given by

$$\begin{aligned}\mathcal{L}_t(\alpha) = &-(m_{01}c^2 + m_{02}c^2)\\ &+ (\tfrac{1}{2}I^{(1)}\omega^{(1)^2} + \tfrac{1}{2}I^{(2)}\omega^{(2)^2} + \cdots)\\ &+ \tfrac{1}{2}\mu v^2 + \tfrac{1}{8}(1 - 3\mu/M)\mu v^4/c^2\\ &- [V_1(\alpha) + V_2(\alpha) + V_{S1} + V_{S2} + V_{S1,S2} + V_{Q1} + V_{Q2}],\end{aligned} \quad (38)$$

where $\mu\vec{v}$ replaces $\vec{P}$ in the potential-energy terms.

IV. PRECESSION OF THE SPIN

The results for the precession of the spin may be derived using the techniques of Ref. 7. For body 1 we find (similar results can be given for body 2)

$$\dot{\vec{n}}^{(1)} = \vec{\Omega}^{(1)} \times \vec{n}^{(1)}, \quad (39)$$

where

$$\vec{\Omega}^{(1)} = \vec{\Omega}^{(1)}_{dS} + \vec{\Omega}^{(1)}_{LT} + \vec{\Omega}^{(1)}_{Q1}, \quad (40)$$

and

$$\vec{\Omega}^{(1)}_{dS} = G\left(2 + \frac{3m_2}{2m_1}\right)\frac{\mu\vec{r}\times\vec{v}}{c^2r^3}, \quad (41)$$

$$\vec{\Omega}^{(1)}_{LT} = \frac{G}{c^2r^3}\left(\frac{3\vec{r}(\vec{S}^{(2)}\cdot\vec{r})}{r^2} - \vec{S}^{(2)}\right), \quad (42)$$

$$\vec{\Omega}^{(1)}_{Q1} = \frac{Gm_2\Delta I^{(1)}}{I^{(1)}\omega^{(1)}r^3}\left(\frac{3\vec{r}(\vec{n}^{(1)}\cdot\vec{r})}{r^2} - \vec{n}^{(1)}\right). \quad (43)$$

The terms $\vec{\Omega}^{(1)}_{dS}$, $\vec{\Omega}^{(1)}_{LT}$, and $\vec{\Omega}^{(1)}_{Q1}$ are determined by the terms V_{S1}, $V_{S1,S2}$, and V_{Q1} in the Hamiltonian of Eq. (29). Equations (41) and (42) can easily be inferred from the results of Sec. II of Ref. 7, while Eq. (43) is given by Eq. (40) of Ref. 13. Note that $\vec{\Omega}^{(1)}_{LT}$ and $\vec{\Omega}^{(1)}_{Q1}$ have the same form as in their large-mass approximation, while $\vec{\Omega}^{(1)}_{dS}$ does not and hence is a new result.

The secular results for the precession of the spin are given by

$$\dot{\vec{n}}^{(1)}_{av} = \vec{\Omega}^{(1)}_{av} \times \vec{n}^{(1)}, \quad (44)$$

where

$$\vec{\Omega}^{(1)}_{dS\,av} = \frac{3G\bar{\omega}(m_2 + \mu/3)}{2c^2a(1-e^2)}\vec{n}, \quad (45)$$

$$\vec{\Omega}^{(1)}_{LT\,av} = \frac{GS^{(2)}}{2c^2a^3(1-e^2)^{3/2}}[\vec{n}^{(2)} - 3(\vec{n}\cdot\vec{n}^{(2)})\vec{n}], \quad (46)$$

$$\vec{\Omega}^{(1)}_{Q1\,av} = \frac{Gm_2\Delta I^{(1)}}{2I^{(1)}\omega^{(1)}a^3(1-e^2)^{3/2}}[\vec{n}^{(1)} - 3(\vec{n}\cdot\vec{n}^{(1)})\vec{n}], \quad (47)$$

and e is the eccentricity, a is the semimajor axis, $\bar{\omega}$ is the average orbital angular velocity, $\vec{L} = \vec{r}\times\vec{P}$ is the orbital angular momentum, and $\vec{n}$ is a unit vector in the $\vec{L}$ direction. Also, we have the relation

$$\frac{L/\mu}{a^2(1-e^2)^{1/2}} = \left(\frac{GM}{a^3}\right)^{1/2} = \frac{2\pi}{T} = \bar{\omega}, \quad (48)$$

where T is the orbital period.

In particular, it is notable that in going from the large-mass approximation ($m_2 \gg m_1$) to the case of arbitrary masses, the results for $\vec{\Omega}^{(1)}_{dS}$ and $\vec{\Omega}^{(1)}_{dS\,av}$ are obtained by the replacement $m_2 \to m_2 + \mu/3$ (whereas, as we shall see in the next section, the corresponding replacement for $\vec{\Omega}^{*(E)}$ is $m_2 \to m_2 + m_1$).

V. PRECESSION OF THE ORBIT

Using the Lagrangian of Eq. (38), we find that the equations of motion are

$$\dot{\vec{v}} + GM\vec{r}/r^3 = \vec{B}(\alpha), \quad (49)$$

where

$$\vec{B}(\alpha) = \vec{B}^{(E)}(\alpha) + \vec{B}^{(1)} + \vec{B}^{(2)} + \vec{B}^{(1,2)} + \vec{B}^{(Q1)} + \vec{B}^{(Q2)}, \quad (50)$$

and

$$\vec{B}^{(E)}(\alpha=-\tfrac{1}{2}\mu/M)=\frac{GM}{c^2r^3}\left[4\left(1+\frac{3}{4}\frac{\mu}{M}\right)\frac{GM\vec{r}}{r}-\left(1+\frac{5}{2}\frac{\mu}{M}\right)v^2\vec{r}+4\left(1-\frac{1}{4}\frac{\mu}{M}\right)(\vec{v}\cdot\vec{r})\vec{v}\right], \tag{51}$$

$$\vec{B}^{(1)}=\frac{G}{c^2r^5}\left(4+\frac{3m_2}{m_1}\right)\{\tfrac{3}{2}[\vec{S}^{(1)}\cdot(\vec{r}\times\vec{v})]\vec{r}+\vec{r}^2\vec{S}^{(1)}\times\vec{v}-\tfrac{3}{2}(\vec{v}\cdot\vec{r})\vec{S}^{(1)}\times\vec{r}\}, \tag{52}$$

$$\vec{B}^{(2)}=\frac{G}{c^2r^5}\left(4+\frac{3m_1}{m_2}\right)\{\tfrac{3}{2}[\vec{S}^{(2)}\cdot(\vec{r}\times\vec{v})]\vec{r}+\vec{r}^2\vec{S}^{(2)}\times\vec{v}-\tfrac{3}{2}(\vec{v}\cdot\vec{r})\vec{S}^{(2)}\times\vec{r}\}, \tag{53}$$

$$\vec{B}^{(1,2)}=\frac{-3G}{c^2r^5\mu}[(\vec{S}^{(2)}\cdot\vec{r})\vec{S}^{(1)}+(\vec{S}^{(1)}\cdot\vec{r})\vec{S}^{(2)}-5(\vec{S}^{(1)}\cdot\vec{r})(\vec{S}^{(2)}\cdot\vec{r})\vec{r}/r^2+(\vec{S}^{(1)}\cdot\vec{S}^{(2)})\vec{r}], \tag{54}$$

$$\vec{B}^{(Q1)}=\frac{-3GJ_2^{(1)}M}{2r^5}\{[1-5(\vec{n}^{(1)}\cdot\vec{r})^2/r^2]\vec{r}+2(\vec{n}^{(1)}\cdot\vec{r})\vec{n}^{(1)}\}, \tag{55}$$

$$\vec{B}^{(Q2)}=\frac{-3GJ_2^{(2)}M}{2r^5}\{[1-5(\vec{n}^{(2)}\cdot\vec{r})^2/r^2]\vec{r}+2(\vec{n}^{(2)}\cdot\vec{r})\vec{n}^{(2)}\}. \tag{56}$$

The term $\vec{B}^{(E)}(\alpha)$ is determined by $\mathcal{H}(\alpha)$ of Eq. (1), while $\vec{B}^{(1)}$, $\vec{B}^{(2)}$, $\vec{B}^{(1,2)}$, $\vec{B}^{(Q1)}$, and $\vec{B}^{(Q2)}$ are determined by V_{S1}, V_{S2}, $V_{S1,S2}$, V_{Q1}, and V_{Q2} of Eq. (29), respectively. Equations (52)–(56) can easily be inferred from the results of Sec. V of Ref. 7, but Eq. (51) cannot and has to be worked out. In Eq. (51) we used the particular value of $\alpha=-\frac{1}{2}\mu/M$ so that this equation would have a very simple form and make it easier to derive $\vec{\Omega}^{*(E)}$ of Eq. (67).

For a Newtonian elliptic orbit for two spherically symmetric bodies, the energy E, the orbital angular momentum $\vec{L}$, and the Runge-Lenz vector $\vec{A}$ are constants of the motion, which can be written as

$$E/\mu=\tfrac{1}{2}v^2-GM/r, \tag{57}$$

$$\vec{L}/\mu=\vec{r}\times\vec{v}, \tag{58}$$

$$\vec{A}/\mu=\vec{v}\times(\vec{r}\times\vec{v})-GM\vec{r}/r. \tag{59}$$

Taking the time derivative of Eqs. (57)–(59) and using Eq. (49) we obtain

$$\dot{E}/\mu=\vec{v}\cdot\vec{B}(\alpha), \tag{60}$$

$$\dot{\vec{L}}/\mu=\vec{r}\times\vec{B}(\alpha), \tag{61}$$

$$\dot{\vec{A}}/\mu=\vec{v}\times[\vec{r}\times\vec{B}(\alpha)]+\vec{B}(\alpha)\times(\vec{r}\times\vec{v}). \tag{62}$$

The secular results for the precession of the orbit are

$$\dot{E}_{\text{av}}=0 \tag{63}$$

$$\dot{\vec{L}}_{\text{av}}=\vec{\Omega}^*\times\vec{L}, \tag{64}$$

$$\dot{\vec{A}}_{\text{av}}=\vec{\Omega}^*\times\vec{A}, \tag{65}$$

where

$$\vec{\Omega}^*=\vec{\Omega}^{*(E)}+\vec{\Omega}^{*(1)}+\vec{\Omega}^{*(2)}+\vec{\Omega}^{*(1,2)}+\vec{\Omega}^{*(Q1)}+\vec{\Omega}^{*(Q2)} \tag{66}$$

and

$$\vec{\Omega}^{*(E)}=\frac{3G\bar{\omega}M}{c^2a(1-e^2)}\vec{n}, \tag{67}$$

$$\vec{\Omega}^{*(1)}=\frac{GS^{(1)}(4+3m_2/m_1)}{2c^2a^3(1-e^2)^{3/2}}[\vec{n}^{(1)}-3(\vec{n}\cdot\vec{n}^{(1)})\vec{n}], \tag{68}$$

$$\vec{\Omega}^{*(2)}=\frac{GS^{(2)}(4+3m_1/m_2)}{2c^2a^3(1-e^2)^{3/2}}[\vec{n}^{(2)}-3(\vec{n}\cdot\vec{n}^{(2)})\vec{n}], \tag{69}$$

$$\vec{\Omega}^{*(1,2)}=\frac{-3GS^{(1)}S^{(2)}/\mu\bar{\omega}}{2c^2a^5(1-e^2)^2}\times\{(\vec{n}\cdot\vec{n}^{(1)})\vec{n}^{(2)}+(\vec{n}\cdot\vec{n}^{(2)})\vec{n}^{(1)}+[\vec{n}^{(1)}\cdot\vec{n}^{(2)}-5(\vec{n}\cdot\vec{n}^{(1)})(\vec{n}\cdot\vec{n}^{(2)})]\vec{n}\}, \tag{70}$$

$$\vec{\Omega}^{*(Q1)}=\frac{-3GMJ_2^{(1)}/\bar{\omega}}{4a^5(1-e^2)^2}\times\{2(\vec{n}\cdot\vec{n}^{(1)})\vec{n}^{(1)}+[1-5(\vec{n}\cdot\vec{n}^{(1)})^2]\vec{n}\}, \tag{71}$$

$$\vec{\Omega}^{*(Q2)}=\frac{-3GMJ_2^{(2)}/\bar{\omega}}{4a^5(1-e^2)^2}\times\{2(\vec{n}\cdot\vec{n}^{(2)})\vec{n}^{(2)}+[1-5(\vec{n}\cdot\vec{n}^{(2)})^2]\vec{n}\}. \tag{72}$$

The terms $\vec{\Omega}^{*(E)}$, $\vec{\Omega}^{*(1)}$, $\vec{\Omega}^{*(2)}$, $\vec{\Omega}^{*(1,2)}$, $\vec{\Omega}^{*(Q1)}$, and $\vec{\Omega}^{*(Q2)}$ are the results corresponding to $\vec{B}^{(E)}(\alpha)$, $\vec{B}^{(1)}$, $\vec{B}^{(2)}$, $\vec{B}^{(1,2)}$, $\vec{B}^{(Q1)}$, and $\vec{B}^{(Q2)}$, respectively. Equations (68)–(72) can easily be inferred from the results of Sec. V of Ref. 7, but Eq. (67) cannot and has to be worked out.

The Einstein term $\vec{\Omega}^{*(E)}$ is independent of the value of α used in $\vec{B}^{(E)}(\alpha)$ and was first given by Robertson.[14]

The terms $\vec{\Omega}^{*(1)}$ and $\vec{\Omega}^{*(2)}$ are new results. The large-mass approximation ($m_2\gg m_1$) of $\vec{\Omega}^{*(2)}$ was first given by Lense and Thirring.[15,16]

The large-mass approximations of $\vec{\Omega}^{*(E)}$, $\vec{\Omega}^{*(1)}$, $\vec{\Omega}^{*(2)}$, $\vec{\Omega}^{*(1,2)}$, and $\vec{\Omega}^{*(Q2)}$ were given in Eqs. (76a)–(76e) of Ref. 7, while the large-mass approximation of $\vec{\Omega}^{*(Q1)}$ was given in Eq. (49) of Ref. 13.

The form of $\vec{B}^{(1)}$ and $\vec{B}^{(2)}$ (though not $\vec{B}^{(1,2)}$) depends on which spin supplementary condition is used. Our results, which are derived from the quantum theory of gravitation, correspond to a specific spin supplementary condition. Since the spin supplementary conditions only determine the location of the center of mass of each of the bodies, the *observable* quantities $\vec{\Omega}^{*(1)}$ and $\vec{\Omega}^{*(2)}$ are independent of which spin supplementary condition is used.[17] For the large-mass approximation ($m_2 \gg m_1$) the explicit forms $\vec{B}^{(1)}_{(CP)}$ and $\vec{B}^{(1)}_{(P)}$ of $\vec{B}^{(1)}$ have been given[17] for the Corinaldesi-Papapetrou and the Pirani spin supplementary conditions, respectively. It was further shown[17] (in the large-mass approximation) that the result from the quantum theory of gravitation for $\vec{B}^{(1)}$ is $\frac{1}{2}(\vec{B}^{(1)}_{(CP)} + \vec{B}^{(1)}_{(P)})$. This corresponds to the supplementary condition of Pryce[18] and Newton and Wigner,[19] which has the advantage that, in the transition to quantum-mechanical operator language, the operators corresponding to the different components of position commute with each other.[18,19,20]

We note that if $\vec{\Omega}^{(2)}_{\text{dS av}}$ represents the de Sitter term for body 2, which can be obtained from Eq. (45) by interchanging m_1 and m_2, we obtain the result

$$\Omega^{*(E)} = \tfrac{6}{7}(\Omega^{(1)}_{\text{dS av}} + \Omega^{(2)}_{\text{dS av}}) + \tfrac{2}{7}[9(\Omega^{(1)}_{\text{dS av}} + \Omega^{(2)}_{\text{dS av}})^2 + 7(\Omega^{(1)}_{\text{dS av}} - \Omega^{(2)}_{\text{dS av}})^2]^{1/2} . \tag{73}$$

In the equal-mass case we note that $\vec{\Omega}^{*(E)} = \frac{24}{7}\vec{\Omega}^{(1)}_{\text{dS av}}$, whereas in the large-mass approximation, as we have previously shown,[7] $\vec{\Omega}^{*(E)} = 2\vec{\Omega}^{(1)}_{\text{dS av}}$.

Letting $\vec{\Omega}^{(2)}_{\text{LT}}$ represent the Lense-Thirring term for body 2, which can be obtained from Eq. (46) by interchanging indices 1 and 2, we find that

$$\vec{\Omega}^{*(1)} = (4 + 3m_2/m_1)\vec{\Omega}^{(2)}_{\text{LT av}} , \tag{74}$$

$$\vec{\Omega}^{*(2)} = (4 + 3m_1/m_2)\vec{\Omega}^{(1)}_{\text{LT av}} . \tag{75}$$

In the equal-mass case we note that $\vec{\Omega}^{*(1)} = 7\vec{\Omega}^{(2)}_{\text{LT av}}$ and $\vec{\Omega}^{*(2)} = 7\vec{\Omega}^{(1)}_{\text{LT av}}$, whereas in the large-mass approximation, as we have previously shown,[7] $\vec{\Omega}^{*(2)} = 4\vec{\Omega}^{(1)}_{\text{LT av}}$.

We now define the total angular momentum

$$\vec{J} \equiv \vec{L} + \vec{S}^{(1)} + \vec{S}^{(2)} . \tag{76}$$

Then, using Eqs. (44)–(48), Eq. (64), and Eqs. (66)–(72), we find that

$$\dot{\vec{J}}_{\text{av}} = \vec{\Omega}^* \times \vec{L} + \vec{\Omega}^{(1)}_{\text{av}} \times \vec{S}^{(1)} + \vec{\Omega}^{(2)}_{\text{av}} \times \vec{S}^{(2)} \equiv 0 , \tag{77}$$

where $\vec{\Omega}^{(2)}_{\text{av}}$ is the precession of the spin of body 2, which can be obtained from the result for $\vec{\Omega}^{(1)}_{\text{av}}$ by interchanging the indices 1 and 2. In other words, the total angular momentum is conserved.

VI. CONCLUSIONS

We have shown that the spin-independent part of the one-graviton exchange interaction given by Barker, Gupta, and Haracz[8] is related to the EIH Hamiltonian by a coordinate transformation. We then found the precession of the spin and the precession of the orbit for the two-body problem in general relativity, with arbitrary masses, spin, and quadrupole moments. In the Hamiltonian of Eq. (29) the potential-energy terms $V_{S1,S2}$, V_{Q1}, and V_{Q2} have the same form as in their large-mass approximation, while the terms V_{S1} and V_{S2} do not. Thus the new results $\vec{\Omega}^{(1)}_{\text{dS}}$ and $\vec{\Omega}^{(1)}_{\text{dS av}}$ along with $\vec{\Omega}^{*(1)}$ and $\vec{\Omega}^{*(2)}$, which were derived from V_{S1} and V_{S2}, are of particular interest. *It will be interesting to see if these results can be derived from a purely classical treatment.*

We have now applied[21] our results to the case of the recently discovered[10] binary pulsar PSR1913+16.

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