Fixed wide-angle scattering of elementary fermions in gauge theories*

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We consider leading logarithmic corrections to the Born amplitude for wide-angle scattering of off-mass-shell quarks through sixth order in non-Abelian gauge theories. We review first the situation for the case of Abelian theories, where the leading logarithmic contributions can be summed to all orders, yielding a simple exponential form $e^{-K\ln^2 t}$. We find that the non-Abelian theory, through fourth order, yields a form corresponding to the perturbation expansion of an exponential. A partial computation of the sixth order suggests that such an expansion may still hold in that order.

I. INTRODUCTION

We examine the leading logarithmic contributions to the scattering of elementary fermions with finite $p^2 \neq m^2$ at very large s, t, and u, in gauge theories. We review first the situation for Abelian gauge theories (Sec. II). The results of this section are not new but we believe our approach can clarify a number of points. The same problem is then tackled in Sec. III for a model where the fermions belong to a multiplet representation of some non-Abelian gauge group (e.g., colored quarks) and interact via a Yang-Mills set of vector gluons. We find that, through fourth order, the leading logarithmic contributions give a result of the form

$$A_4 = \frac{g^2}{t}\{T_a\}\{T_a\} \times\left[1 - \frac{2C_N g^2}{8\pi^2}\ln^2\frac{t}{p^2} + \frac{1}{2!}\left(\frac{2C_N g^2}{8\pi^2}\ln^2\frac{t}{p^2}\right)^2\right], \quad (1.1)$$

where T_a is the group matrix on the fermion representation and $T_a T_a = C_N$. The terms in the curly brackets are to be understood as group operators in each of the two fermion lines. The calculational method used in this paper and the form-factor result obtained by that method have been discussed in a previous paper.[1] Some further calculational details are given in the two appendixes of this paper.

We note that our result is of interest for the consideration of the naive scaling laws in composite hadron-hadron scattering.[2] It has been remarked by a number of authors[3] that a rapid falloff of the quark-quark scattering of large s and t but finite p^2 would suffice to maintain the naive scaling laws. Further, it has been commented[4] that the falloff of this scattering could be as rapid as $e^{-\ln^2 t}$, as is true for the leading logarithmic terms in an Abelian theory. Our calculations are an encouraging, but nevertheless weak, sign that this could be the case. We should also stress that if this is true then the whole question of the nonleading terms becomes of crucial importance. At this moment, we have no insight to contribute on this matter.

Since the breakdown of the exponential series occurs first in sixth order it is by no means clear whether this contribution or nonleading logarithmic terms of lower order in g^2 are dominant. We have no insight to contribute on this matter.

II. WIDE-ANGLE SCATTERING OF ELEMENTARY FERMIONS IN ABELIAN GAUGE THEORIES

We begin by examining the Abelian case in some detail. The situation has been discussed correctly and at length by Halliday, Huskins, and Sachrajda,[5] which we shall refer to as HHS in subsequent discussion. We feel, however, that our analysis gives a clearer understanding of several points, and that calculational methods are more transparent and less cumbersome. It is therefore worth reviewing the situation before tackling the more complicated non-Abelian theories.

A. Identification of leading logarithmic contributions

Figure 1(a) defines the momentum labeling we will use throughout this paper. We are interested in the region

$$\begin{aligned} s &= (p_1 + p_2)^2 \gg p_i^2\,, \\ -t &= q^2 = -(p_1 - p_3)^2 \gg p_i^2\,, \\ -u &= -(p_1 - p_4)^2 \gg p_i^2\,, \end{aligned} \quad (2.1)$$

for all i; s/t and s/u fixed and finite. The diagrams of Fig. 1(b), 1(c), and 1(d) show the first-order corrections to the Born term of Fig. 1(a). Evaluation of the leading logarithmic contributions from these diagrams gives[5,6]

$$\frac{-g^2}{t}\,\frac{g^2}{8\pi^2}\,(2\ln^2 t+2\ln^2 u-2\ln^2 s)\{\gamma_\mu\}\{\gamma_\mu\}\ , \tag{2.2}$$

where the curly brackets represent the insertions in the two fermion lines. The successive terms come from Figs. 1(b), 1(c), and 1(d) in that order. The evaluation of Fig. 1(b) corresponds exactly to the form-factor case; that for Figs. 1(c) and 1(d) is discussed in detail in Appendix A, and also in HHS. The significant feature is that one can readily identify the regions of momentum space which can contribute leading logarithms. One obtains two powers of logarithms from a loop integration only when the integration becomes infrared singular if either of two external momenta is set to zero. Because of the linear nature of fermion propagators the loop integrations of Figs. 1(c) and 1(d) only have infrared divergences (for zero p_i^2) if either one of the exchanged photons is soft, and the other carries all the large momentum transfer q as well as the loop momentum.[7] (See Fig. 2.) This feature can be clearly seen to persist to all orders. Leading logarithmic contributions in higher orders arise only if every loop integrations is infrared singular in such a way as to produce two powers of logarithms. If any two exchanged photons both carry a finite fraction of the large momentum q then successive integrations will eventually produce a loop integration involving two hard bosonlike propagators and two fermion lines—such an integration is not infrared singular and, therefore, does not give leading logarithmic contributions. This feature of the calculation was noticed by HHS but not explained.

One can also readily see that diagrams where a single photon leaves and returns to the same fermion line do not generate leading logarithms. Similarly, the self-energy corrections can be seen to be nonleading. There remains the question of

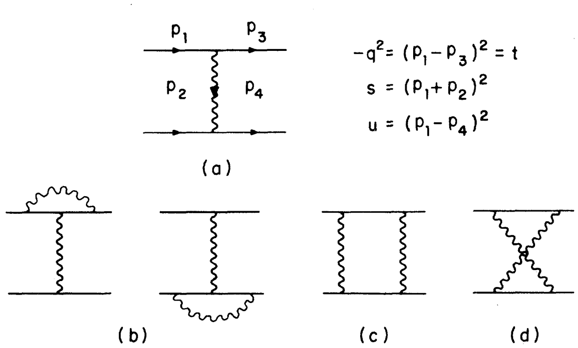

FIG. 1. (a) The Born term showing the momentum labeling used throughout this paper; (b), (c), and (d) first-order corrections to the Born term.

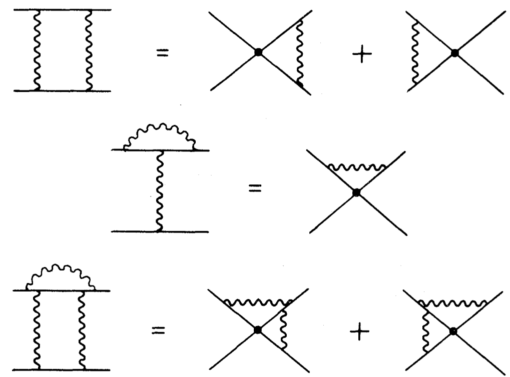

FIG. 2. Examples of reduced graphs determined by the single-hard-gluon approximation. The heavy dot indicates contraction of the hard gluon.

whether in higher orders diagrams involving light-by-light scattering might not also give leading logarithms. Examination of many examples leads to the prejudice that this is not the case—we are unaware of any definitive argument about such terms.[8]

B. Cancellation of pinch singularities

A further result noted but not fully explained by HHS is that pinch singularities originating from certain nonplanar diagrams cancel when a gauge-invariant set of such graphs is summed. These pinches occur in diagrams such as those of Fig. 3, and give contributions which are more infrared singular (by powers) than is indicated by naive power counting. The kinematical origin of these pinches has been examined by Coleman and Norton.[9] Using their analysis one can readily see that the singularity must vanish when a gauge-invariant set of diagrams is summed. The pinch in Fig. 3 arises when the momenta k_i are proportional to the momenta p_i; $k_i = x_i p_i$. In Fig. 4 the blobs represent the complete meson-fermion scattering amplitudes $T_{\alpha\delta}$ and $T_{\beta\delta}$. All diagrams included in Fig. 4 may have pinch singularities occurring for $k_i = x_i p_i$. However, in this region the fermion numerators, for example on the incoming fermion line labeled p_1, yield a factor

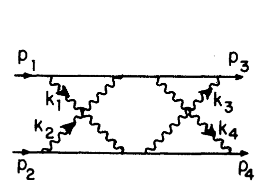

FIG. 3. Nonplanar diagram that contains pinch singularities.

$$\left(\frac{1-x_1}{x_1}\right)k_{1\alpha}+O(p_1{}^2)\ .$$

The same occurs for each external fermion. The pinch contributions from the gauge-invariant sum of diagrams represented by Fig. 3 thus vanish because $k_{3\delta}k_{1\alpha}T_{\alpha\delta}(k_1, k_3)=0$. The terms not eliminated by gauge invariance are suppressed by numerous powers of fermion momenta, and hence are irrelevant. This same argument can be applied to eliminate pinch singularities in any order of perturbation theory. (A very similar argument also works to explain the cancellation of the pinch singularities in scalar-meson electrodynamics.)

C. Exponentiation

The exponentiation to all orders of the Abelian leading logarithmic contributions can be shown by a generalization of the argument originally used in the form-factor case. This has already been discussed by Cardy.[6] It is amusing to see how simply this result is obtained by our method.[1] A general diagram with a single hard photon and N soft photons can be described by six parameters

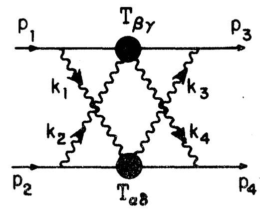

FIG. 4. Gauge-invariant sum of diagrams of the type of Fig. 3 which is free of pinch singularities. The blobs represent full meson-fermion scattering amplitudes.

$n_1, n_2, \ldots, n_6$, where (see Fig. 5 for an example)

n_1 = number of photons joining line 1 to line 2,

n_2 = number of photons joining line 3 to line 4,

n_3 = number of photons joining line 1 to line 3,

n_4 = number of photons joining line 2 to line 4,

n_5 = number of photons joining line 1 to line 4,

n_6 = number of photons joining line 2 to line 3.

Clearly, $n_1+n_2+n_3+n_4+n_5+n_6=N$. Straightforward loop-by-loop integration[1] gives a weight

$$\left(\frac{g^2}{8\pi^2}\right)^N \frac{[\ln^2(s/p^2)]^{n_1+n_2}[-\ln^2(t/p^2)]^{n_3+n_4}[-\ln^2(u/p^2)]^{n_5+n_6}}{(n_1+n_3+n_5)!\,(n_1+n_4+n_6)!\,(n_2+n_3+n_6)!\,(n_2+n_4+n_5)!} \tag{2.3}$$

for such a diagram. The numerator factors of fermion momenta give $p_1\cdot p_3$ or $p_2\cdot p_4$ for each t-channel line and similarly $p_1\cdot p_4$ or $p_2\cdot p_3$ for each u-channel line and $p_1\cdot p_2$ or $p_3\cdot p_4$ for each s-channel line. The denominators always combine to give factors $+s$, $+t$, and $+u$ which cancel these numerator factors (up to terms of order $p_i{}^2/t$) but leave minus signs for each t- or u-channel line. It is a simple counting problem to show that there are

$$\frac{(n_1+n_3+n_5)!\,(n_1+n_4+n_6)!\,(n_2+n_3+n_6)!\,(n_2+n_4+n_5)!}{n_1!\,n_2!\,n_3!\,n_4!\,n_5!\,n_6!} \tag{2.4}$$

distinct diagrams with $n_1, \ldots, n_6$ lines. Thus, the contribution of all diagrams with N soft photons is

$$\left(\frac{g^2}{8\pi^2}\right)^N \sum_{n_1n_2n_3n_4n_5n_6} \delta(N-n_1-n_2-n_3-n_4-n_5-n_6)\frac{[\ln^2(s/p^2)]^{n_1+n_2}[-\ln^2(t/p^2)]^{n_3+n_4}[-\ln^2(u/p^2)]^{n_5+n_6}}{n_1!\,n_2!\,n_3!\,n_4!\,n_5!\,n_6!}$$
$$=\frac{1}{N!}\left(\frac{g^2}{8\pi^2}\right)^N[2\ln^2(s/p^2)-2\ln^2(t/p^2)-2\ln^2(u/p^2)]^N\ . \tag{2.5}$$

The simplicity of the combinatorics here is in marked contrast to the situation in the non-Abelian case, to which we now turn.

III. NON-ABELIAN THEORIES

The technique we have developed allows us to tackle this more complicated situation graph-by-graph, but not to make a simple sum to all orders. We find the situation is very similar to that for the form factor.[1] Up to terms of order g^6 (fourth-order corrections to the Born term) a number of cancellations occur and one obtains a simple re-

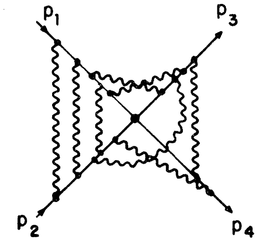

FIG. 5. Example of a general Abelian graph with $n_1=3$, $n_2=1$, $n_3=2$, $n_4=1$, $n_5=0$, $n_6=1$, and $N=8$.

sult which looks like the beginning of an exponential series. We have examined all graphs that could contribute leading logarithms to the scattering in this order. We find that there is a possibility that a continuation of the simple exponential series is given by this sum of graphs. We will return to amplify this comment after we have discussed the fourth-order corrections in some detail.

A. Corrections to the Born term through fourth order

In a non-Abelian gauge theory for the scattering

$$q_1 q_2 \to q_3 q_4$$

there are in general two possible Born terms, depending on the quark types. These correspond to two of the three distinct possibilities for the continuity of the quark line shown in Fig. 6. The diagrams formed by adding higher-order corrections to any one of these terms are distinct. We will present our discussion only in terms of corrections to the diagram of Fig. 6(a), but clearly, it applies equally to sequence built on either of the other possible Born diagrams with the obvious modifications of channel label. We write this Born term as $A_B = (g^2/t)\{T_a\}\ \{T_a\}$, where the terms in curly brackets represent the group operators to be inserted in the fermion lines.

The diagrams corresponding to lowest-order corrections to the Born term are the same as those of the Abelian theory, Figs. 1(b), 1(c), and 1(d). The new diagram in this order, Fig. 7, does not contribute leading logarithms as the loop integration is infrared finite in this case.[1] We find the leading contribution to be

$$\frac{-g^2}{8\pi^2}\left[\ln^2\left(\frac{t}{p^2}\right)(\{T_a T_b T_a\}\{T_b\}+\{T_b\}\{T_a T_b T_a\}) + 2\ln^2\left(\frac{u}{p^2}\right)\{T_a T_b\}\{T_b T_a\} - 2\ln^2\left(\frac{s}{p^2}\right)\{T_a T_b\}\{T_a T_b\}\right]\frac{g^2}{t} \quad (3.1)$$

Since $s \propto t \propto u$ this can be rewritten as

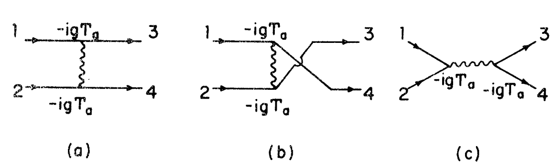

FIG. 6. Possible quark-quark scattering Born terms. The variables s and t are defined with the conventions of Fig. 1. The fermion group matrices T_a are shown explicitly.

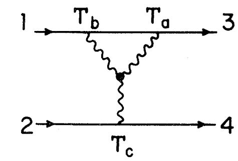

FIG. 7. Lowest-order non-Abelian correction. This diagram contributes only nonleading logarithms.

$$\frac{-g^2}{8\pi^2}\ln^2\left(\frac{t}{p^2}\right)[2(C_N - C_A)\{T_b\}\{T_b\} - 2\{T_a T_b\}\{T_a T_b\}]\frac{g^2}{t}. \quad (3.2)$$

The first term in (3.2) is obtained from the t-channel term of (3.1) using the definitions of the Casimir operators[1]

$$T_a T_a = C_N\,, \qquad if_{abc} T_b T_c = -C_A T_a\,. \quad (3.3)$$

This simply corresponds to lowest order form-factor corrections at the hard-meson vertex.

The second term, which is the sum of the s- and u-channel pieces, can also be rewritten using (3.3)

$$\{T_a T_b\}\{T_c\} if_{abc} = -C_A\{T_c\}\{T_c\}\,. \quad (3.4)$$

Thus, the entire second-order correction becomes

$$\frac{-2g^2}{8\pi^2} C_N\left(\ln^2\frac{t}{p^2}\right)A_B\,. \quad (3.5)$$

The fourth-order corrections correspond to the diagrams of Figs. 8 through 11. The contributions of Fig. 8(a) correspond to second-order form-factor corrections at either of the hard-gluon vertices and give a contribution[1]

$$A_{8(a)} = \frac{2}{2!}\left(\frac{-g^2}{8\pi^2}(C_N - C_A)\ln^2\frac{t}{p^2}\right)^2 A_B\,. \quad (3.6a)$$

The contributions of Fig. 8(b) correspond to first-order form-factor corrections at both hard-gluon vertices and yield an additional

$$A_{8(b)} = \left(\frac{-g^2}{8\pi^2}(C_N - C_A)\ln^2\frac{t}{p^2}\right)^2 A_B\,. \quad (3.6b)$$

The diagrams of Fig. 9 are identical to Abelian theory diagrams up to group-theoretic weights. Letting each photon in turn be hard, and evaluating the resultant diagrams we find

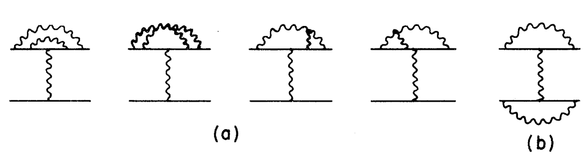

FIG. 8. Two "t-loop" corrections to the Born term.

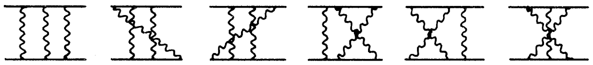

FIG. 9. Graphs that generate $2s$, $2u$, and $1s$-$1u$ Abelian corrections to the Born term.

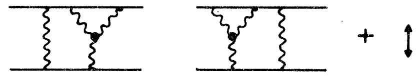

FIG. 10. $2s$ non-Abelian corrections.

$$A_9=\left[\frac{+g^2}{8\pi^2}\ln^2\left(\frac{t}{p^2}\right)\right]^2\left[(1+\tfrac{2}{4})\{T_aT_bT_c\}\{T_aT_bT_c\}+(\tfrac{1}{4}-\tfrac{2}{2})\{T_aT_bT_c\}\{T_aT_cT_b\}\right.$$
$$+(-\tfrac{2}{2}+\tfrac{1}{4})\{T_aT_bT_c\}\{T_bT_aT_c\}+(\tfrac{1}{4}-\tfrac{2}{2})\{T_aT_bT_c\}\{T_bT_cT_a\}$$
$$\left.+(\tfrac{1}{4}-\tfrac{2}{2})\{T_aT_bT_c\}\{T_cT_bT_a\}+(1+\tfrac{2}{4})\{T_aT_bT_c\}\{T_cT_bT_a\}\right]\frac{g^2}{t}$$
$$=3\,\frac{C_A{}^2}{2!}\left[\frac{g^2}{8\pi^2}\ln^2\left(\frac{t}{p^2}\right)\right]^2 A_B\ . \tag{3.7}$$

The diagrams of Fig. 10 can readily be seen to contribute leading logarithms only when the large momentum transfer is carried across the diagram by the gluon with no Yang-Mills vertex correction. This is the extension to higher orders of the observation that Fig. 7 is nonleading and shows that again in the non-Abelian case the single-hard-photon approximation correctly gives all leading logarithms. The contributions of Fig. 10 are then

$$\left(\frac{g^2}{8\pi^2}\ln^2(t/p^2)\right)^2\frac{4}{8}\,C_A{}^2A_B\ . \tag{3.8}$$

The contributions of Fig. 11(a) give leading logarithms when either gluon is hard and yield

$$A_{11(a)}=\left(\frac{g^2}{8\pi^2}\ln^2(t/p^2)\right)^2 2(C_N-C_A)\,C_AA_B\ . \tag{3.9}$$

For Fig. 11(b) only the gluon which has the vertex correction can be hard; the other choice gives nonleading logarithms (just as in the Abelian case). The net contribution of these terms is thus identical to that of Fig. 11(a). The diagrams of Fig. 11(c) are examples of those which give nonleading contributions.

The diagrams of Fig. 12 constitute the only new feature of this calculation (everything else can be evaluated by inspection with the methods developed for our form-factor calculation). We find there is no surviving leading logarithmic contribution from these diagrams. Let us examine Fig. 12(a) using the momentum routings shown:

$$A_{12(a)}=\frac{g^6}{(2\pi)^8t}I\{T_aT_bT_c\}\{T_bT_d\}\,if_{acd}\ , \tag{3.10}$$

where

$$I=\int d^4k\int d^4r\,\frac{(2k\cdot p_1p_3\cdot p_4-k\cdot p_4p_1\cdot p_3-k\cdot p_3p_1\cdot p_4)+(2r\cdot p_4p_1\cdot p_3-r\cdot p_1p_3\cdot p_4-r\cdot p_3p_1\cdot p_4)}{k^2r^2(k-r)^2(p_1+r)^2(p_4+k)^2(p_3-k+r)^2}\ . \tag{3.11}$$

We notice that the substitution $k\leftrightarrow r$, $p_1\leftrightarrow p_4$, $p_3\leftrightarrow -p_3$ leaves the denominator unchanged but converts the second term in the numerator into the negative of the first. Thus, we can write

$$I=F(s,t,u)-F(-t,-s,u)\ , \tag{3.12}$$

where the quantity $F(s,t,u)$ is obtained by evaluating the first bracket in the numerator of (3.11). Using our usual techniques[1] we find

$$F(s,t,u)=K\ln^4(t/p^2)+\text{nonleading logarithms}, \tag{3.13}$$

where K is a constant independent of s, t, u. Hence, the contribution of Fig. 12(a) vanishes. The argument for Fig. 12(b) is identical.

Collecting all fourth-order corrections we find

$$A_8=\frac{1}{2!}\,4(C_N-C_A)^2\left(\frac{-g^2}{8\pi^2}\ln^2(t/p^2)\right)^2A_B\ ,$$
$$A_9+A_{10}=\frac{1}{2!}\,4C_A{}^2\left(\frac{-g^2}{8\pi^2}\ln^2(t/p^2)\right)^2A_B\ , \tag{3.14}$$
$$A_{11}=\frac{1}{2!}\,8C_A(C_N-C_A)\left(\frac{-g^2}{8\pi^2}\ln^2(t/p^2)\right)^2A_B\ ,$$
$$A_{12}=0\ ,$$

which gives a resulting fourth-order correction

$$\frac{1}{2!}\left(-2C_N\frac{g^2}{8\pi^2}\ln^2(t/p^2)\right)^2 A_B \ . \qquad (3.15)$$

B. Comments on sixth-order corrections to the Born term

We notice that the simple form of (3.15) is due to two important features:

(i) The fourth-order form-factor result gives a simple form for the t-channel corrections of Fig. 11;

(ii) The vanishing of the contributions of Fig. 12 means that all the contributing terms can be viewed as products of s-, t-, and u-channel vertex corrections to the effective four-fermion vertex produced by extracting the single-hard-photon line.

The first of these remarks is also true in sixth order, but the second no longer applies. Identifying a single hard gluon line labels each segment of fermion line as initial state if it occurs before the hard-gluon vertex or final state if it occurs after that vertex. In sixth-order corrections there are leading logarithmic terms arising from diagrams where more than two fermion lines are connected via tri-gluon interactions. This destroys the simple pattern denoted (ii) above. We have not calculated these diagrams, but we have examined their group structure. Combining these terms with others we obtain the result

$$A_6 = -\frac{1}{3!}\left(\frac{2g^2}{8\pi^2}\ln^2(t/p^2)\right)^3 C_N{}^3 + \cdots \ , \qquad (3.16)$$

where the $+\cdots$ represents only terms proportional to $C_A{}^3$ and higher-order Casimir operators. The details of this discussion are given in Appendix B. It is quite plausible that the term denoted $+\cdots$ in (3.16) is in fact zero. We have not, at this stage, carried out the calculation. (Using the method given in Ref. 1 this would be a "straightforward but tedious" exercise.) An amusing exercise which we found helpful in reaching this conclusion is to consider the rather unphysical problem of the strong (non-Abelian) corrections to electromagnetic (Abelian) quark-quark scattering. Dynamically the problem is identical to the one discussed here (only as regards leading logarithms, of course). The only difference is that the single hard vector meson couples to the fermion line with a group matrix $\underline{1}$ in place of T_a. This simplifies the group structure of each diagram sufficiently that one can more readily examine the question of possible cancellations. Through fourth order the net result is the same in this case as for the fully non-Abelian case. This is no longer true in sixth order, but the exercise is still instructive in that case.

IV. SUMMARY

We have reviewed the leading logarithmic calculation of wide-angle scattering in Abelian gauge theories to all orders. We have then carried out a calculation of the fourth-order corrections to the Born term in non-Abelian gauge theories. We have checked that certain cancellations occur in sixth-order corrections, but we have not performed a complete calculation. However, one is left with the strong possibility that the leading logarithms sum to give a severely damped contribution both in Abelian and non-Abelian theories. A number of questions remain that must be examined before one draws physical conclusions from these results. Principal among these are the behavior of nonleading logarithms and the question of whether these leading logarithms are

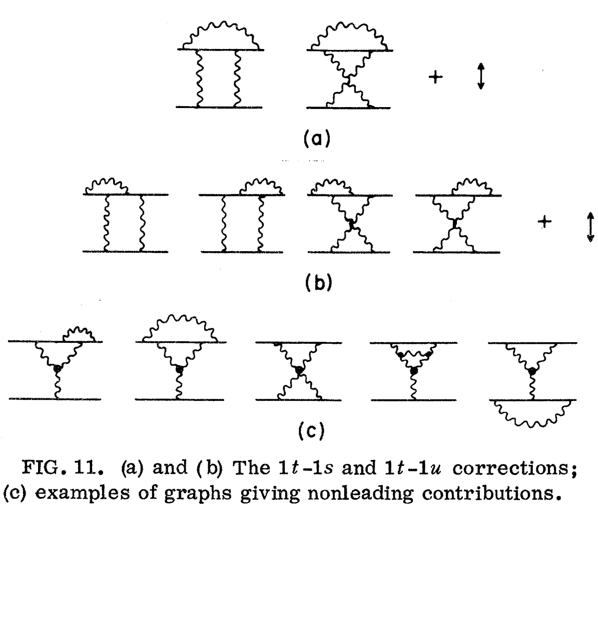

FIG. 11. (a) and (b) The $1t$-$1s$ and $1t$-$1u$ corrections; (c) examples of graphs giving nonleading contributions.

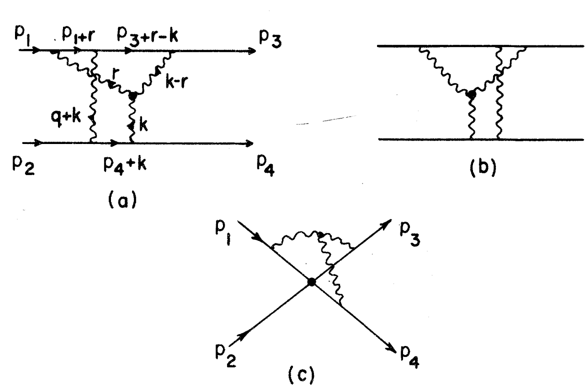

FIG. 12. (a) and (b) Two loop corrections that cannot be classified as belonging to any of the types described in Figs. 8–11; (c) the hard-gluon-reduced form of Fig. 12(a). We call it a "pyramid graph."

removed by inclusion of real emission of massless vectors as they are in the Abelian case. Cornwall and Tiktopoulos[1] have given speculative answers to both these questions which lead to a number of very strong conclusions. However, we feel that more systematic calculations (or rigorous general arguments) are needed before such conclusions can be justified. Hence we present this work as simply a calculation of leading logarithms, which suggests that an interesting feature of Abelian gauge theories may be shared by the non-Abelian theories.

APPENDIX A: EVALUATION OF FIG. 1(c)

We consider first Fig. 1(b). We will examine this first by standard Feynman parameter techniques and then briefly discuss the application of our simplified techniques. The amplitude for Fig. 1(b) is

$$A_1 = \frac{g^4}{(2\pi)^4}\int d^4k\, \frac{\{\gamma_\alpha(\not{p}_3-\not{k})\gamma_\beta\}\{\gamma^\alpha(\not{p}_4+\not{k})\gamma^\beta\}}{k^2(k+p_1-p_3)^2(k+p_4)^2(p_3-k)^2}\,. \tag{A1}$$

The numerator factors in (A1) are enclosed between projection operators

$$\overline{P}(p_3)\{\cdots\}P(p_1)\,\overline{P}(p_4)\{\cdots\}P(p_2)\,,$$

where

$$\not{p}_i P(p_i) = \overline{P}(p_i)\,p_i = O(p_i^2)\,.$$

Since we are looking only for leading terms we will drop any numerator term that is manifestly of order p_i^2. Using the Feynman parameter technique on (A1) and setting all p_i^2 equal for simplicity gives

$$A_1 = \frac{g^4 3!}{(2\pi)^4}\int d^4k'\,d\alpha_1\cdots d\alpha_4\,\frac{\delta(1-\sum_i\alpha_i)\{\gamma_\alpha N_1\gamma_\beta\}\{\gamma^\alpha N_2\gamma^\beta\}}{\{k'^2-[\alpha_1\alpha_2 t+\alpha_3\alpha_4 s+(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)p^2]\}^4}\,, \tag{A2a}$$

where

$$N_1 = \{\not{p}_3(1-\alpha_2-\alpha_3)+\alpha_4\not{p}_4+\alpha_2\not{p}_1-\not{k}'\} \tag{A2b}$$

and

$$N_2 = \{\not{p}_4(1-\alpha_4-\alpha_2)+\alpha_2\not{p}_2+\alpha_3\not{p}_3+\not{k}'\}\,. \tag{A2c}$$

The numerator terms linear in k' vanish by symmetry and the term quadratic in k' gives no leading logarithms (it is only an infrared-finite part of the integral). Thus, we find

$$A_1 = \frac{g^4}{(2\pi)^4}\left(\frac{\pi^2}{i}\right)\int d\alpha_1\cdots d\alpha_4\,\frac{\delta(1-\sum\alpha_i)\{\gamma_\alpha\overline{N}_1\gamma_\beta\}\{\gamma^\alpha\overline{N}_2\gamma^\beta\}}{\{\alpha_1\alpha_2 t+\alpha_3\alpha_4 s+(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)p^2\}^2}\,, \tag{A3}$$

where $\overline{N}_1$ and $\overline{N}_2$ are equal to N_1 and N_2 defined in (A2b) and (A2c), respectively, with k' set to zero. Naively examining the denominator would suggest that one should get terms of order $(1/st)\ln^2 t$ from each of the regions $\alpha_1\to 1$, $\alpha_2, \alpha_3, \alpha_4\to 0$; $\alpha_2\to 1$, $\alpha_1, \alpha_3, \alpha_4\to 0$; $\alpha_3\to 1, \alpha_1, \alpha_2, \alpha_4\to 0$, and $\alpha_4\to 1$, $\alpha_1, \alpha_2, \alpha_3\to 0$. However, one readily sees that in the last two cases there is a numerator factor $1-\alpha_3$ (or $1-\alpha_4$, respectively) which will remove one power of a logarithm. Thus, the leading logarithmic contributions come only from $\alpha_1\to 1$, that is, the region $k\to 0$, or from $\alpha_2\to 1$, that is, $k\to -q$. One readily sees that each of these regions corresponds to one hard-gluon and one soft-gluon exchange. Furthermore, the numerator in (A3) becomes

$$\{\gamma_\alpha\not{p}_3\gamma_\beta\}\{\gamma^\alpha\not{p}_4\gamma^\beta\}=s\{\gamma_\alpha\}\{\gamma^\alpha\}\text{ for }\alpha_1\to 1,\ \alpha_2, \alpha_3, \alpha_4\to 0$$

and

$$\{\gamma_\alpha\not{p}_1\gamma_\beta\}\{\gamma^\alpha\not{p}_2\gamma^\beta\}=s\{\gamma_\alpha\}\{\gamma^\alpha\}\text{ for }\alpha_2\to 1,\ \alpha_1, \alpha_3, \alpha_4\to 0\,.$$

This shows that the leading logarithmic terms from the evaluation of (A3) all have a simple numerator structure identical to that of the Born term. This result does not agree with that of Fishbane and Simmons[10] who obtain a much more complicated form. Correcting a small algebraic error in their Eq. (2.16) we find that their result [their Eq. (2.18)] is identically zero. (The contributions which they isolate do correspond to the single-hard-gluon region, though they do not seem to realize this fact.) One obtains the result of HHS by straightforward integration of (A3).

We can readily obtain the same result by the method of Ref. 1 where we seek to isolate the regions of momentum space which give leading logarithms by using the Feynman parameter technique on only three denominators at a time. We find

$$A=\frac{g^4}{(2\pi)^4}\int d^4k'\int d\alpha_1\cdots d\alpha_4 3!\,\delta(1-\textstyle\sum\alpha_i)\Bigg\{\frac{N(k'\to q)}{(k'-\alpha_1 q-\alpha_2 p_4+\alpha_3 p_3)^2[k'^2+\alpha_1(\alpha_2+\alpha_3)p^2+\alpha_2\alpha_3 s]^3}$$
$$+\frac{N(k'\to 0)}{(k'+q-\alpha_2 p_4+\alpha_3 p_3)^2[k'^2+\alpha_2\alpha_3 s+\alpha_1(\alpha_2+\alpha_3)p^2]^3}$$
$$+\frac{N(k'\to -p_4)\to 0}{(p_3-k'+\alpha_2 q+\alpha_3 p_4)^2[k'^2+\alpha_1\alpha_2 t+\alpha_3(\alpha_1+\alpha_2)p^2]^3}$$
$$+\frac{N(k'\to p_3)\to 0}{(p_4+k'-\alpha_2 q+\alpha_3 p_3)^2[k'^2+\alpha_1\alpha_2 t+\alpha_3(\alpha_1+\alpha_2)p^2]^3}\Bigg\}. \tag{A4}$$

The four terms correspond to isolating the denominator k'^2, $(k'+q)^2$, $(p_3-k')^2$, $(p_4+k')^2$ in that order. Each cubic denominator gives double logarithms only for one possible choice of parameters. The notation $N(k'\to x)$ means the numerator in (A1) is evaluated with the substitution $k'=x$, since that corresponds to the singular region of the cubic denominator in that term. The first two terms then are clearly one-hard-gluon (and one-soft-gluon) contributions. The second two terms give no leading logarithms as the numerator vanishes in the singular region of the cubic denominator.

The analysis of Fig. 1(d) proceeds in exactly the same fashion and one can once again convince oneself of the validity of the single-hard-gluon approximation.

APPENDIX B: CALCULATION OF THE SIXTH-ORDER CORRECTIONS

1. Classification of the eighth-order graphs

In this appendix we present a calculation of the sixth-order corrections to the Born term, and show that the exponential series breaks down. We have found that to exhibit this it is not necessary to carry out an explicit computation of all the contributing graphs.

We sort all graphs into two classes. Shrinking the hard-gluon exchange to an effective four-fermion vertex we identify these as the following:

I. Two-by-two graphs, in which any complex of exchanged gluons connects only two fermion lines. A complex of gluons means gluons which are interconnected via tri-gluon interactions. This class includes graphs in which there are more than one such complex connecting different pairs of fermion lines. Some examples are shown in Fig. 13.

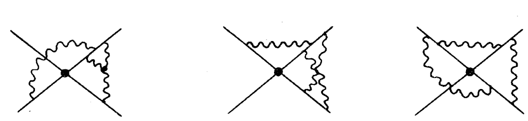

FIG. 13. Examples of "two-by-two" graphs.

II. Graphs in which more than two fermion lines are connected by a single complex of gluons. Figure 12 is an example of such a graph in sixth order which we refer to as a pyramid graph (a rather loosely chosen name). The eighth-order graphs shown in Fig. 14 are all formed by adding a single disconnected gluon to Fig. 12. Also in this class are the graphs of Fig. 15 which we refer to as nonplanar H graphs.

Our analysis proceeds by examining first the contributions of all graphs of class I in some detail. We find they give a contribution of the correct form for the continuation of the exponential series of (1.1) plus an additional piece 2Δ due to the nonexponentiation of the form-factor result, and some other pieces of distinct group structure. We then examine the graphs of class II and show that they also have a distinct group structure from the term Δ and hence, cannot in general cancel this piece.

2. Two-by-two graphs–class I

We find it convenient to subdivide the graphs of class I into four subclasses, which we identify

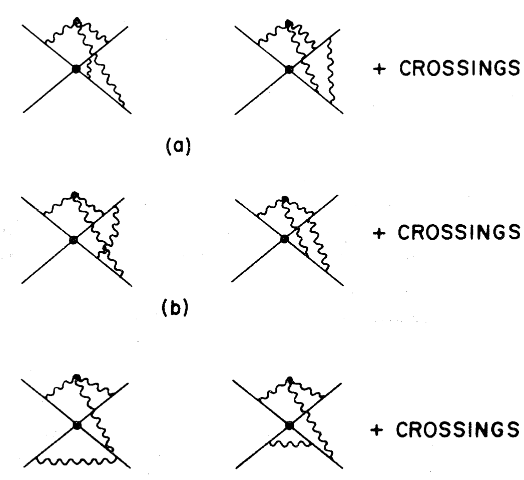

FIG. 14. Sixth-order "pyramid graphs."

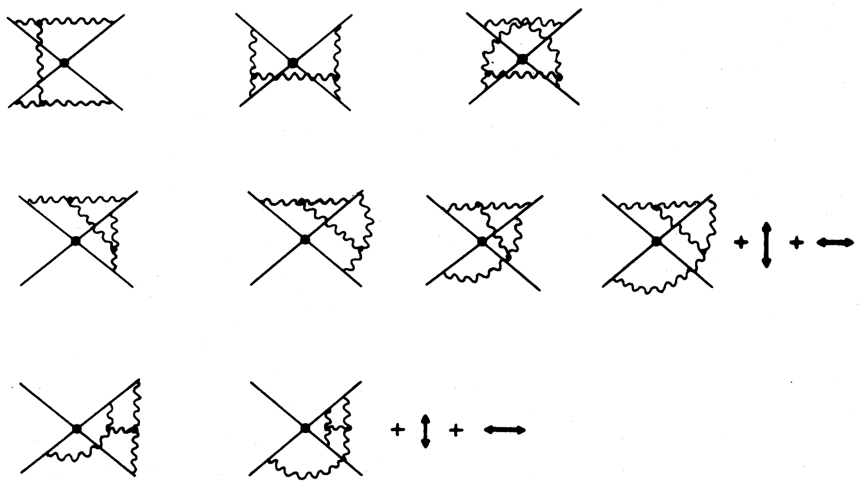

FIG. 15. Nonplanar H graphs.

as the following:

(i) graphs with three t-channel loops;

(ii) graphs with two t-channel loops;

(iii) graphs with one t-channel loop;

(iv) graphs with no t-channel loops.

We express all contributions in terms of a weight which multiplies the factor xA_B where

$$x=\frac{1}{3!}\left(\frac{g^2}{8\pi^2}\ln^2(t/p^2)\right)^3 . \tag{B1a}$$

For graphs of class I (i) we can use the form-factor results directly to obtain

$$w_{3t}=-8(C_N-C_A)^3 . \tag{B1b}$$

This contribution includes sixth-order form-factor corrections at either hard-gluon-fermion vertex and also fourth-order corrections of one vertex multiplied by a second-order correction of the other.

Turning now to graphs with two t-channel loops we find

$$w_{2t}=-3.8C_A(C_N-C_A)^2+\cdots , \tag{B2}$$

where the symbols $+\cdots$ indicate terms which do not involve the quantity C_N. These may include terms which cannot be reduced to the simple $\{T_a\}\{T_a\}$ form of the Born term, or they may be proportional to A_B but with a proportionality constant depending only on C_A and a higher-order Casimir operator which we write as H.[1] The graphs involved are those of Figs. 16 and 17 with all suitable choices of the labels, that is, A and D can be 1 and 3 or 2 and 4 (in either order). For each choice of A and D there are two possible choices for B and C. If we look at the diagrams of Fig. 16(a) we find that the first diagram is clearly proportional to $(C_N-C_A)^2\{T_aT_b\}\{T_aT_b\}$ (for the choice $A=P_1$, $D=P_3$, $B=P_2$, and $C=P_4$). The second diagram can be cast in this same form by commuting two of the group matrices corresponding to t-channel gluons. The additional piece coming from the commutator is canceled by the remaining two graphs. (This is, of course, just the second-order form-factor correction pattern.) The graphs of Fig. 16(b) add in a similar fashion, as do those of Fig. 16(c). Evaluating each set and adding we find for Fig. 16 the contributions, with x defined by (B1a),

$$\frac{24x}{(2!)}(C_N-C_A)^2\{T_aT_b\}\{T_aT_b\}\frac{g^2}{t}+\cdots , \tag{B3}$$

when the single gluon is in the s channel and

$$\frac{-24x}{2!}(C_N-C_A)^2\{T_aT_b\}\{T_bT_a\}\frac{g^2}{t}+\cdots , \tag{B4}$$

when the single gluon is in the u channel. The graphs of Fig. 17 give

$$\frac{8(3!)x}{(2!)^2}(C_N-C_A)^2\{T_aT_b\}\{T_aT_b\}\frac{g^2}{t}+\cdots \tag{B5}$$

for a single s-channel gluon and

$$\frac{-8(3!)x}{(2!)^2}(C_N-C_A)^2\{T_aT_b\}\{T_bT_a\}\frac{g^2}{t}+\cdots \tag{B6}$$

for a single u-channel gluon. Combining similar s- and u-channel terms yields a commutator, $\{T_aT_b\}\{[T_a,T_b]\}=-C_A\{T_a\}\{T_a\}$, and hence, the sum of (B3) through (B6) gives the result stated in (B2).

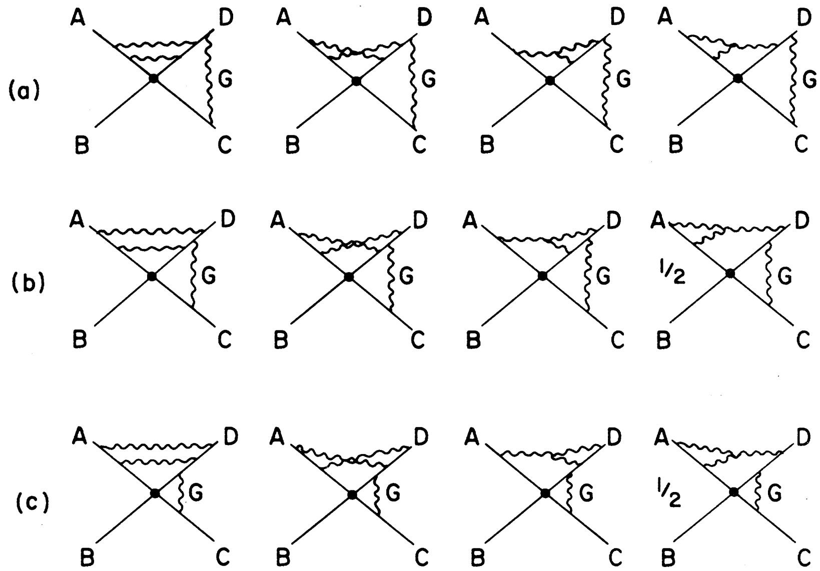

FIG. 16. Reduced graphs representing those three-loop corrections to the Born term which have been explicitly computed in Appendix B. Case where two loops are in one sector of one channel.

The graphs with one t-channel loop are again those of Fig. 16 and 17, but now with the choices A and D equal to 1 and 2, 3 and 4, 1 and 4, or 2 and 3. In addition, we must include the graphs of Fig. 18, where one gluon is exchanged in each of the three channels. It is again a matter of straightforward computation to obtain

$$w_{1t} = 3.8\, C_A^2 (C_N - C_A) + \cdots . \quad \text{(B7)}$$

The graphs of Fig. 16 give

$$\frac{+24x}{(2!)} (C_N - C_A)\{T_a T_b T_c\}(\{T_a T_b T_c\} + \{T_c T_b T_a\}) \frac{g^2}{t} + \cdots . \quad \text{(B8)}$$

The first term is the $1t$, $2s$ contribution while the second one is the $1t$, $2u$. Similarly, those of Fig. 17 contribute

$$\frac{+(3!)\,8x}{(2!)^2} (C_N - C_A)\{T_a T_b T_c\}(\{T_a T_b T_c\} + \{T_c T_b T_a\}) \frac{g^2}{t} + \cdots . \quad \text{(B9)}$$

The contribution from Figs. 18(a) and 18(b) are

$$- \frac{(3!)\,16x}{(2!)^3} (C_N - C_A)\{T_a T_b T_c\}(\{T_b T_a T_c\} + \{T_b T_c T_a\}) \frac{g^2}{t} + \cdots \quad \text{(B10)}$$

and

$$- 12x (C_N - C_A)\{T_a T_b T_c\}(\{T_b T_a T_c\} + \{T_b T_c T_a\}) \frac{g^2}{t} + \cdots , \quad \text{(B11)}$$

respectively. Adding (B8) and (B9) we get, after performing various commutations,

$$3!x(C_N - C_A)(8\{T_a T_b T_c\}\{T_a T_b T_c\} + 12 C_A \{T_a T_b\}\{T_a T_b\} + 8 C_A^2 \{T_a\}\{T_a\} + F) \frac{g^2}{t} , \quad \text{(B12)}$$

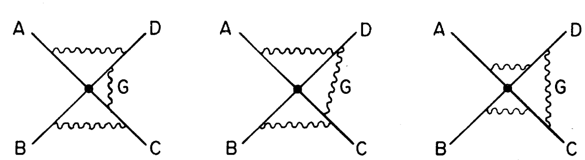

FIG. 17. Case where two loops are in the two different sectors of the same channel.

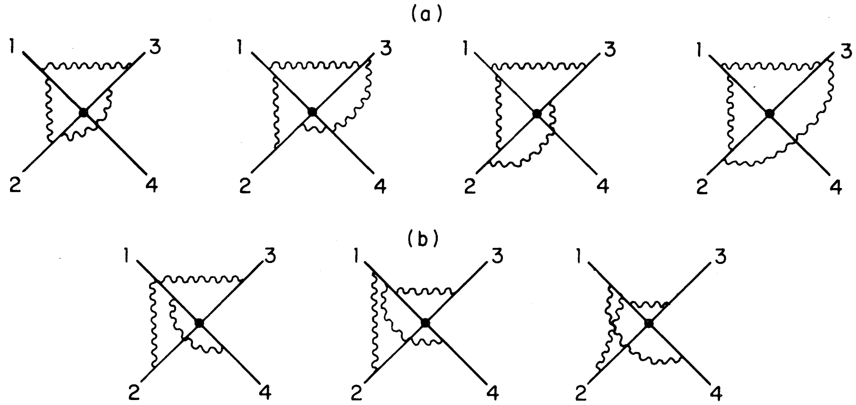

FIG. 18. Case where one loop is in every channel.

where

$$F \equiv 4\{T_p T_c\}\{T_q T_b\} if_{pab}\, if_{qca} . \tag{B13}$$

Adding (B10) and (B11) we obtain

$$+3!\, x(C_N - C_A)(8\{T_a T_b T_c\}\{T_a T_b T_c\} + 12C_A\{T_a T_b\}\{T_a T_b\} + 4C_A^2\{T_a\}\{T_a\} + F)\frac{g^2}{t}. \tag{B14}$$

Addition of (B12) and (B14) gives (B7).

The graphs with no t-channel loops can be evaluated similarly. One readily finds that the commutators which arise on adding s- and u-channel graphs yield only terms proportional to C_A or a higher-order Casimir operator times the Born amplitude, or terms such as the F of (B13) above.

3. Graphs of class II

The graphs of Figs. 14 and 15 remain to be discussed. The graphs of Figs. 14(a) and 14(b) do not give leading logarithmic contributions. The argument is the same as that used to eliminate the pyramid graphs (Fig. 12) in fourth order. None of the remaining graphs [Figs. 14(c) and 15] can be reduced to give contributions which cancel the additional $(C_N - C_A)C_A^2\{T_a\}\{T_a\}$ term of the form-factor corrections. We have verified this statement by examining the group-theoretic weights of these diagrams. It is simpler to do this first for the rather unphysical problem of strong corrections to electromagnetic quark-quark scattering. The dynamics of this problem is identical to the one we are studying, the only change is that the single hard vector meson couples with a factor 1 replacing the group matrix T_x. This simplifies the group algebra. In lower orders it restores the property of the purely Abelian theory that there is an exact cancellation between s- and u-channel contributions and the full correction comes from the t-channel form-factor corrections. A similar simplification results for sixth order. The graphs discussed previously add up to

$$\left(\frac{C_N^3}{3!}\right)\{1\}\{1\} + \text{terms proportional to } \{T_a\}\{T_a\} \text{ and } f^{abz}f^{cdz}\{T_a T_c\}\{T_b T_d\} . \tag{B15}$$

The additional graphs all give contributions which have group weights which vanish or give terms corresponding to the extra terms in (B14). For the real problem with a hard exchanged gluon the situation is similar. The additional terms look somewhat more complicated but again cannot be reduced to terms proportional to C_N.

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†Alfred P. Sloan Foundation Fellow.

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[7]It is clear that in scalar ϕ^3 theory this argument will not apply, as all propagators are equivalent. In this case, any side of a box graph can become soft and give infrared singularities. This is observed in the calculations of HHS. The single-hard-gluon approximation has been questioned by Fishbane and Simmons (see Ref. 10); their paper is discussed further in Appendix A. We disagree with its conclusions.
[8]The discussion of Cardy (Ref. 6) on this point is not correct. If a complete (gauge-invariant) light-by-light scattering amplitude is included, there are no remaining ultraviolet divergences, so the terms isolated by Cardy do not appear.
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