Fermion-fermion scattering in a Yang-Mills theory at high energy: Sixth-order perturbation theory*

Barry M. McCoy†
Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794

Tai Tsun Wu
Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138

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We study the high-energy behavior of the elastic scattering of two isospin-$\frac{1}{2}$ fermions interacting through the Yang-Mills field. The Higgs mechanism is invoked so that there is no infrared divergence. In the sixth order, the amplitude for isovector exchange is found to behave as $s\ln^2 s$ multiplied by a function of t, while the amplitude for isoscalar exchange behaves as $s\ln s$ multiplied by another function of t. These results are qualitatively different from the answers previously given in the literature. In particular, there is no contribution to the leading terms from large transverse momentum transfers; this is the reason why we obtain one less factor of $\ln s$ in sixth order.

I. INTRODUCTION

Recently the study of the high-energy limit of non-Abelian gauge fields was initiated by Nieh and Yao.[1] They consider the theory where the Yang-Mills boson[2] interacts with a fermion doublet and study the fermion-fermion elastic scattering at high energies. To avoid the infrared problem, a complex scalar doublet is introduced and the Higgs mechanism[3] is invoked to give masses to the vector mesons. In terms of the usual Mandelstam invariants s (the square of the center-of-mass energy) and t ($=-\vec{\Delta}^2$, where $\vec{\Delta}$ is the momentum transfer), Nieh and Yao study the limit $s\to\infty$ with $t\leq 0$ fixed and state that in sixth-order perturbation theory the amplitude behaves as $s\ln^3 s$ and that in eighth order the amplitude behaves as $s\ln^5 s$.

The purpose of this present paper is to demonstrate explicitly that in sixth order this previous result is incorrect. Instead we show that, for $s\to\infty$ with fixed $t\leq 0$, the sixth-order non-spin-flip amplitude is given by

$$\mathfrak{M}^{(6)}\sim -2^{-4}g^6m^{-2}s[(\ln^2 s-\pi i\ln s)\,\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}f_1(t)+3\pi i\ln s\,I^{(1)}I^{(2)}f_2(t)]\ , \tag{1.1}$$

where

$$f_1(t)=(\lambda^2+\vec{\Delta}^2)\left\{\int\frac{d^2k_\perp}{(2\pi)^3}\left[\left(\vec{k}_\perp-\frac{\vec{\Delta}}{2}\right)^2+\lambda^2\right]^{-1}\left[\left(\vec{k}_\perp+\frac{\vec{\Delta}}{2}\right)^2+\lambda^2\right]^{-1}\right\}^2, \tag{1.2}$$

and

$$f_2(t)=(\tfrac{5}{4}\lambda^2+\vec{\Delta}^2)\left\{\int\frac{d^2k_\perp}{(2\pi)^3}\left[\left(\vec{k}_\perp-\frac{\vec{\Delta}}{2}\right)^2+\lambda^2\right]^{-1}\left[\left(\vec{k}_\perp+\frac{\vec{\Delta}}{2}\right)^2+\lambda^2\right]^{-1}\right\}^2$$
$$-\int\frac{d^2k_{1\perp}}{(2\pi)^3}\frac{d^2k_{2\perp}}{(2\pi)^3}(\vec{k}_{1\perp}{}^2+\lambda^2)^{-1}(\vec{k}_{2\perp}{}^2+\lambda^2)^{-1}[(\vec{k}_{1\perp}+\vec{k}_{2\perp}+\vec{\Delta})^2+\lambda^2]^{-1}\ . \tag{1.3}$$

In (1.1)–(1.3), g is the coupling constant, m is the mass of the fermion, λ is the mass of the Yang-Mills boson, $\vec{\tau}^{(1)}$ and $\vec{\tau}^{(2)}$ are the Pauli matrices for isotopic spin of first and second fermions, and $I^{(1)}$ and $I^{(2)}$ are the corresponding 2×2 unit matrices.

For comparison, the second-order non-spin-flip amplitude, which comes from the Born diagram of Fig. 1(a), is

$$\mathfrak{M}^{(2)}\sim -2^{-3}g^2m^{-2}s\,\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}(\vec{\Delta}^2+\lambda^2)^{-1}\ ; \tag{1.4}$$

and the fourth-order non-spin-flip amplitude, which comes from the two diagrams of Fig. 1(b), is[1]

$$\mathfrak{M}^{(4)}\sim 2^{-5}g^4m^{-2}s[2(2\ln s-\pi i)\,\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}+3\pi iI^{(1)}I^{(2)}]\int\frac{d^2k_\perp}{(2\pi)^3}\left[\left(\vec{k}_\perp-\frac{\vec{\Delta}}{2}\right)^2+\lambda^2\right]^{-1}\left[\left(\vec{k}_\perp+\frac{\vec{\Delta}}{2}\right)^2+\lambda^2\right]^{-1}. \tag{1.5}$$

We make the following remarks about the result (1.1):

1. In this final result all integrals over the transverse momenta converge. Therefore, all $\ln s$ factors arise from integrations over longitudinal momenta. This is similar to the situation in quantum electrodynamics.[4]

2. In order to get (1.1), Feynman diagrams that contain the scalar particle must be included. In particular, the scalar particle is needed to produce the factor $\lambda^2 + \vec{\Delta}^2$ in the isovector amplitude (1.2). Thus this isovector amplitude vanishes when $t = \lambda^2$. This is similar to the situation for fermion exchange in quantum electrodynamics.[5]

3. Neither the Higgs ghost nor the Faddeev-Popov ghost[6] contribute.

4. We particularly wish to call attention to the last term in the isoscalar amplitude (1.3). The analogous term in quantum electrodynamics appears when three photons are exchanged between two electrons. However, in that case there is more cancellation among diagrams and the sum of the six graphs with three photons exchanged is of order s instead of $s \ln s$. *This is a profound difference between the cases of Abelian and non-Abelian gauge theories.* On the basis of this lack of cancellation of logarithms in the non-Abelian case, we speculate that in the leading-logarithm approximation the Yang-Mills theory is unitary in both the s and t channels.

II. METHOD OF CALCULATION

The relevant Feynman rules for this theory are given in Fig. 2. We operate exclusively in 't Hooft's R gauge[7] where there are no $k_\mu k_\nu$ terms in the propagator for the Yang-Mills field. In this gauge the propagator for the Yang-Mills boson is

$$\frac{-i\delta_{ab}\, g_{\mu\nu}}{k^2 - \lambda^2 + i\epsilon}, \tag{2.1}$$

where a and b (=1, 2, 3) are isotopic-spin indices and μ and ν (=0, 1, 2, 3) are space-time indices. [The metric is (+ − − −).] The fermion propagator is

$$i\,\frac{\not{k} + m}{k^2 - m^2 + i\epsilon}, \tag{2.2}$$

where $\not{k} = \gamma_\nu k_\nu$ and Dirac γ matrices obey

$$[\gamma_\mu, \gamma_\nu] = 2g_{\mu\nu}. \tag{2.3}$$

We also need the propagator of the scalar field of mass M:

$$\frac{i}{k^2 - M^2 + i\epsilon}. \tag{2.4}$$

The vertices which we need are (Fig. 2): the three-boson vertex

$$g\epsilon_{abc}[(p_1 - p_2)_\rho g_{\mu\nu} + (p_2 - p_3)_\mu g_{\nu\rho} + (p_3 - p_1)_\nu g_{\rho\mu}], \tag{2.5}$$

where ϵ_{abc} is the totally antisymmetric symbol with $\epsilon_{123} = 1$; the fermion-boson vertex

$$\tfrac{1}{2} i g \tau_a \gamma_\mu, \tag{2.6}$$

where the Pauli matrices τ_a obey

$$\tau_a^2 = 1, \quad \tau_a \tau_b = i\epsilon_{abc}\tau_c; \tag{2.7}$$

the scalar-boson vertex

$$ig\lambda g_{\mu\nu}\delta_{ab}; \tag{2.8}$$

and the four-boson vertex

$$\begin{aligned} -ig^2[&\delta_{ab}\delta_{cd}(2g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \\ &+ \delta_{ac}\delta_{bd}(2g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\nu\rho}) \\ &+ \delta_{ad}\delta_{bc}(2g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma})]. \end{aligned} \tag{2.9}$$

The derivation of these Feynman rules from a Lagrangian is a somewhat subtle process and many more propagators and vertices are needed for a complete theory[6,8] than are given here. However, for sixth order, it may be shown that none of the omitted propagators (such as the Higgs ghost or the Faddeev-Popov ghost) and the associated vertices contribute to (1.1).

We will study the sixth-order scattering amplitude by means of the momentum-space techniques which have proven useful in previous studies of massive quantum electrodynamics.[9] For a detailed discussion of these methods, we refer the reader to Ref. 9.

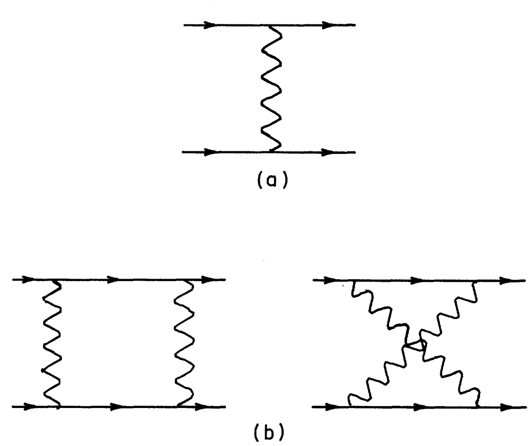

FIG. 1. (a) The Born approximation. (b) The two fourth-order Feynman diagrams which contribute to leading order as $s \to \infty$.

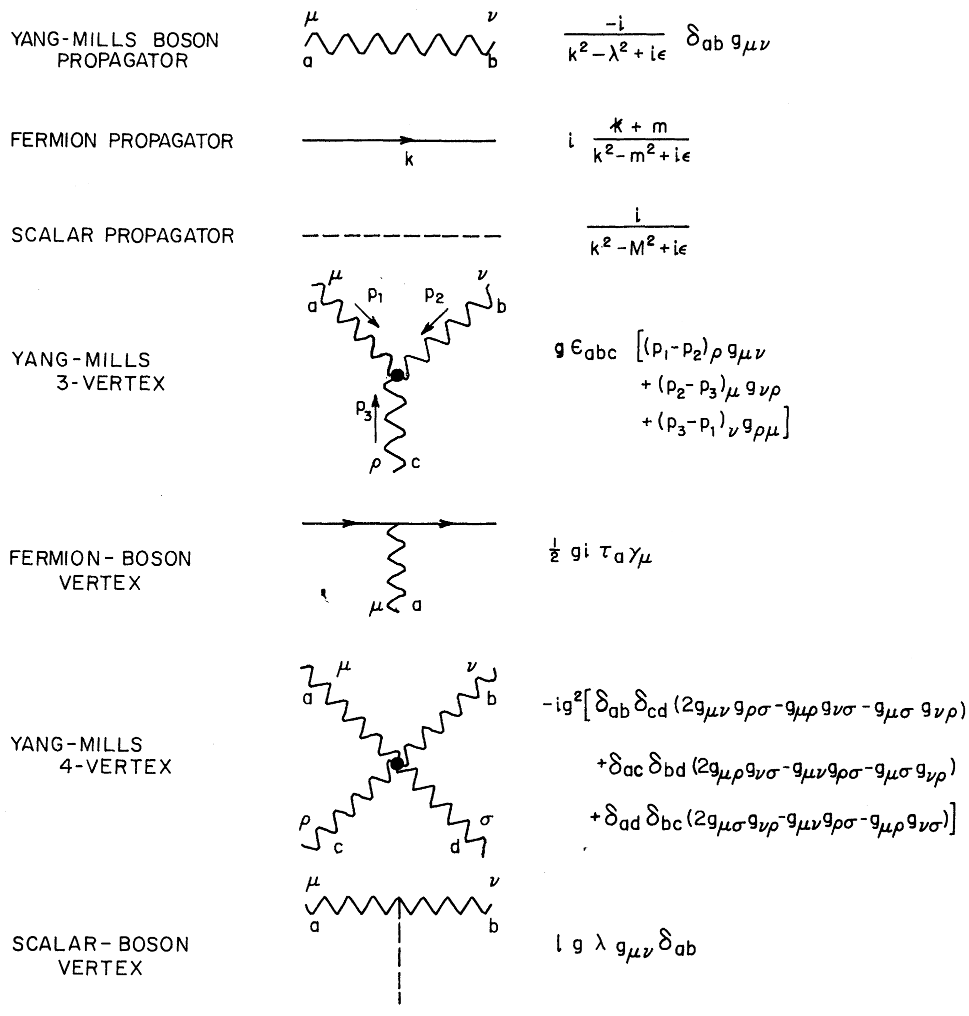

FIG. 2. The Feynman rules for a Yang-Mills boson interacting with a fermion doublet which are needed to study the high-energy behavior of fermion-fermion scattering in leading order.

The essence of the momentum-space method is that in the $s\to\infty$ limit we choose a coordinate system where the large components of the momenta of the incoming and outgoing particles are along the z axis. Let r_2+r_1 and r_3-r_1 be the momenta of the incoming fermions, r_2-r_1 and r_3+r_1 be those of the outgoing fermions as shown in Fig. 3, then we have approximately

$$r_2\sim(\omega,\omega,\vec{0})\ ,\qquad r_3\sim(\omega,-\omega,\vec{0})\ , \tag{2.10a}$$

where $\omega=\frac{1}{2}\sqrt{s}$. In such a coordinate system,

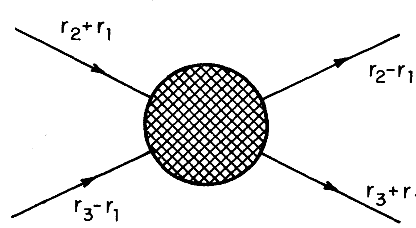

FIG. 3. The kinematics for fermion-fermion scattering.

$$r_1 = (0, 0, \vec{r}_\perp) \ . \tag{2.10b}$$

We make use of (2.10) by introducing the variables[9]

$$k_+ = k_0 + k_3$$

and (2.11)

$$k_- = k_0 - k_3 \ ,$$

so that

$$dk_0 dk_3 = \tfrac{1}{2}\, dk_+ dk_- \ . \tag{2.12}$$

Thus

$$r_{2+} = 2\omega + O\left(\frac{1}{2\omega}\right), \quad r_{2-} = O\left(\frac{1}{2\omega}\right),$$

$$r_{3+} = O\left(\frac{1}{2\omega}\right), \qquad r_{3-} = 2\omega + O\left(\frac{1}{2\omega}\right), \tag{2.13}$$

$$r_{1\pm} = 0 \ ,$$

and the basic approximation is to drop r_{2-}, r_{3+}, and $r_{1\pm}$ wherever they occur in the Feynman integral.

We will also impose a cutoff $k_{\perp \max}$ on all transverse momentum components which appear in the Feynman integrals in the intermediate stages of the calculation, and for each Feynman diagram we compute the $s \to \infty$ behavior in the presence of this cutoff. For the individual Feynman diagrams the remaining integrals over transverse momenta will not converge when the cutoff is removed. However, we find that after all contributing diagrams are added together the cutoff may finally be removed. This situation is entirely similar to the case of quantum electrodynamics.[9]

The twenty diagrams which contribute (with the cutoff) are given in Fig. 4. Of these diagrams we need only consider explicitly diagrams 1, 3, 7, 15, and 19 since the remaining diagrams are obtained from these by symmetry considerations.

In the final answer, the real part of the amplitude is larger than the imaginary part by a factor of ln$s$. We will compute both the leading real and the leading imaginary parts. The result (1.1) should be interpreted in this sense.

We remark that in the diagrams of Fig. 4 there are no diagrams with four boson vertices. In particular, the diagram of Fig. 5 does not contribute to leading order with a cutoff imposed. It is known that, without a cutoff, by itself the diagram of Fig. 5 is of order s^2, but that this unphysical behavior is cancelled by similar s^2 terms from diagrams 1 and 2. With our momenta-space approximation technique these terms, which can never appear in the final answer, are suppressed from the beginning. This situation is again entirely similar to the case of quantum electrodynamics.[10]

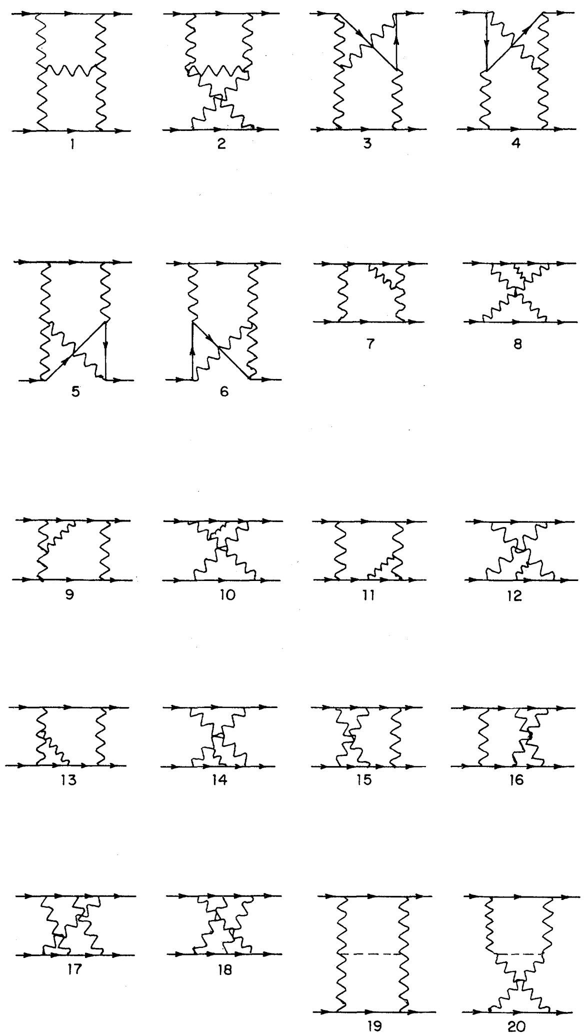

FIG. 4. The twenty sixth-order Feynman diagrams which contribute to leading order with a transverse cutoff. The s channel is from left to right.

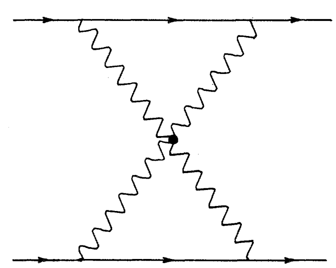

FIG. 5. A Feynman diagram which does not contribute to leading order with a cutoff imposed in momentum space even though without the cutoff it is of order s^2.

III. FEYNMAN DIAGRAM 1

From the Feynman rules of Fig. 2 we find that the amplitude for Feynman diagram 1 shown in Fig. 6 is

$$\mathfrak{M}_1^{(6)} = +(\tfrac{1}{2})^4 g^6 \int \frac{d^4k_1}{(2\pi)^4}\, \frac{d^4k_2}{(2\pi)^4}\, N_1 D_1^{-1} , \tag{3.1}$$

$$\begin{aligned} N_1 = {} & [\bar{u}(r_2 - r_1)\,\gamma_\lambda \tau_a^{(1)}(\not{r}_2 - \not{k}_1 + m)\,\gamma_\mu \tau_b^{(1)}\, u(r_2 + r_1)][\bar{u}(r_3 + r_1)\,\gamma_\nu\, \tau_e^{(2)}(\not{r}_3 + \not{k}_2 + m)\,\gamma_\sigma \tau_d^{(2)} u(r_3 - r_1)] \\ & \times \epsilon_{bcd}[(2k_1 - k_2 + r_1)_\sigma g_{\mu\rho} + (2k_2 - k_1 + r_1)_\mu g_{\rho\sigma} + (-k_2 - k_1 - 2r_1)_\rho g_{\mu\sigma}] \\ & \times \epsilon_{aec}[(-k_1 - k_2 + 2r_1)_\rho g_{\lambda\nu} + (2k_2 - k_1 - r_1)_\lambda g_{\nu\rho} + (2k_1 - k_2 - r_1)_\nu g_{\rho\lambda}] \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} D_1 = {} & [(r_2 - k_1)^2 - m^2 + i\epsilon][(k_1 - r_1)^2 - \lambda^2 + i\epsilon][(k_1 - k_2)^2 - \lambda^2 + i\epsilon] \\ & \times [(k_1 + r_1)^2 - \lambda^2 + i\epsilon][(k_2 - r_1)^2 - \lambda^2 + i\epsilon][(k_2 + r_1)^2 - \lambda^2 + i\epsilon][(r_3 + k_2)^2 - m^2 + i\epsilon] . \end{aligned} \tag{3.3}$$

We first use (2.13) to approximate D_1 as

$$\begin{aligned} D_1 \sim \overline{D}_1 = {} & [(-2\omega + k_{1+})\,k_{1-} - \vec{k}_{1\perp}^{\,2} - m^2 + i\epsilon][k_{1+}k_{1-} - (\vec{k}_{1\perp} - \vec{r}_\perp)^2 - \lambda^2 + i\epsilon] \\ & \times [(k_{1+} - k_{2+})\,(k_{1-} - k_{2-}) - (\vec{k}_{1\perp} - \vec{k}_{2\perp})^2 - \lambda^2 + i\epsilon][k_{1+}k_{1-} - (\vec{k}_{1\perp} + \vec{r}_\perp)^2 - \lambda^2 + i\epsilon] \\ & \times [k_{2+}k_{2-} - (\vec{k}_{2\perp} - \vec{r}_\perp)^2 - \lambda^2 + i\epsilon][k_{2+}\,k_{2-} - (\vec{k}_{2\perp} + \vec{r}_\perp)^2 - \lambda^2 + i\epsilon][(2\omega + k_{2-})\,k_{2+} - \vec{k}_{2\perp}^{\,2} - m^2 + i\epsilon] . \end{aligned} \tag{3.4}$$

We will then carry out the k_{1+} and k_{2+} integrations by closing on the poles

$$k_{2+} = C_1(2\omega + k_{2-})^{-1} \tag{3.5a}$$

and

$$k_{1+} - k_{2+} = C_2(k_{1-} - k_{2-})^{-1} , \tag{3.5b}$$

with

$$C_1 = \vec{k}_{2\perp}^{\,2} + m^2 - i\epsilon \tag{3.6a}$$

and

$$C_2 = (\vec{k}_{1\perp} - \vec{k}_{2\perp})^2 + \lambda^2 - i\epsilon . \tag{3.6b}$$

Define

$$q_1 = -k_{1-} \tag{3.7a}$$

and

$$q_2 = k_{1-} - k_{2-} . \tag{3.7b}$$

Then the integral over k_{1+} and k_{2+} gives zero unless

$$0 < q_1, \quad 0 < q_2, \quad q_1 + q_2 < 2\omega . \tag{3.8}$$

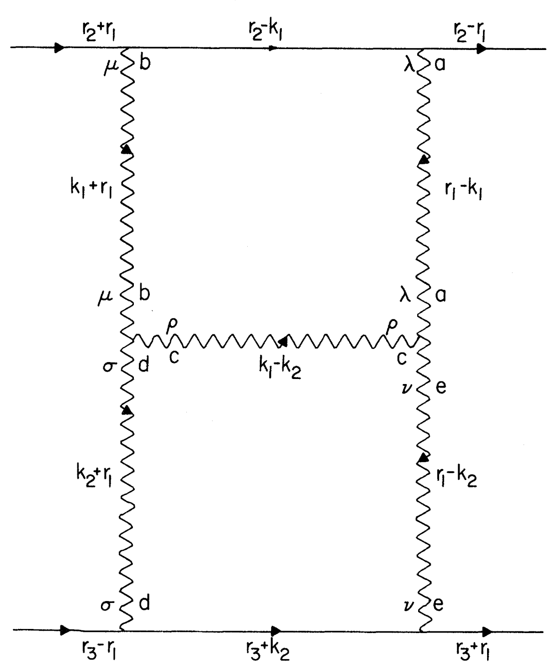

FIG. 6. Feynman diagram 1.

Therefore, we obtain

$$\mathfrak{M}_1^{(6)} \sim -(\tfrac{1}{2})^4 g^6 (2\omega)^{-1} \int \frac{d^2k_{1\perp}}{2(2\pi)^3} \int \frac{d^2k_{2\perp}}{2(2\pi)^3} \int dq_1\, dq_2\, N_1 \tilde{D}_1^{-1}\,, \tag{3.9}$$

where

$$\tilde{D}_1 = q_2[(\vec{k}_{2\perp}+\vec{r}_\perp)^2+\lambda^2][(\vec{k}_{2\perp}-\vec{r}_\perp)^2+\lambda^2](2\omega q_1 - \vec{k}_{1\perp}{}^2 - m^2 + i\epsilon)$$
$$\times[-q_1 C_2 q_2^{-1} - (\vec{k}_{1\perp}-\vec{r}_\perp)^2 - \lambda^2 + i\epsilon][-q_1 C_2 q_2^{-1} - (\vec{k}_{1\perp}-\vec{r}_\perp)^2 - \lambda^2 + i\epsilon]\ . \tag{3.10}$$

We must now approximate the numerator. First of all the isospin factor is reduced using the identity

$$\epsilon_{bcd}\,\epsilon_{aec} = -(\delta_{ba}\delta_{de} - \delta_{be}\delta_{da}) \tag{3.11}$$

to obtain

$$\tau_a^{(1)}\tau_b^{(1)}\tau_e^{(2)}\tau_d^{(2)}\epsilon_{bcd}\,\epsilon_{aec} = -(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(1)}\,\vec{\tau}^{(2)}\cdot\vec{\tau}^{(2)} - \tau_a^{(1)}\tau_b^{(1)}\tau_b^{(2)}\tau_a^{(2)})\ . \tag{3.12}$$

Now using (2.7) we obtain

$$\vec{\tau}^{(1)}\cdot\vec{\tau}^{(1)}\,\vec{\tau}^{(2)}\cdot\vec{\tau}^{(2)} = 9\,I^{(1)}I^{(2)} \tag{3.13}$$

and

$$\tau_a^{(1)}\tau_b^{(1)}\tau_b^{(2)}\tau_a^{(2)} = 3\,I^{(1)}I^{(2)} + 2[\tau_1^{(1)}\tau_2^{(1)}\tau_2^{(2)}\tau_1^{(2)} + \tau_2^{(1)}\tau_3^{(1)}\tau_3^{(2)}\tau_2^{(2)} + \tau_3^{(1)}\tau_1^{(1)}\tau_1^{(2)}\tau_3^{(2)}]$$
$$= 3\,I^{(1)}I^{(2)} + 2\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}\ , \tag{3.14}$$

and hence

$$\tau_a^{(1)}\tau_b^{(1)}\tau_e^{(2)}\tau_d^{(2)}\epsilon_{bcd}\,\epsilon_{aec} = 2\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)} - 6\,I^{(1)}I^{(2)}\ . \tag{3.15}$$

We now approximate

$$\not{r}_2 - \not{k}_1 + m \sim \not{r}_2 \qquad \not{r}_3 + \not{k}_2 + m \sim \not{r}_3\ . \tag{3.16}$$

Then we anticommute $\not{r}_2$ and $\not{r}_3$ past the γ matrices. The terms where $\not{r}_2$ or $\not{r}_3$ act on the free spinors may be dropped to leading order by use of the Dirac equation. In this way we may expand (3.2) into a sum of nine terms. Two of these terms may be shown not to contribute and we are left with

$$N_1 \sim 4(2\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)} - 6\,I^{(1)}I^{(2)})\,\bar{u}(r_2-r_1)\gamma_\lambda u(r_2+r_1)\,\bar{u}(r_3+r_1)\gamma_\lambda u(r_3-r_1)\,\overline{N}_1\,, \tag{3.17}$$

where

$$\overline{N}_1 = r_2\cdot r_3(-k_2-k_1-2r_1)\cdot(-k_1-k_2+2r_1) + r_3\cdot(2k_1-k_2)\,r_2\cdot(-k_1-k_2) + r_3\cdot(-k_1-k_2)\,r_2\cdot(2k_2-k_1)$$
$$+ r_2\cdot(-k_2-k_1)\,r_3\cdot(2k_1-k_2) + r_2\cdot(2k_2-k_1)\,r_3\cdot(-k_2-k_1) + r_3\cdot(2k_1-k_2)\,r_2\cdot(2k_2-k_1) + r_2\cdot(2k_2-k_1)\,r_3\cdot(2k_1-k_2)\ . \tag{3.18}$$

To simplify N_1 further we use

$$\bar{u}(r_2-r_1)\gamma_\lambda u(r_2+r_1) \sim r_{2\lambda} m^{-1}\delta_{1,1'}\,, \quad \bar{u}(r_3+r_1)\gamma_\lambda u(r_3-r_1) \sim r_{3\lambda} m^{-1}\delta_{2,2'}\ , \tag{3.19}$$

where $\delta_{a,a'}$ is 1 if the spin of fermion a is the same in the incoming and outgoing state and is zero if the spin of the fermion is flipped. Therefore,

$$N_1 \sim 2(2\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)} - 6\,I^{(1)}I^{(2)})\,s\,m^{-2}\delta_{1,1'}\delta_{2,2'}\overline{N}_1\ . \tag{3.20}$$

Moreover,

$$\overline{N}_1 = (k_1+k_2+2r_1)\cdot(k_1+k_2-2r_1)\,r_2\cdot r_3$$
$$+2[r_3\cdot(2k_1-k_2)\,r_2\cdot(-k_1-k_2) + r_3\cdot(-k_1-k_2)\,r_2\cdot(2k_2-k_1) + r_3\cdot(2k_1-k_2)\,r_2\cdot(2k_2-k_1)]$$
$$= (k_1+k_2+2r_1)\cdot(k_1+k_2-2r_1)\,r_2\cdot r_3 + 2[r_3\cdot(2k_1-k_2)\,r_2\cdot(-k_1-k_2) + r_3\cdot(k_1-2k_2)\,r_2\cdot(2k_2-k_1)]$$
$$\sim 2\omega^2\{(k_{1+}+k_{2+})(k_{1-}+k_{2-}) - (\vec{k}_{1\perp}+\vec{k}_{2\perp}+2\vec{r}_\perp)\cdot(\vec{k}_{1\perp}+\vec{k}_{2\perp}-2\vec{r}_\perp)$$
$$+[(2k_{1+}-k_{2+})(-k_{1-}-k_{2-}) + (k_{1+}-2k_{2+})(2k_{2-}-k_{1-})]\}$$
$$\sim 2\omega^2\{C_2 q_2^{-1}(-q_2-2q_1) - (\vec{k}_{1\perp}+\vec{k}_{2\perp}+2\vec{r}_\perp)\cdot(\vec{k}_{1\perp}+\vec{k}_{2\perp}-2\vec{r}_\perp) + [2C_2 q_2^{-1}(q_2+2q_1) + C_2 q_2^{-1}(-2q_2-q_1)]\}\ . \tag{3.21}$$

The only regions of the (q_1, q_2) space which can contribute to the maximum number of logarithms in (3.9) (i.e., $\ln^2 s$ and $i\ln s$) are

$$0<q_1 \ll q_2 \ll 2\omega \tag{3.22a}$$

and

$$0<q_2 \ll q_1 \ll 2\omega \ . \tag{3.22b}$$

But if (3.22b) holds, the factor $\tilde{D}_1^{-1}$ (3.10) behaves as $q_1^{-3}q_2$ which can give two logarithms only if multiplied by $q_1^2 q_2^{-2}$. Such a term with four factors can clearly never be produced by $\overline{N}_1$ of (3.21). Therefore, we only need consider the region (3.22a).

When (3.22a) holds, $\tilde{D}_1^{-1}$ of (3.10) behaves as $q_1^{-1}q_2^{-1}$. Furthermore, the second term in (3.21) behaves as $q_1 q_2^{-1}$ which does not produce two logarithms. Thus $\overline{N}_1$ may be approximated as

$$\overline{N}_1 \sim 2\omega^2[-C_2-(\vec{k}_{1\perp}+\vec{k}_{2\perp}+2\vec{r}_\perp)\cdot(\vec{k}_{1\perp}+\vec{k}_{2\perp}-2\vec{r}_\perp)] \tag{3.23}$$

which is independent of q_1 and q_2. Then, since

$$\int_0^{2\omega} dq_2 \int_0^{q_2} dq_1\, q_2^{-1}(2\omega q_1-\vec{k}_{1\perp}^{\,2}-m^2+i\epsilon)^{-1} \sim \frac{1}{2\omega}\,\frac{1}{2}\,(\ln^2 s-2\pi i\ln s)\ , \tag{3.24}$$

we obtain the final result:

$$\mathfrak{M}_1^{(6)} \sim 2^{-7} g^6 m^{-2} s(\ln^2 s-2\pi i \ln s)(2\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}-6I^{(1)}I^{(2)})\,\delta_{1,1'}\,\delta_{2,2'}\int\frac{d^2k_{1\perp}}{(2\pi)^3}\int\frac{d^2k_{2\perp}}{(2\pi)^3}\,N_{1\perp}D_{1\perp}^{-1}, \tag{3.25}$$

where

$$D_{1\perp}=[(\vec{k}_{1\perp}-\vec{r}_\perp)^2+\lambda^2][(\vec{k}_{1\perp}+\vec{r}_\perp)^2+\lambda^2][(\vec{k}_{2\perp}-\vec{r}_\perp)^2+\lambda^2][(\vec{k}_{2\perp}+\vec{r}_\perp)^2+\lambda^2] \tag{3.26}$$

and

$$N_{1\perp}=(\vec{k}_{1\perp}-\vec{k}_{2\perp})^2+\lambda^2+(\vec{k}_{1\perp}+\vec{k}_{2\perp}+2\vec{r}_\perp)\cdot(\vec{k}_{1\perp}+\vec{k}_{2\perp}-2\vec{r}_\perp)\ . \tag{3.27}$$

IV. FEYNMAN DIAGRAM 2

Feynman diagram 2 as given in Fig. 7 is obtained from Feynman diagram 1 by $s \leftrightarrow u$ interchanged. Its calculation differs from Feynman diagram 1 in only two respects. First of all, the isospin factor is

$$\begin{aligned}\tau_a^{(1)}\tau_b^{(1)}\epsilon_{dbc}\epsilon_{aec}\tau_d^{(2)}\tau_e^{(2)} &= \tau_a^{(1)}\tau_b^{(1)}(\delta_{da}\delta_{be}-\delta_{de}\delta_{ba})\tau_d^{(2)}\tau_e^{(2)}\\ &= \tau_a^{(1)}\tau_b^{(1)}\tau_a^{(2)}\tau_b^{(2)}-9I^{(1)}I^{(2)}\\ &= -6I^{(1)}I^{(2)}-2\vec{\tau}^{(1)}\cdot\tau^{(2)}\ .\end{aligned} \tag{4.1}$$

Secondly, (3.24) is replaced by

$$\int_0^{2\omega} dq_2\int_0^{q_2} dq_1\, q_2^{-1}(-2\omega q_1-\vec{k}_{1\perp}^{\,2}-m^2+i\epsilon)^{-1}\sim -(2\omega)^{-1}\tfrac{1}{2}\ln^2 s\ . \tag{4.2}$$

Therefore

$$\mathfrak{M}_1^{(6)}+\mathfrak{M}_2^{(6)}\sim 2^{-6}g^6 m^{-2}s\,\delta_{1,1'}\,\delta_{2,2'}[6I^{(1)}I^{(2)}\pi i\ln s+2\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}(\ln^2 s-\pi i\ln s)]\int\frac{d^2k_{1\perp}}{(2\pi)^3}\,\frac{d^2k_{2\perp}}{(2\pi)^3}\,N_{1\perp}D_{1\perp}^{-1}\ . \tag{4.3}$$

V. FEYNMAN DIAGRAM 3

The amplitude for Feynman diagram 3 given by Fig. 8 is

$$\mathfrak{M}_3^{(6)}=-\left(\frac{i}{2}\right)^5 g^6\int\frac{d^4k_1}{(2\pi)^4}\,\frac{d^4k_2}{(2\pi)^4}\,N_3 D_3^{-1}\ , \tag{5.1}$$

where

$$\begin{aligned}N_3 &= [\overline{u}(r_2+r_1)\gamma_\nu\tau_a^{(1)}(\not r_2-\not k_1-\not k_2+\not r_1+m)\gamma_\lambda\tau_b^{(1)}(\not r_2-\not k_1+m)\gamma_\mu\tau_c^{(1)}u(r_2-r_1)]\\ &\quad\times[\overline{u}(r_3-r_1)\gamma_\lambda\tau_b^{(2)}(\not r_3-\not k_2+m)\gamma_\rho\tau_d^{(2)}u(r_3+r_1)]\\ &\quad\times\epsilon_{adc}[g_{\nu\rho}(-k_1-2k_2-r_1)_\mu+g_{\rho\mu}(k_2-k_1+2r_1)_\nu+g_{\mu\nu}(2k_1+k_2-r_1)_\rho]\end{aligned} \tag{5.2}$$

and

$$D_3 = [(r_2 - k_1)^2 - m^2 + i\epsilon][(r_2 - k_1 - k_2 + r_1)^2 - m^2 + i\epsilon][(k_1 + k_2)^2 - \lambda^2 + i\epsilon][(k_1 - r_1)^2 - \lambda^2 + i\epsilon]$$
$$\times[(k_2 + r_1)^2 - \lambda^2 + i\epsilon][(k_2 - r_1)^2 - \lambda^2 + i\epsilon][(r_3 - k_2)^2 - m^2 + i\epsilon] \ . \tag{5.3}$$

We use (2.13) to approximate D_3 as

$$D_3 \sim \overline{D}_3 = [-(2\omega - k_{1+})k_{1-} - \vec{k}_{1\perp}{}^2 - m^2 + i\epsilon][-(2\omega - k_{1+} - k_{2+})(k_{1-} + k_{2-}) - (\vec{k}_{1\perp} + \vec{k}_{2\perp} - \vec{r}_\perp)^2 - m^2 + i\epsilon]$$
$$\times[(k_{1+} + k_{2+})(k_{1-} + k_{2-}) - (\vec{k}_{1\perp} + \vec{k}_{2\perp})^2 - \lambda^2 + i\epsilon][k_{1+}k_{1-} - (\vec{k}_{1\perp} - \vec{r}_\perp)^2 - \lambda^2 + i\epsilon]$$
$$\times[k_{2+}k_{2-} - (\vec{k}_{2\perp} + \vec{r}_\perp)^2 - \lambda^2 + i\epsilon][k_{2+}k_{2-} - (\vec{k}_{2\perp} - \vec{r}_\perp)^2 - \lambda^2 + i\epsilon][-k_{2+}(2\omega - k_{2-}) - \vec{k}_{2\perp}{}^2 - m^2 + i\epsilon] \ . \tag{5.4}$$

We will carry out the k_{1+} and k_{2+} integrations by closing on the poles

$$k_{2+} = -C_1(2\omega - k_{2-})^{-1} \tag{5.5a}$$

and

$$k_{1+} + k_{2+} = C_3(k_{1-} + k_2)^{-1} \tag{5.5b}$$

or

$$k_{1+} + k_{2+} = 2\omega + C_4(k_{1-} + k_{2-})^{-1} \ , \tag{5.5c}$$

with

$$C_3 = (\vec{k}_{1\perp} + \vec{k}_{2\perp})^2 + \lambda^2 - i\epsilon \tag{5.6a}$$

and

$$C_4 = (\vec{k}_{1\perp} + \vec{k}_{2\perp} - \vec{r}_\perp)^2 + m^2 - i\epsilon \ . \tag{5.6b}$$

However, the pole (5.5c) is suppressed by the factor in the numerator coming from the upper fermion. Therefore, only the poles (5.5a) and (5.5b) need be considered.

Define, instead of (3.7),

$$q_1 = -k_{1-} \tag{5.7a}$$

and

$$q_2 = k_{1-} + k_{2-} \ . \tag{5.7b}$$

Then the integral over k_{1+} and k_{2+} gives zero unless (3.8) holds. Therefore, we obtain

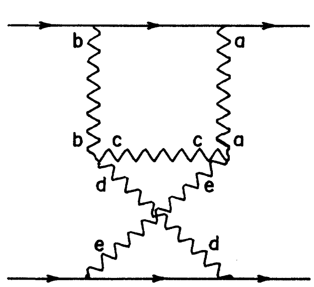

FIG. 7. Feynman diagram 2.

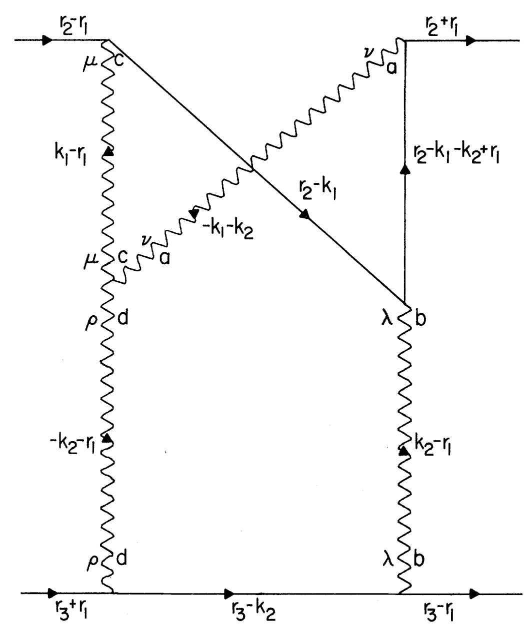

FIG. 8. Feynman diagram 3.

$$\mathfrak{M}_3^{(6)}=\left(\frac{i}{2}\right)^5 g^6(2\omega)^{-1}\int\frac{d^2k_{1\perp}}{2(2\pi)^3}\,\frac{d^2k_{2\perp}}{2(2\pi)^3}\int dq_1\,dq_2\,N_3\,\tilde{D}_3^{-1}\ , \tag{5.8}$$

where

$$\tilde{D}_3=q_2[(\vec{k}_{2\perp}+\vec{r}_\perp)^2+\lambda^2][(\vec{k}_{2\perp}-\vec{r}_\perp)^2+\lambda^2][-q_1C_3q_2^{-1}-(\vec{k}_{1\perp}-\vec{r}_\perp)^2-\lambda^2+i\epsilon]$$
$$\times(2\omega q_1-\vec{k}_{1\perp}{}^2-m^2+i\epsilon)[-2\omega q_2-(\vec{k}_{1\perp}+\vec{k}_{2\perp}-\vec{r}_\perp)^2-m^2+i\epsilon]\ . \tag{5.9}$$

We next reduce N_3. First we note that

$$\tau_a^{(1)}\tau_b^{(1)}\tau_c^{(1)}\tau_b^{(2)}\tau_d^{(2)}\epsilon_{adc}=(-\tau_b^{(1)}\tau_a^{(1)}\tau_c^{(1)}+2\delta_{ab}\tau_c^{(1)})\,\epsilon_{adc}(-\tau_d^{(2)}\tau_b^{(2)}+2\delta_{bd})$$
$$=-\tau_b^{(1)}\tau_a^{(1)}\tau_c^{(1)}\epsilon_{acd}\tau_d^{(2)}\tau_b^{(2)}+2\tau_b^{(1)}\tau_a^{(1)}\tau_c^{(1)}\epsilon_{bac}I^{(2)}+2\tau_c^{(1)}\epsilon_{cda}\tau_d^{(2)}\tau_a^{(2)}$$
$$=2i(-\tau_b^{(1)}\tau_d^{(1)}\tau_d^{(2)}\tau_b^{(2)}+2\tau^{(1)}\cdot\tau^{(2)}+6I^{(1)}I^{(2)})$$
$$=6iI^{(1)}I^{(2)}\ , \tag{5.10}$$

where to obtain the last line we used (3.14). Therefore, if we also approximate $\not{r}_2-\not{k}_1-\not{k}_2+\not{r}_1+m$ by $\not{r}_2$ and anticommute $\not{r}_2$ to the left, approximate $\not{r}_2-\not{k}_1+m$ by $\not{r}_2$ and anticommute $\not{r}_2$ to the right, approximate $\not{r}_3-\not{k}_2+m$ by $\not{r}_3$ and anticommute $\not{r}_3$ to the right, and then use (3.19), we find

$$N_3\sim 6iI^{(1)}I^{(2)}\,4s\,m^{-2}\,\delta_{1,1'}\,\delta_{2,2'}\,\overline{N}_3\ , \tag{5.11}$$

with

$$\overline{N}_3=2\omega^3(-2k_{1-}-k_{2-})=2\omega^3(q_1-q_2)\ . \tag{5.12}$$

Using this numerator with the approximate denominator $\tilde{D}_3$ (5.9) in (5.8), we see that both the regions $0<q_1\ll q_2\ll 2\omega$ and $0<q_2\ll q_1\ll 2\omega$ contribute. Therefore, using (3.24) and (4.2), we obtain

$$\mathfrak{M}_3^{(6)}\sim+2^{-6}g^6m^{-2}s(3I^{(1)}I^{(2)})\,\delta_{1,1'}\,\delta_{2,2'}$$
$$\times\int\frac{d^2k_{1\perp}}{(2\pi)^3}\,\frac{d^2k_{2\perp}}{(2\pi)^3}[(\vec{k}_{2\perp}+\vec{r}_\perp)^2+\lambda^2]^{-1}[(\vec{k}_{2\perp}-\vec{r}_\perp)^2+\lambda^2]^{-1}$$
$$\times\{\tfrac{1}{2}(\ln^2 s-2\pi i\ln s)[(\vec{k}_{1\perp}-\vec{r}_\perp)^2+\lambda^2]^{-1}-\tfrac{1}{2}\ln^2 s[(\vec{k}_{1\perp}+\vec{k}_{2\perp})^2+\lambda^2]^{-1}\}\ . \tag{5.13}$$

VI. FEYNMAN DIAGRAMS 4, 5, AND 6

Feynman diagram 4 is the same as diagram 3 reversed right for left. It is also obtained from diagram 3 by $s\to u$. It is easily verified that the isospin factor for diagram 4 is the same as that for diagram 3 [namely, (5.10)]. We thus may obtain the expansion for $\mathfrak{M}_4^{(6)}$ from (5.13) by the replacement

$$s\tfrac{1}{2}(\ln^2 s-2\pi i\ln s)\to-s\tfrac{1}{2}\ln^2 s\ . \tag{6.1}$$

Therefore,

$$\mathfrak{M}_3^{(6)}+\mathfrak{M}_4^{(6)}\sim-2^{-6}g^6m^{-2}s\,\delta_{1,1'}\,\delta_{2,2'}\,(3I^{(1)}I^{(2)})\,\pi i\ln s$$
$$\times\int\frac{d^2k_{1\perp}}{(2\pi)^3}\int\frac{d^2k_{2\perp}}{(2\pi)^3}[(\vec{k}_{2\perp}+\vec{r}_\perp)^2+\lambda^2]^{-1}[(\vec{k}_{2\perp}-\vec{r}_\perp)^2+\lambda^2]^{-1}$$
$$\times\{[(\vec{k}_{1\perp}-\vec{r}_\perp)^2+\lambda^2]^{-1}+[(\vec{k}_{1\perp}+\vec{k}_{2\perp})^2+\lambda^2]^{-1}\}\ . \tag{6.2}$$

Feynman diagrams 5 and 6 are Feynman diagrams 3 and 4 turned upside down. Therefore, the sum is obtained from (6.2) by the replacement $1\leftrightarrow 2$ everywhere, and we find

$$\mathfrak{M}_5^{(6)}+\mathfrak{M}_6^{(6)}\sim\mathfrak{M}_3^{(6)}+\mathfrak{M}_4^{(6)}\ . \tag{6.3}$$

VII. FEYNMAN DIAGRAM 7

Feynman Diagram 7 is given in Fig. 9. The amplitude is given by

$$\mathfrak{M}_7^{(6)}=-\left(\frac{i}{2}\right)^5 g^6\int\frac{d^4k_1}{(2\pi)^4}\,\frac{d^4k_2}{(2\pi)^4}\,N_7\,D_7^{-1}\ , \tag{7.1}$$

where

$$N_7=[\bar{u}(r_2+r_1)\gamma_\mu\tau_a^{(1)}(\not{r}_2-\not{k}_1-\not{k}_2+m)\gamma_\nu\tau_b^{(1)}(\not{r}_2-\not{k}_1+m)\gamma_\lambda\tau_c^{(1)}u(r_2-r_1)]$$
$$\times[\bar{u}(r_3-r_1)\gamma_\sigma\tau_d^{(2)}(\not{r}_3+\not{k}_1+m)\gamma_\lambda\tau_c^{(2)}u(r_3+r_1)]$$
$$\times\epsilon_{adb}[g_{\mu\sigma}(-2k_1-k_2-2r_1)_\nu+g_{\sigma\nu}(k_1-k_2+r_1)_\mu+g_{\nu\mu}(2k_2+k_1+r_1)_\sigma] \qquad (7.2)$$

and

$$D_7=[(r_2-k_1)^2-m^2+i\epsilon][(r_2-k_1-k_2)^2-m^2+i\epsilon]$$
$$\times[(k_1-r_1)^2-\lambda^2+i\epsilon](k_2^2-\lambda^2+i\epsilon)[(k_1+k_2+r_1)^2-\lambda^2+i\epsilon][(k_1+r_1)^2-\lambda^2+i\epsilon][(r_3+k_1)^2-m^2+i\epsilon]\ . \qquad (7.3)$$

We use (2.13) to approximate D_7 as

$$D_7\sim\bar{D}_7=[(-2\omega+k_{1+})k_{1-}-\vec{k}_{1\perp}^{\,2}-m^2+i\epsilon][(-2\omega+k_{1+}+k_{2+})(k_{1-}+k_{2-})-(\vec{k}_{1\perp}+\vec{k}_{2\perp})^2-m^2+i\epsilon]$$
$$\times[k_{1+}k_{1-}-(\vec{k}_{1\perp}-\vec{r}_{1\perp})^2-\lambda^2+i\epsilon][k_{2+}k_{2-}-\vec{k}_{2\perp}^{\,2}-\lambda^2+i\epsilon]$$
$$\times[(k_{1+}+k_{2+})(k_{1-}+k_{2-})-(\vec{k}_{1\perp}+\vec{k}_{2\perp}+\vec{r}_\perp)^2-\lambda^2+i\epsilon]$$
$$\times[k_{1+}k_{1-}-(\vec{k}_{1\perp}+\vec{r}_\perp)^2-\lambda^2+i\epsilon][(2\omega+k_{1-})k_{1+}-\vec{k}_{1\perp}^{\,2}-m^2+i\epsilon]\ . \qquad (7.4)$$

We carry out the k_{1+} and k_{2+} integrals by closing on the poles

$$k_{1+}=C_5(2\omega+k_{1-})^{-1} \qquad (7.5)$$

and

$$k_{2+}=C_6k_{2-}^{-1}\ , \qquad (7.6)$$

where

$$C_5=\vec{k}_{1\perp}^{\,2}+m^2-i\epsilon \qquad (7.7a)$$

and

$$C_6=\vec{k}_{2\perp}^{\,2}+\lambda^2-i\epsilon\ . \qquad (7.7b)$$

Define this time

$$q_1=-k_{1-}-k_{2-} \qquad (7.8a)$$

and

$$q_2=k_{2-}\ . \qquad (7.8b)$$

Then the integral over k_{1+} and k_{2+} gives zero unless (3.8) holds, and we obtain

$$\mathfrak{M}_7^{(6)}\sim+\left(\frac{i}{2}\right)^5g^6(2\omega)^{-1}\int\frac{d^2k_{1\perp}}{2(2\pi)^3}\frac{d^2k_{2\perp}}{2(2\pi)^3}\int dq_1\,dq_2\,N_7\,\tilde{D}_7^{-1}\ , \qquad (7.9)$$

where

$$\tilde{D}_7=q_2[(\vec{k}_{1\perp}-\vec{r}_\perp)^2+\lambda^2][(\vec{k}_{1\perp}+\vec{r}_\perp)^2+\lambda^2][2\omega(q_1+q_2)-\vec{k}_{1\perp}^{\,2}-m^2+i\epsilon][2\omega q_1-(\vec{k}_{1\perp}+\vec{k}_{2\perp})^2-m^2+i\epsilon]$$
$$\times[-q_1C_6q_2^{-1}-(\vec{k}_{1\perp}+\vec{k}_{2\perp}+\vec{r}_\perp)^2-\lambda^2+i\epsilon]\ . \qquad (7.10)$$

We next approximate N_7. First we note that

$$\tau_a^{(1)}\tau_b^{(1)}\tau_c^{(1)}\epsilon_{adb}\tau_d^{(2)}\tau_c^{(2)}=-2i\tau_d^{(1)}\tau_c^{(1)}\tau_d^{(2)}\tau_c^{(2)}$$
$$=-2i(3I^{(1)}I^{(2)}-2\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})\ . \qquad (7.11)$$

Then we approximate $\not{r}_2-\not{k}_1+m$ by $\not{r}_2$ and anticommute $\not{r}_2$ to the left; approximate $\not{r}_2-\not{k}_1-\not{k}_2+m$ by $\not{r}_2$ and anticommute $\not{r}_2$ to the left; and approximate $\not{r}_3+\not{k}_1+m$ by $\not{r}_3$ and anticommute $\not{r}_3$ to the left. Therefore, if we also use (3.19) we obtain

$$N_7\sim-2i(3I^{(1)}I^{(2)}-2\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})\,4s\,m^{-2}\,\delta_{1,1'}\,\delta_{2,2'}\,\bar{N}_7\ , \qquad (7.12)$$

with

$$\bar{N}_7=2\omega^3(-k_{1-}-2k_{2-})=2\omega^3(q_1-q_2)\ . \qquad (7.13)$$

Using (7.12) and (7.10) in (7.9), we find that the only region which contributes the maximum number of logarithms is $0<q_1 \ll q_2 \ll 2\omega$. Therefore, we obtain

$$\mathfrak{M}_7^{(6)} \sim +2^{-6} g^6 m^{-2} \delta_{1,1'} \delta_{2,2'} (3 I^{(1)} I^{(2)} - 2\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}) \times s \tfrac{1}{2}(\ln^2 s - 2\pi i \ln s) \int \frac{d^2 k_{1\perp}}{(2\pi)^3} \frac{d^2 k_{2\perp}}{(2\pi)^3} [(\vec{k}_{1\perp} - \vec{r}_\perp)^2 + \lambda^2]^{-1} [(\vec{k}_{1\perp} + \vec{r}_\perp)^2 + \lambda^2]^{-1} (\vec{k}_{2\perp}{}^2 + \lambda^2)^{-1} . \tag{7.14}$$

VIII. FEYNMAN DIAGRAMS 8–14

Feynman diagram 8 is obtained from diagram 7 by $s \leftrightarrow u$. We thus obtain the amplitude for diagram 8 from (7.14) by the replacements

$$\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)} \rightarrow -\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)} \tag{8.1}$$

and (6.1).

Feynman diagrams 9 and 10 are just diagrams 7 and 8 reversed right for left and hence

$$\mathfrak{M}_7^{(6)} \sim \mathfrak{M}_9^{(6)}, \quad \mathfrak{M}_8^{(6)} \sim \mathfrak{M}_{10}^{(6)} . \tag{8.2}$$

In addition, diagrams 11–14 are equal to diagrams 7–10, respectively. Therefore, we find

$$\sum_{i=7}^{14} \mathfrak{M}_i^{(6)} \sim 2^{-6} g^6 m^{-2} \delta_{1,1'} \delta_{2,2'} 4[s\pi i \ln s (3 I^{(1)} I^{(2)}) + s(\ln^2 s - \pi i \ln s)(2\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)})] \times \int \frac{d^2 k_{1\perp}}{(2\pi)^3} \frac{d^2 k_{2\perp}}{(2\pi)^3} [(\vec{k}_{1\perp} - \vec{r}_\perp)^2 + \lambda^2]^{-1} [(\vec{k}_{1\perp} + \vec{r}_\perp)^2 + \lambda^2] (\vec{k}_{2\perp}{}^2 + \lambda^2)^{-1} . \tag{8.3}$$

IX. SUMMATION OF FEYNMAN DIAGRAMS 1–14

None of the integrals appearing in the asymptotic expansions of Feynman diagrams 1–14 converge when $k_{\perp \max} \rightarrow \infty$. However, when all 14 are added together the cutoff may in fact be removed.

Consider first $\mathfrak{M}_3^{(6)} + \mathfrak{M}_4^{(6)}$ given by (6.2). The integral over $\vec{k}_{1\perp}$ diverges logarithmically. Therefore, in the second term we let

$$\vec{k}_{1\perp} + \vec{k}_{2\perp} \rightarrow \vec{k}_{1\perp} - \vec{r}_\perp \tag{9.1}$$

to obtain

$$\mathfrak{M}_3^{(6)} + \mathfrak{M}_4^{(6)} \sim -2^{-6} g^6 m^{-2} s \delta_{1,1'} \delta_{2,2'} (3 I^{(1)} I^{(2)}) \times 2\pi i \ln s \int \frac{d^2 k_{1\perp}}{(2\pi)^3} \int \frac{d^2 k_{2\perp}}{(2\pi)^3} [(\vec{k}_{2\perp} + \vec{r}_\perp)^2 + \lambda^2]^{-1} [(\vec{k}_{2\perp} - \vec{r}_\perp)^2 + \lambda^2]^{-1} [(\vec{k}_{1\perp} - \vec{r}_\perp)^2 + \lambda^2]^{-1} . \tag{9.2}$$

Thus we find from (9.2), (6.3), and (8.3) that

$$\sum_{i=3}^{14} \mathfrak{M}_i^{(6)} \sim -2^{-6} g^6 m^{-2} s \delta_{1,1'} \delta_{2,2'} 4[\pi i \ln s \, 6 I^{(1)} I^{(2)} + (\ln^2 s - \pi i \ln s) 2\tau^{(1)} \cdot \tau^{(2)}] \times \int \frac{d^2 k_{1\perp}}{(2\pi)^3} \frac{d^2 k_{2\perp}}{(2\pi)^3} [(\vec{k}_{1\perp} - \vec{r}_\perp)^2 + \lambda^2]^{-1} [(\vec{k}_{1\perp} + \vec{r}_\perp)^2 + \lambda^2]^{-1} [(\vec{k}_{2\perp} - \vec{r}_\perp)^2 + \lambda^2]^{-1} . \tag{9.3}$$

The isospin factor in front of the integral is now the same as that in (4.3). Therefore, we add (4.3) and (9.3) to obtain

$$\sum_{i=1}^{14} \mathfrak{M}_i^{(6)} \sim -2^{-6} g^6 m^{-2} \delta_{1,1'} \delta_{2,2'} s[\pi i \ln s \, 6 I^{(1)} I^{(2)} + (\ln^2 s - \pi i \ln s) 2\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}] \times \int \frac{d^2 k_{1\perp}}{(2\pi)^3} \frac{d^2 k_{2\perp}}{(2\pi)^3} [(\vec{k}_{1\perp} - \vec{r}_\perp)^2 + \lambda^2]^{-1} [(\vec{k}_{1\perp} + \vec{r}_\perp)^2 + \lambda^2]^{-1} \times [(\vec{k}_{2\perp} - \vec{r}_\perp)^2 + \lambda^2]^{-1} [(\vec{k}_{2\perp} + \vec{r}_\perp)^2 + \lambda^2]^{-1} (3\lambda^2 + 8\vec{r}_\perp{}^2) . \tag{9.4}$$

X. FEYNMAN DIAGRAM 15

The amplitude for Feynman diagram 15, given by Fig. 10 is

$$\mathfrak{M}_{15}^{(6)} = \left(\frac{i}{2}\right)^6 g^6 \int \frac{d^4k_1}{(2\pi)^4}\,\frac{d^4k_2}{(2\pi)^4}\, N_{15}\, D_{15}^{\;-1}\,, \tag{10.1}$$

where

$$\begin{aligned} N_{15} = {} & [\bar{u}(r_2 - r_1)\,\gamma_\lambda\,\tau_a^{(1)}(\not{r}_2 + \not{r}_1 - \not{k}_1 - \not{k}_2 + m)\,\gamma_\mu\,\tau_b^{(1)}(\not{r}_2 + \not{r}_1 - \not{k}_1 + m)\,\gamma_\nu\,\tau_c^{(1)} u(r_2 + r_1)] \\ & \times [\bar{u}(r_3 + r_1)\,\gamma_\lambda\,\tau_a^{(2)}(\not{r}_3 - \not{r}_1 + \not{k}_1 + \not{k}_2 + m)\,\gamma_\nu\,\tau_c^{(2)}(\not{r}_3 - \not{r}_1 + \not{k}_2 + m)\,\gamma_\mu\,\tau_b^{(2)} u(r_3 - r_1)] \end{aligned} \tag{10.2}$$

and

$$\begin{aligned} D_{15} = {} & [(r_2 + r_1 - k_1)^2 - m^2 + i\epsilon]\,[(r_2 + r_1 - k_1 - k_2)^2 - m^2 + i\epsilon]\,(k_1^{\,2} - \lambda^2 + i\epsilon)\,(k_2^{\,2} - \lambda^2 + i\epsilon) \\ & \times [(k_1 + k_2 + 2r_1)^2 - \lambda^2 + i\epsilon]\,[(r_3 - r_1 + k_2)^2 - m^2 + i\epsilon]\,[(r_3 - r_1 + k_1 + k_2)^2 - m^2 + i\epsilon]\ . \end{aligned} \tag{10.3}$$

We use (2.13) to approximate D_{15} as

$$\begin{aligned} D_{15} \sim \bar{D}_{15} = {} & [(-2\omega + k_{1+})\,k_{1-} - (\vec{r}_\perp - \vec{k}_{1\perp})^2 - m^2 + i\epsilon]\,[(-2\omega + k_{1+} + k_{2+})\,(k_{1-} + k_{2-}) - (\vec{r}_\perp - \vec{k}_{1\perp} - \vec{k}_{2\perp})^2 - m^2 + i\epsilon] \\ & \times (k_{1+}k_{1-} - \vec{k}_{1\perp}^{\,2} - \lambda^2 + i\epsilon)\,[(k_{1+} + k_{2+})\,(k_{1-} + k_{2-}) - (\vec{k}_{1\perp} + \vec{k}_{2\perp} + 2\vec{r}_\perp)^2 - \lambda^2 + i\epsilon] \\ & \times [(2\omega + k_{1-})\,k_{1+} - (\vec{r}_\perp - \vec{k}_{1\perp})^2 - m^2 + i\epsilon]\,[(2\omega + k_{1-} + k_{2-})\,(k_{1+} + k_{2+}) - (\vec{r}_\perp - \vec{k}_{1\perp} - \vec{k}_{2\perp})^2 - m^2 + i\epsilon]\,. \end{aligned} \tag{10.4}$$

The integrals over k_{1+}, k_{2+} will be zero unless

$$k_{1-} < 0, \quad k_{2-} < 0, \quad \text{and } k_{1-} + k_{2-} > -2\omega \tag{10.5a}$$

or

$$k_{1-} > 0, \quad -2\omega < k_{1-} + k_{2-} < 0\ . \tag{10.5b}$$

When (10.5a) holds we close on the poles

$$\begin{aligned} & k_{2+} = C_7\, k_{2-}^{\;-1}\,, \\ & k_{1+} + k_{2+} = C_8 (2\omega + k_{1-} + k_{2-})^{-1}\,, \end{aligned} \tag{10.6a}$$

where

$$C_7 = \vec{k}_{2\perp}^{\,2} + \lambda^2 - i\epsilon$$

and (10.7a)

$$C_8 = (\vec{k}_{1\perp} + \vec{k}_{2\perp} - \vec{r}_\perp)^2 + m^2 - i\epsilon\,,$$

and when (10.5b) holds we close on the poles

$$k_{1+} + k_{2+} = C_9 (k_{1-} + k_{2-})^{-1}$$

and (10.6b)

$$k_{1+} = C_{10} (2\omega + k_{1-})^{-1}\,,$$

where

$$\begin{aligned} & C_9 = (\vec{k}_{1\perp} + \vec{k}_{2\perp} + 2\vec{r}_\perp)^2 + \lambda^2 - i\epsilon\,, \\ & C_{10} = (\vec{k}_{1\perp} - \vec{r}_\perp)^2 + m^2 - i\epsilon\,. \end{aligned} \tag{10.7b}$$

In this latter case the contribution from the pole at

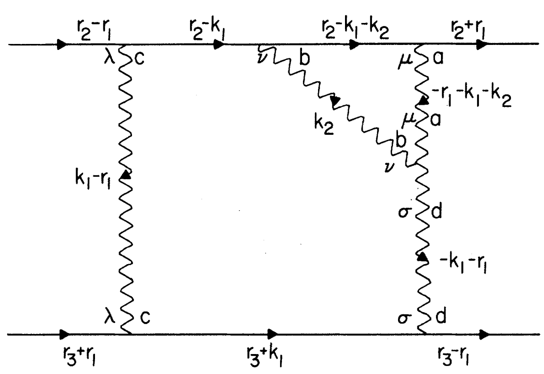

FIG. 9. Feynman diagram 7.

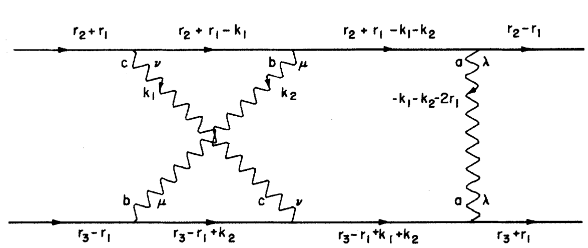

FIG. 10. Feynman diagram 15.

$$k_{1+}+k_{2+}-2\omega=[(\vec{r}_\perp-\vec{k}_{1\perp}-\vec{k}_{2\perp})^2+m^2-i\epsilon](k_{1-}+k_{2-})^{-1} \tag{10.8}$$

is suppressed by the numerator.

For both regions (10.5) we may approximate the numerator by first using

$$\begin{aligned}\tau_a^{(1)}\tau_b^{(1)}\tau_c^{(1)}\tau_a^{(2)}\tau_c^{(2)}\tau_b^{(2)}&=\tau_a^{(1)}\tau_a^{(2)}(3I^{(1)}I^{(2)}+2\tau_b^{(1)}\tau_b^{(2)})\\&=6I^{(1)}I^{(2)}-\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}.\end{aligned} \tag{10.9}$$

Then, dropping $\not r_1$, $\not k_1$, $\not k_2$, and m compared with $\not r_2$ or $\not r_3$ we obtain

$$\begin{aligned}N_{15}\sim\overline{N}_{15}&=(6I^{(1)}I^{(2)}-\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})[\overline{u}(r_2-r_1)\gamma_\lambda\not r_2\gamma_\mu\not r_2\gamma_\nu u(r_2+r_1)][\overline{u}(r_3+r_1)\gamma_\lambda\not r_3\gamma_\nu\not r_3\gamma_\mu u(r_3-r_1)]\\&\sim16(2\omega^2)^4s\tfrac{1}{2}m^{-2}\delta_{1,1'}\delta_{2,2'}(6I^{(1)}I^{(2)}-\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}).\end{aligned} \tag{10.10}$$

Since no k-dependent factors appear in the numerator the rest of this calculation is identical with the sixth-order electrodynamic calculation. In particular,

$$\mathfrak{M}_{15}^{(6)}\sim-\left(\frac{i}{2}\right)^6g^6\overline{N}_{15}(I_a+I_b), \tag{10.11}$$

where

$$I_a=\int\frac{d^2k_{1\perp}}{2(2\pi)^3}\int\frac{d^2k_{2\perp}}{2(2\pi)^3}\int dk_{1-}dk_{2-}D_a^{-1}, \tag{10.12a}$$

with

$$\begin{aligned}D_a=&(2\omega)^2[(\vec{k}_{1\perp}+\vec{k}_{2\perp}+2\vec{r}_\perp)^2+\lambda^2](k_{1-}C_7k_{2-}^{-1}-\vec{k}_{1\perp}^2-\lambda^2)C_7\\&\times[-2\omega k_{1-}-(\vec{r}_\perp-\vec{k}_{1\perp})^2-m^2+i\epsilon][-2\omega(k_{1-}+k_{2-})-(\vec{r}_\perp-\vec{k}_{1\perp}-\vec{k}_{2\perp})^2-m^2+i\epsilon],\end{aligned} \tag{10.13a}$$

and the k_{1-}, k_{2-} integral is over (10.5a), and where

$$I_b=\int\frac{d^2k_{1\perp}}{2(2\pi)^3}\int\frac{d^2k_{2\perp}}{2(2\pi)^3}\int dk_{1-}dk_{2-}D_b^{-1}, \tag{10.12b}$$

with

$$\begin{aligned}D_b=&(2\omega)^2(k_1+k_2)[\vec{k}_{1\perp}^2+\lambda^2][k_{2-}C_9(k_{1-}+k_{2-})^{-1}-\vec{k}_{2\perp}^2-\lambda^2]C_9\\&\times[-2\omega k_{1-}-(\vec{r}_\perp-\vec{k}_{1\perp})^2-m^2+i\epsilon][-2\omega(k_{1-}+k_{2-})-(\vec{r}_\perp-\vec{k}_{1\perp}-\vec{k}_{2\perp})^2-m^2+i\epsilon].\end{aligned} \tag{10.13b}$$

In I_a we use

$$q_{1a}=-k_{1-},\quad q_{2a}=-k_{2-}. \tag{10.14a}$$

The only region which contributes is $0<q_{1a}\ll q_{2a}\ll2\omega$, and hence

$$I_a=-(2\omega)^{-4}\tfrac{1}{2}[\ln^2s-2\pi i\ln s]\int\frac{d^2k_{1\perp}}{2(2\pi)^3}\int\frac{d^2k_{2\perp}}{2(2\pi)^3}(\vec{k}_{1\perp}^2+\lambda^2)^{-1}(\vec{k}_{2\perp}^2+\lambda^2)^{-1}[(\vec{k}_{1\perp}+\vec{k}_{2\perp}+2\vec{r}_\perp)^2+\lambda^2]^{-1}. \tag{10.15a}$$

Similarly, in I_b we use

$$q_{1b}=-k_{1-},\quad q_{2b}=-k_{1-}-k_{2-}. \tag{10.14b}$$

The only contributing region is $0<q_{1b}\ll q_{2b}\ll2\omega$ and hence

$$I_b=(2\omega)^{-4}\tfrac{1}{2}\ln^2s\int\frac{d^2k_{1\perp}}{2(2\pi)^3}\frac{d^2k_{2\perp}}{2(2\pi)^3}(\vec{k}_{1\perp}^2+\lambda^2)^{-1}(\vec{k}_{2\perp}^2+\lambda^2)^{-1}[(\vec{k}_{1\perp}+\vec{k}_{2\perp}+2\vec{r}_\perp)^2+\lambda^2]^{-1}. \tag{10.15b}$$

Therefore,

$$\begin{aligned}\mathfrak{M}_{15}^{(6)}\sim&2^{-7}g^6m^{-2}\delta_{1,1'}\delta_{2,2'}s\pi i\ln s(6I^{(1)}I^{(2)}-\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})\\&\times\int\frac{d^2k_{1\perp}}{(2\pi)^3}\frac{d^2k_{2\perp}}{(2\pi)^3}(\vec{k}_{1\perp}^2+\lambda^2)^{-1}(\vec{k}_{2\perp}^2+\lambda^2)^{-1}[(\vec{k}_{1\perp}+\vec{k}_{2\perp}+2\vec{r}_\perp)^2+\lambda^2]^{-1}.\end{aligned} \tag{10.16}$$

The cutoff $k_{\perp\,\max}$ is not needed here.

XI. FEYNMAN DIAGRAMS 16–18

Feynman diagram 16 is obtained from diagram 15 by a right-left reversal, and it is easily verified that

$$\mathfrak{M}_{16}^{(6)} \sim \mathfrak{M}_{15}^{(6)} . \tag{11.1}$$

Moreover, Feynman diagrams 17 and 18 are the $s \leftrightarrow u$ crossed diagrams obtained from 15 and 16 and these amplitudes are obtained from (10.16) by the replacements

$$\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)} \to -\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)} . \tag{11.2}$$

Therefore,

$$\sum_{i=15}^{18} \mathfrak{M}_i^{(6)} \sim 2^{-5} g^6 m^{-2} \delta_{1,1'} \delta_{2,2'} s \pi i \ln s \, 6 I^{(1)} I^{(2)}$$

$$\times \int \frac{d^2k_{1\perp}}{(2\pi)^3} \frac{d^2k_{2\perp}}{(2\pi)^3} (\vec{k}_{1\perp}^{\,2} + \lambda^2)^{-1} (\vec{k}_{2\perp}^{\,2} + \lambda^2)^{-1} [(\vec{k}_{1\perp} + \vec{k}_{2\perp} + 2\vec{r}_\perp)^2 + \lambda^2]^{-1} . \tag{11.3}$$

XII. FEYNMAN DIAGRAMS 19 AND 20

Feynman diagram 19, Fig. 11, gives the amplitude

$$\mathfrak{M}_{19}^{(6)} = \left(\frac{i}{2}\right)^4 g^6 \lambda^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} N_{19} D_{19}^{-1} , \tag{12.1}$$

where the denominator D_{19} is obtained from D_1 of (3.3) by using M as the mass of the scalar particle of momentum $k_1 - k_2$ and

$$N_{19} = [\bar{u}(r_2 - r_1) \gamma_\lambda \tau_a^{(1)} (\not{r}_2 - \not{k}_1 + m) \gamma_\mu \tau_b^{(1)} u(r_2 + r_1)] \, [\bar{u}(r_3 + r_1) \gamma_\lambda \tau_a^{(2)} (\not{r}_3 + \not{k}_2 + m) \gamma_\mu \tau_b^{(2)} u(r_3 + r_1)] , \tag{12.2}$$

which is approximated as

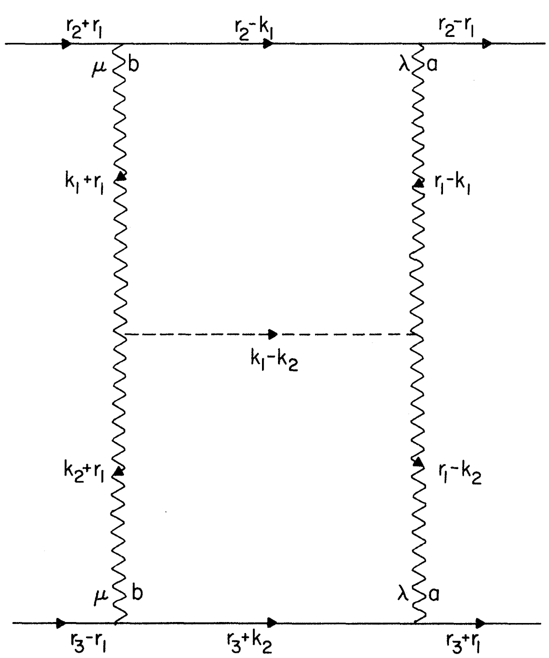

FIG. 11. Feynman diagram 19.

$$N_{19} \sim 4(2\omega^2)\tfrac{1}{2} s m^{-2}\delta_{1,1'}\delta_{2,2'}(3I^{(1)}I^{(2)} - 2\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}) . \quad (12.3)$$

This numerator does not contain any k-dependent factors, and hence we find

$$\mathfrak{M}_{19}^{(6)} \sim -2^{-6} g^6 m^{-2}\delta_{1,1'}\delta_{2,2'} s\tfrac{1}{2}(\ln^2 s - 2\pi i \ln s)\lambda^2(3I^{(1)}I^{(2)} - 2\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})$$
$$\times\int\frac{d^2k_{1\perp}}{(2\pi)}\frac{d^2k_{2\perp}}{(2\pi)^3}[(\vec{k}_{1\perp}-\vec{r}_\perp)^2+\lambda^2]^{-1}[(\vec{k}_{1\perp}+\vec{r}_\perp)^2+\lambda^2]^{-1}[(\vec{k}_{2\perp}-\vec{r}_\perp)^2+\lambda^2]^{-1}[(\vec{k}_{2\perp}+\vec{r}_\perp)^2+\lambda^2]^{-1} . \quad (12.4)$$

Here again the cutoff $k_{\perp\,\max}$ is not needed.

Feynman diagram 20 is obtained from Feynman diagram 19 by $s \leftrightarrow u$ crossing. Therefore,

$$\mathfrak{M}_{19}^{(6)} + \mathfrak{M}_{20}^{(6)} \sim 2^{-6} g^6 m^{-2}\delta_{1,1'}\delta_{2,2'} s[\pi i \ln s(3I^{(1)}I^{(2)}) + (\ln^2 s - \pi i \ln s)2\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}]$$
$$\times\lambda^2\int\frac{d^2k_{1\perp}}{(2\pi)^3}\frac{d^2k_{2\perp}}{(2\pi)^3}[(\vec{k}_{1\perp}-\vec{r}_\perp)^2+\lambda^2]^{-1}[(\vec{k}_{1\perp}+\vec{r}_\perp)^2+\lambda^2]^{-1}[(\vec{k}_{2\perp}-\vec{r}_\perp)^2+\lambda^2]^{-1}[(\vec{k}_{2\perp}+\vec{r}_\perp)^2+\lambda^2]^{-1} . \quad (12.5)$$

The final result (1.1) is obtained by combining (9.4), (11.3), and (12.5).

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