Scattering in parabolic coordinates and a new representation for the scattering amplitude

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We derive and investigate an exact integral representation of the scattering amplitude that results from the description of scattering in parabolic coordinates. The spectral function $a(s,\nu)$ of this representation turns out to be an entire function of order 1 in the ν variable provided that the partial-wave amplitude $A_l \sim l^{-1/2}e^{-l\xi}$, $\xi > 0$ for l large. We also briefly discuss the counterpart of the partial-wave series.

I. INTRODUCTION

The usual derivation of the scattering amplitude using spherical coordinates leads to the partial-wave expansion and, by means of a Sommerfeld-Watson transformation, to the Regge representation and Regge poles. Khuri[1] investigated crossing-symmetric power-series expansions and the related Sommerfeld-Watson representations of the scattering amplitude and found, in addition to Regge poles, "satellite" poles. He remarked[2] that any expansion in a set of polynomials $\phi_n(z)$ which are such that $\phi_n(z)\sim z^n$ as $z\to\infty$ would give rise to at least the set of Regge poles; thus, under this assumption the angular momentum l is the variable in which we have the least number of poles.

In this paper[3] we derive rigorously and investigate an exact representation of the scattering amplitude that results from the description of scattering in parabolic coordinates. This investigation was suggested by the facts that scattering by a central potential is axially symmetric, that Coulomb scattering is most naturally described using parabolic coordinates, and that at high energy the Yukawa potential becomes Coulomb-like. We deduce an integral representation [Eq. (1)] and a corresponding series expansion [Eq. (32)] related by a Sommerfeld-Watson transformation.

Unlike the Regge representation and the Sommerfeld-Watson representations of Khuri,[1] our integral representation (1) has no associated poles or cuts. In fact, we show that the spectral function $a(s, \nu)$ is *entire* in the ν plane. This does not contradict Khuri's remark,[2] since our expansion turns out to be not in terms of polynomials, but rather a power series in the variable $\tau=(1-z)/(1+z)$. In addition, we show that $a(s, \nu)\sim\xi^{\nu}$; therefore, the integral in (1) converges rapidly (exponentially). Finally, the angular dependence is simple and explicit. For these reasons we believe that representation (1) is a useful alternative, valid at *all* angles, to the various impact-parameter representations.[4]

In Sec. II we first give a derivation of the integral representation without the use of parabolic coordinates. Then we prove that $a(s, \nu)$ is an entire function in the variable ν, provided that the partial-wave amplitude A_l has the asymptotic behavior as $l\to\infty$, $A_l\sim l^{-1/2}e^{-l\xi}$, $\xi>0$. We also show that as $\nu\to\infty$, $a(s, \nu)\sim\xi^{\nu}$; that is, a is of order 1. Hence, we can use Hadamard's factorization theorem to write representation (22). Finally, we derive elastic unitarity in terms of $a(s, \nu)$.

In Sec. III we obtain the series expansion of the scattering amplitude by performing the inverse of a Sommerfeld-Watson transformation.

The connection of representation (1) and the description of scattering in terms of parabolic coordinates is discussed in Sec. IV. Here we also discuss, as an example, the Coulomb potential.

In Appendix A we derive an asymptotic representation for the Bateman functions F_l that appear in some of our equations for $a(s, \nu)$. Although we have made no use of this asymptotic representation, we give it because we are not aware of its existence in the published literature.

Appendixes B and C reproduce some useful definitions and relations regarding Bateman and Buchholz functions, respectively.

II. INTEGRAL REPRESENTATION OF THE SCATTERING AMPLITUDE

A. Derivation of the representation

We begin by deriving the following integral representation for the scattering amplitude A:[5]

$$A(s,\tau)=(1+\tau)\frac{i}{2}\int_{-1/2-i\infty}^{-1/2+i\infty}\frac{\tau^{\nu}a(s,\nu)d\nu}{\sin(\pi\nu)}, \quad (1)$$

where

$$a(s,\nu)=\sum_{l=0}^{\infty}(2l+1)A_l(s)F_l(-2\nu-1), \quad (2)$$

and the functions $F_l(-2\nu-1)$ are Bateman polynomials for integer l.[6] We can obtain (1) by substituting the representation[7]

$$P_l\left(\frac{1-\tau}{1+\tau}\right)=(1+\tau)\frac{i}{2}\int_{-1/2-i\infty}^{-1/2+i\infty}\frac{\tau^{\nu}F_l(-2\nu-1)d\nu}{\sin(\pi\nu)}, \quad (3)$$

for the Legendre functions P_l in the partial-wave series and interchanging the order of summation and integration. We wish to show now that this interchange is rigorously permitted. We begin by rewriting (1) for convenience as a Fourier transform using the change of variable $\nu=-\frac{1}{2}(1+iy)$; thus,

$$A=\frac{(1+\tau)}{4\sqrt{\tau}}\int_{-\infty}^{\infty}\frac{\tau^{-iy/2}\sum(2l+1)A_lF_l(iy)dy}{\cosh(\frac{1}{2}\pi y)}. \quad (4)$$

According to a well-known theorem,[8] the interchange of the order of summation and integration is permitted if

$$\sum_{l=0}^{\infty}\int_{-\infty}^{\infty}\left|(2l+1)A_l(s)\tau^{-iy/2}\frac{F_l(iy)}{\cosh(\frac{1}{2}\pi y)}\right|dy<\infty. \quad (5)$$

We shall make use of the following two bounds:

Bound I.

$$\frac{|F_l(iy)|}{\cosh(\frac{1}{2}\pi y)}\leq 1, \quad \text{for all real } y.$$

Proof. We have the representation[6]

$$\frac{F_l(iy)}{\cosh(\frac{1}{2}\pi y)}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-ixy}}{\cosh x}P_l(\tanh x)dx. \quad (6)$$

Therefore,

$$\frac{|F_l(iy)|}{\cosh(\frac{1}{2}\pi y)}\leq\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|P_l(\tanh x)|}{\cosh x}dx \leq\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dx}{\cosh x}=1,$$

where we used the fact that for x real $-1\leq\tanh x\leq+1$, and, consequently, $|P_l(\tanh x)|\leq 1$ (l integer).

Bound II.

$$\frac{|F_l(iy)|}{\cosh(\frac{1}{2}\pi y)}\leq\frac{p(l)}{y^2}, \quad \text{for all real } y,$$

where $p(l)=2l^2+\frac{9}{2}l+\frac{3}{2}$.

Proof. Our first step is to integrate by parts representation (6) twice. Making use of the relation[9]

$$P_l'=\frac{-l\tanh x P_l(\tanh x)+lP_{l-1}(\tanh x)}{1-\tanh^2 x},$$

we get

$$\frac{F_l(iy)}{\cosh(\frac{1}{2}\pi y)}=\frac{-1}{\pi y^2}\int_{-\infty}^{\infty}e^{-ixy}\left\{[(l+1)^2\tanh^2x-l^2]\frac{P_l(\tanh x)}{\cosh x}-2l\tanh x\frac{P_{l-1}(\tanh x)}{\cosh x}-(l+1)\frac{P_l(\tanh x)}{\cosh^3 x}\right\}dx.$$

As above, $|\tanh x|\leq 1$, and $|P_l(\tanh x)|\leq 1$, so that

$$\frac{|F_l(iy)|}{\cosh(\frac{1}{2}\pi y)}\leq\frac{1}{\pi y^2}\int_{-\infty}^{\infty}\left[\frac{(l+1)^2+l^2+2l}{\cosh x}+\frac{l+1}{\cosh^3 x}\right]dx=\frac{p(l)}{y^2}.$$

Now we proceed with the proof of (5). Since[6]

$$F_l(-w)=(-1)^lF_l(w), \quad (7)$$

the integrand in (5) is even; thus we consider the convergence of

$$\sum_{l=0}^{\infty}(2l+1)|A_l|\int_0^{\infty}\frac{|F_l(iy)|}{\cosh(\frac{1}{2}\pi y)}dy=\sum_{l=0}^{\infty}(2l+1)|A_l|\left[\int_0^a\frac{|F_l(iy)|}{\cosh(\frac{1}{2}\pi y)}dy+\int_a^{\infty}\frac{|F_l(iy)|}{\cosh(\frac{1}{2}\pi y)}dy\right] \quad (a>0). \quad (8)$$

Substituting bound I and bound II, respectively, in the first and second integrals of the right-hand side of Eq. (8), we have

$$\sum_{l=0}^{\infty}(2l+1)|A_l|\int_0^{\infty}\frac{|F_l(iy)|}{\cosh(\frac{1}{2}\pi y)}\,dy \le \sum_{l=0}^{\infty}(2l+1)|A_l|\left[a+\frac{p(l)}{a}\right]<\infty,$$

assuming, as usual, that for large l (Ref. 10)

$$A_l(s)\sim h(s)l^{-1/2}e^{-l\xi} \quad (\xi>0). \tag{9}$$

B. Properties of $a(s,\nu)$

A remarkable feature of the integral representation (1) is that, *if (9) holds, the spectral function $a(s,\nu)$ is an entire function on the ν plane.* To prove this statement we will use the following theorem[11]:

Theorem. Suppose that the functions

$$\{w_l(\nu)\,|\,l=0,1,2,\ldots\}$$

are analytic in a fixed domain D, and suppose that the series

$$\sum_{l=0}^{\infty}w_l(\nu)\equiv W(\nu)$$

converges uniformly in any compact subset of D; then $W(\nu)$ is analytic in D, except possibly at $\nu=\infty$ if ∞ belongs to D. The series may be differentiated term by term as often as we please. The p-times-differentiated series converges to $W^{(p)}(\nu)$ uniformly on compact sets.

Thus, we have to prove that series (2) converges uniformly everywhere on the finite ν plane; to do this we will apply the Weierstrass M test.

It was shown by Rice[7] that for l a positive integer, and $l\gg\max\{1,|\nu|\}$,

$$F_l(-2\nu-1)=-\frac{\sin(\pi\nu)}{\pi}\times\left[\frac{\Gamma(-\nu)}{\Gamma(\nu+1)}l^{2\nu}+(-1)^l\frac{\Gamma(\nu+1)}{\Gamma(-\nu)}l^{-2\nu-2}\right]\times\left[1+O\left(\frac{1}{l^2}\right)\right]. \tag{10}$$

Let us consider for simplicity a circular domain D on the ν plane centered about $\nu=0$; then, using the maximum modulus theorem, we see that from some l onwards

$$(2l+1)|A_l|\,|F_l(-2\nu-1)|\le M_l, \quad \text{for all } \nu\in D,$$

where

$$M_l=(2l+1)|A_l|\left[2\max_{|\nu|=r}\left(\left|\frac{\sin(\pi\nu)}{\pi}\frac{\Gamma(-\nu)}{\Gamma(\nu+1)}\right|l^{2|\nu|}\right)\right], \tag{11}$$

and $|\nu|=r$ is the circular boundary of D. Now, as long as (9) holds, it is evident that $\sum_{l=0}^{\infty}M_l$ converges, and therefore series (2) converges uniformly in D. Since we can repeat these arguments for arbitrary circular domains D with boundary $|\nu|=r<\infty$, we have fulfilled the conditions of our theorem, and under assumption (9) $a(s,\nu)$ is entire in ν.

A very important property that characterizes an entire function is its rate of growth for large values of its variable. To deduce the asymptotic behavior of $a(s,\nu)$ as $|\nu|\to\infty$, we shall make use of a number of known results, which we present below for the sake of completeness.

(a) For l positive integer (or zero)[6]

$$F_l(w)>0 \quad \text{if } w<0. \tag{12}$$

(b) Bateman[6] found that the coefficients in the power series for $F_{2n}(w)$ and $-F_{2n+1}(w)$ are all positive. Thus, we write

$$F_l(w)=(-1)^l(a_0+a_1w+\cdots+a_lw^l), \quad a_i\ge 0.$$

Furthermore, Eq. (7) implies that for even l we have only even powers and a constant, while for odd l we have only odd powers and no constant. If we let $w=-(2\nu+1)$, we have

$$F_l(-2\nu-1)=b_0+b_1\nu+\cdots+b_l\nu^l, \quad b_i\ge 0.$$

Let $\nu=re^{i\theta}$, then

$$|F_l(-2\nu-1)|=\left|\sum_{n=0}^{l}b_nr^ne^{in\theta}\right|\le\sum_{n=0}^{l}b_nr^n.$$

Note that the equality holds for $\theta=0$; therefore,

$$\max_{|\nu|=r}|F_l(-2\nu-1)|=|F_l(-2r-1)|. \tag{13}$$

(c) Again from Bateman[6] we have

$$\sum_{l=0}^{\infty}(2l+1)Q_l(p)F_l(w)=\frac{1}{p-1}\left(\frac{p-1}{p+1}\right)^{(w+1)/2}, \tag{14}$$

valid for all w and $p\ne -1,+1$.

(d) For l large and positive the Legendre function of the second kind is given by[9]

$$Q_l(z)=\left(\frac{\pi}{2}\right)^{1/2}\frac{[z+(z^2-1)^{1/2}]^{-1/2}}{(z^2-1)^{1/4}}\frac{e^{-l\xi}}{\sqrt{l}}\times[1+O(l^{-1})], \tag{15}$$

uniformly for $1<z<\infty$, where $\xi=\ln[z+(z^2-1)^{1/2}]$; the domain of validity of (15) can, of course, be extended to complex z and l.[12]

It follows from (15) that for sufficiently large l,

$$Q_l(z)>0. \tag{16}$$

To derive an asymptotic bound for $|a(s,\nu)|$ as $|\nu|\to\infty$, we use (2) and write

$$|a(s,\nu)| = \left|\sum_{l=0}^{\infty}(2l+1)A_l(s)F_l(-2\nu-1)\right|$$
$$\leq \sum_{l=0}^{\infty}(2l+1)|A_lF_l|$$
$$= \sum_{l=0}^{L-1}(2l+1)|A_lF_l| + \sum_{l=L}^{\infty}(2l+1)|A_lF_l|. \quad (17)$$

Let us assume for the moment that the nearest z-plane singularity of the scattering amplitude is z_0. Then for sufficiently large l, and s physical, we have[13]

$$|A_l(s)| \leq q(s)Q_l(|z_0|), \quad (18)$$

where we made use of (16), and $q(s)$ is some positive function of s. Substituting (18) in (17), provided that L is sufficiently large, we obtain

$$|a(s,\nu)| \leq \sum_{l=0}^{L-1}(2l+1)|A_lF_l| + q(s)\sum_{l=L}^{\infty}(2l+1)Q_l(|z_0|)|F_l|$$
$$= \sum_{l=0}^{L-1}(2l+1)[|A_l| - q(s)Q_l(|z_0|)]|F_l| + q(s)\sum_{l=0}^{\infty}(2l+1)Q_l(|z_0|)|F_l(-2\nu-1)|. \quad (19)$$

The first sum in (19) is a polynomial in ν of degree $L-1$. Consider the last sum in (19) and let $\nu>0$; then from (12) and (14) we have that

$$\sum_{l=0}^{\infty}(2l+1)Q_l(|z_0|)|F_l(-2\nu-1)| = \sum_{l=0}^{\infty}(2l+1)Q_l(|z_0|)F_l(-2\nu-1) = \frac{1}{|z_0|-1}\left(\frac{|z_0|+1}{|z_0|-1}\right)^{\nu}. \quad (20)$$

It follows from (13) and (20) that for $\nu\to\infty$ along any ray from the origin on the ν plane,

$$|a(s,\nu)| \leq R(s)\left(\frac{|z_0|+1}{|z_0|-1}\right)^{|\nu|}, \quad (21)$$

where $R(s)$ is a positive function of s.[14]

Since $a(s,\nu)$ is an entire function of ν of order 1, it follows from Hadamard's factorization theorem[15] that $a(s,\nu)$ can be represented as

$$a(s,\nu) = a(s,0)e^{c(s)\nu}P\left(\frac{\nu}{\nu_n}\right), \quad (22)$$

where $P(\nu/\nu_n)$ is a canonical product of order $\rho\leq 1$ (if $c=0$, then $\rho=1$) formed with the ν plane zeros ν_n of $a(s,\nu)$.[16] We would expect the exact $a(s,\nu)$ to have an infinite number of ν plane zeros, some of which will be functions of s.[3] The contribution of Regge amplitudes to $a(s,\nu)$ was discussed in Ref. 3, where it was found that a t-channel Regge pole gave rise in $a(s,\nu)$ to the left real axis set of zeros of $1/\Gamma(\nu+1)$, while a u-channel Regge pole gave rise to the right real axis set of zeros of $1/\Gamma(-\nu)$.

C. Elastic unitarity

Before closing this section we shall discuss briefly unitarity in terms of $a(s,\nu)$. We use again the change of variable $\nu=-\frac{1}{2}(1+iy)$, which gives $F_l(-2\nu-1)=F_l(iy)$; thus

$$a(s,-\tfrac{1}{2}(1+iy)) = \sum_{l=0}^{\infty}(2l+1)A_l(s)F_l(iy). \quad (23)$$

From the discussion leading to Eq. (13) we have that $F_l(iy)$ is real for even l and pure imaginary for odd l; therefore,

$$F_l^*(-iy) = F_l(iy), \quad (24)$$

and

$$a^*(s,-\tfrac{1}{2}(1-iy)) = \sum_{l=0}^{\infty}(2l+1)A_l^*F_l^*(-iy) = \sum_{l=0}^{\infty}(2l+1)A_l^*F_l(iy). \quad (25)$$

Bateman[17] obtained the orthogonality relation

$$\int_{-\infty}^{\infty}\frac{F_l(iy)F_n(iy)dy}{\cosh^2(\frac{1}{2}\pi y)} = \frac{4\delta_{ln}}{\pi(2n+1)}; \quad (26)$$

therefore, using (25) and (26), we have

$$\int_{-\infty}^{\infty}\frac{a(s,-\frac{1}{2}(1+iy))a^*(s,-\frac{1}{2}(1-iy))dy}{\cosh^2(\frac{1}{2}\pi y)} = \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}(2l+1)A_l(s)(2n+1)A_n^*(s)\int_{-\infty}^{\infty}\frac{F_l(iy)F_n(iy)dy}{\cosh^2(\frac{1}{2}\pi y)}$$
$$= \frac{4}{\pi}\sum_{l=0}^{\infty}(2l+1)|A_l(s)|^2, \quad (27)$$

or, in terms of the variable ν,

$$\int_{-1/2-i\infty}^{-1/2+i\infty} \frac{a(s,\nu)a^*(s,-\nu-1)d\nu}{\sin^2(\pi\nu)} = \frac{2i}{\pi}\sum_{l=0}^{\infty}(2l+1)|A_l(s)|^2. \quad (28)$$

Since $F_l(-1)=1$, we write

$$a(s_\pm,0)=\sum_{l=0}^{\infty}(2l+1)A_l(s_\pm), \quad (29)$$

where $s_\pm = s\pm i\epsilon$. From (28) and (29) we can now express elastic s-channel unitarity in terms of $a(s,\nu)$ by the relation

$$a(s_+,0)-a(s_-,0) = \frac{\pi k}{\sqrt{s}}\int_{-1/2-i\infty}^{-1/2+i\infty}\frac{a(s,\nu)a^*(s,-\nu-1)}{\sin^2(\pi\nu)}d\nu. \quad (30)$$

III. SERIES EXPANSION OF THE SCATTERING AMPLITUDE

From the asymptotic behavior of $a(s,\nu)$ [Eq. (21)] we see that we can close the contour of representation (1) to the left or to the right, thus obtaining essentially a power-series expansion in τ for the scattering amplitude.

Let us suppose that for $|\nu|\to\infty$,[14]

$$a(s,\nu)\sim f(s)[\xi(s)]^{\nu}. \quad (31)$$

Then, if $\tau\xi<1$, we close the contour on the right and obtain

$$A(s,\tau)=(1+\tau)\sum_{n=0}^{\infty}(-\tau)^n a_n(s) = a_0+\sum_{n=1}^{\infty}(-1)^n(a_n-a_{n-1})\tau^n, \quad (32)$$

where

$$a_n(s)=\sum_{l=0}^{\infty}(2l+1)A_l(s)F_l(-2n-1), \quad (33)$$

and[6]

$$F_l(-2n-1)=1+\binom{n}{1}\frac{l(l+1)}{(1!)^2}+\binom{n}{2}\frac{(l-1)l(l+1)(l+2)}{(2!)^2}+\cdots. \quad (34)$$

Similarly, if $\tau\xi>1$, we close the contour on the left and obtain

$$A(s,\tau)=-(1+\tau)\sum_{n=-1}^{-\infty}(-\tau)^n a_n(s) = a_{-1}+\sum_{n=-1}^{-\infty}(-1)^n(a_{n-1}-a_n)\tau^n, \quad (35)$$

with $a_n(\nu)$ given again by (33). In this case we can make use of (7) to transform the F_l in (35) so that (34) is still applicable. [In (32) and (35), a_0 and a_{-1} are respectively the forward and backward scattering amplitudes.] Bateman[6] and Pasternack[18] give the relation

$$F_l(w-2)-F_l(w) = l(l+1)\,{}_3F_2(-l+1,\, l+2,\, \tfrac{1}{2}(w+1);\; 2,2;\; 1), \quad (36)$$

which is good also for noninteger l, and in that case converges for $\mathrm{Re}\,w<1$. We can use this to rewrite (a_n-a_{n-1}) in (32) and (35).

Series (32) may also be obtained by substituting[6]

$$P_l\left(\frac{1-\tau}{1+\tau}\right)=(1+\tau)\sum_{n=0}^{\infty}(-\tau)^n F_l(-2n-1), \quad |\tau|<1 \quad (37)$$

in the partial-wave series and interchanging the order of the two summations, keeping in mind the two cases $\tau\xi\gtrless 1$; [although series (37) does not converge at $\tau=1$, series (32) does converge there if $\xi<1$, while (35) converges there if $\xi>1$].

IV. CONNECTION WITH POTENTIAL SCATTERING IN PARABOLIC COORDINATES

The scattering solutions of the Schrödinger equation for a spherically symmetric potential are required to have the following asymptotic behavior at infinity:

$$\psi\sim e^{ikz}+\frac{e^{ikr}}{r}f(k^2,\cos\theta), \quad (38)$$

where $f(k^2,\cos\theta)$ is the scattering amplitude.

We introduce the parabolic coordinates defined by the formulas

$$\xi=r+z,\quad \eta=r-z,\quad \phi=\tan^{-1}(y/x); \quad (39)$$

ξ and η take values from 0 to ∞, and ϕ from 0 to 2π.

In a rigorous treatment of the scattering problem in parabolic coordinates, it is natural to assume an expansion of the wave function $\psi(\xi,\eta)$[19] in terms of combinations of Buchholz functions[20] (see Appendix C) of the type $u_\alpha(-ik\xi)u_\beta(-ik\eta)$, where u_γ stands for any of the two functions w_γ and m_γ. Since, however, we are mainly interested in the form of the scattering amplitude, we can make the following heuristic arguments.

Let

$$\psi(\xi,\eta)-e^{ik(\xi-\eta)/2}$$
$$=k\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{a(k^2,\nu)}{\sin(\pi\nu)}w_{-\nu-1/2}(-ik\xi)w_{\nu+1/2}(-ik\eta)d\nu, \tag{40}$$
$$-1<\sigma<0.$$

We have chosen the combination $w_{-\nu-1/2}(-ik\xi)w_{\nu+1/2}(-ik\eta)$ because it is the only simple combination which leads to the asymptotic condition (38). In other words, although representation (40) may not be valid for all η and ξ, if other terms are present they must tend to zero faster than $1/r$.

Using the asymptotic expansions (C4), we have

$$w_{-\nu-1/2}(-ik\xi)w_{\nu+1/2}(-ik\eta)\underset{\substack{\xi\to\infty\\ \eta\to\infty}}{\sim}\left(\frac{i}{2k}\right)\frac{e^{ikr}}{r}\tau^{\nu}(1+\tau), \tag{41}$$

where $\tau=(\eta/\xi)=(1-\cos\theta)/(1+\cos\theta)$. Substituting (41) in (40), we obtain

$$f(k^2,\tau)=(1+\tau)\frac{i}{2}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\tau^{\nu}a(k^2,\nu)d\nu}{\sin(\pi\nu)},\quad -1<\sigma<0, \tag{42}$$

which is essentially representation (1). The $\sin(\pi\nu)$ appearing in the denominator in (40) and (42) was introduced by comparison with (C5); in this way if the function a is independent of ν, then f is independent of τ.

As an example, we now evaluate $a(s,\nu)$ for the case of the Coulomb potential. We consider an attractive Coulomb potential and let $s\equiv k^2$, $[t=-2k^2(1-\cos\theta)$ as usual$]$; then

$$f(s,\tau)=-\frac{e^2m}{2\hbar^2 s}\frac{\Gamma(-\alpha)}{\Gamma(2+\alpha)}\left(\frac{\tau}{1+\tau}\right)^{\alpha}, \tag{43}$$

where

$$\frac{\tau}{1+\tau}=-\frac{t}{4s},\ \text{and}\ \alpha=-1+\frac{ie^2m}{\hbar^2\sqrt{s}}.$$

Series (2) does not converge for the Coulomb case, so we shall obtain $a(s,\nu)$ by inverting (42). We thus obtain the Mellin transform

$$a(s,\nu)=-\frac{\sin(\pi\nu)}{\pi}\int_0^{\infty}\frac{f(s,\tau)d\tau}{\tau^{\nu+1}(1+\tau)},\quad -1<\mathrm{Re}\nu<0. \tag{44}$$

Substituting (43) in (44), we note that the integral does not converge, so we have to introduce a suitable convergence parameter. We let $\mathrm{Re}\alpha=-1+\epsilon$ such that $\mathrm{Re}\alpha+1>\mathrm{Re}\alpha-\mathrm{Re}\nu>0$. Performing the integration, we obtain

$$a(s,\nu)=\left(\frac{i}{2\sqrt{s}}\right)\frac{\Gamma(-\alpha)}{[\Gamma(\alpha+1)]^2}\ \frac{\Gamma(\alpha-\nu)}{\Gamma(-\nu)}. \tag{45}$$

This is exact, and if substituted in (42), it will give us back (with the use of a convergence parameter) Eq. (43).

Unlike the short-range interaction case for which relation (9) holds, in the Coulomb case $a(s,\nu)$ is meromorphic in the ν plane with poles at $\nu=n+\alpha$, $n=0,1,2,\ldots$. These correspond to the set of Regge poles. It is interesting to note that the zeros of $a(s,\nu)$ coming from the $1/\Gamma(-\nu)$ factor are the same as the zeros coming from a u-channel Regge-pole contribution to $a(s,\nu)$.[3] This seems to confirm our belief[3] that right-hand zeros of $a(s,\nu)$ are associated with backward (large momentum transfer) scattering, while left-hand zeros are associated with forward (small momentum transfer) scattering.

In conclusion, we emphasize that in this paper our intention has been to derive rigorously representations (1), (32), (35), and the main properties of $a(s,\nu)$. More needs to be learned about $a(s,\nu)$, for example, the distribution and movement of the zeros in the ν plane. To this end a number of theorems dealing with the zeros of entire functions can be found in the mathematics literature. Also much can be learned from specific examples such as the Regge-pole cases of Ref. 3. From such studies phenomenological representations of $a(s,\nu)$ can be written and tested with the high-energy data.

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APPENDIX A: AN ASYMPTOTIC REPRESENTATION FOR THE BATEMAN FUNCTION F_l

We shall derive here an asymptotic representation of $F_l(-2\nu-1)$ for large $|\nu|$.

The representation

$$F_l(-2\nu-1)={}_3F_2(-l,\,l+1,\,-\nu;\ 1,1;\ 1), \tag{A1}$$

where ${}_3F_2$ is the generalized hypergeometric function, was given by Bateman[6] and is valid for arbitrary l, but $\mathrm{Re}\nu>-1$ (unless l is integer). Using a generalization of Dixon's theorem,[21] we obtain the following relation:

$$F_l(-2\nu-1)=\frac{\Gamma(\nu+1)}{\Gamma(-l)\Gamma(\nu+l+2)}\times{}_3F_2(l+1,\,l+1,\,\nu+1;\ \nu+l+2,\,1;\ 1), \tag{A2}$$

and we must have $\mathrm{Re}\nu>-1$ and $\mathrm{Re}\,l<0$ in order that the series be convergent. Thus

$$F_l(-2\nu-1)=\frac{\Gamma(\nu+1)}{\Gamma(-l)\Gamma(\nu+l+2)}\left[1+\sum_{n=1}^{\infty}\left(\frac{\Gamma(l+n+1)}{\Gamma(l+1)}\right)^2\frac{1}{(n!)^2}\left\{\frac{(\nu+1)(\nu+2)\cdots(\nu+n)}{(\nu+l+2)(\nu+l+3)\cdots(\nu+l+n+1)}\right\}\right]. \quad \text{(A3)}$$

Clearly, for $|\nu|\gg|l|$ and $\text{Re}\nu+2>-\text{Re}l$ (a condition which is satisfied as $|\nu|\to\infty$ along a ray from the origin), the factors in the curly brackets are $O(1)$, while

$$\sum_{n=0}^{\infty}\left(\frac{\Gamma(l+n+1)}{\Gamma(l+1)}\right)^2\frac{1}{(n!)^2}=F(l+1,\,l+1;\,1;\,1),$$

and[22] (A4)

$$F(l+1,\,l+1;\;1;\;1)=\frac{\Gamma(-2l-1)}{[\Gamma(-l)]^2},\quad \text{Re}l<-\tfrac{1}{2}.$$

Hence, we can write

$$F_l(-2\nu-1)\sim\frac{\Gamma(-2l-1)}{[\Gamma(-l)]^3}\,\nu^{-l-1}, \quad \text{(A5)}$$

for $|\nu|\gg|l|$, $\text{Re}\nu>-1$, $\text{Re}l<-\frac{1}{2}$.

The same relation (Dixon's theorem)[21] that gave us (A2) can be used to obtain

$$F_l(-2\nu-1)=\frac{\Gamma(\nu+1)}{\Gamma(l+1)\Gamma(\nu-l+1)}\times{}_3F_2(-l,\,-l,\,\nu+1;\;1,\,\nu-l+1;\;1), \quad \text{(A6)}$$

where now we must require $\text{Re}\nu>-1$, $\text{Re}l>-1$, in order that the series be convergent. Following the steps leading to (A5), we obtain this time

$$F_l(-2\nu-1)\sim\frac{\Gamma(2l+1)}{[\Gamma(l+1)]^3}\,\nu^{l}, \quad \text{(A7)}$$

for $|\nu|\gg|l|$, $\text{Re}\nu>-1$, $\text{Re}l>-\frac{1}{2}$.

Combining (A5) and (A7), we can write

$$F_l(-2\nu-1)\sim\frac{\Gamma(2l+1)}{[\Gamma(l+1)]^3}\,\nu^{l}+\frac{\Gamma(-2l-1)}{[\Gamma(-l)]^3}\,\nu^{-l-1}. \quad \text{(A8)}$$

APPENDIX B: RELATIONS INVOLVING THE BATEMAN FUNCTION F_l

In this appendix we summarize for easy access a number of useful results involving the Bateman functions.

First, in addition to (A1), we have representations (B1) and (B2).

$$F_l(-2\nu-1)=\frac{1}{2\pi i}\int_L(1-t)^{-\nu-1}P_l\left(\frac{2}{t}-1\right)\frac{dt}{t}, \quad \text{(B1)}$$

$$\text{Re}\nu>-1,$$

where the contour L may be any contour starting and terminating at infinity that can be deformed into the straight line joining $\frac{1}{2}-i\infty$ and $\frac{1}{2}+i\infty$ without passing over the points $t=0$ and $t=1$.[7] This representation can be shown to be valid for all l.

If l is integer [cf. Eq. (34)] we have[6]

$$F_l(w)=1-\frac{l(l+1)}{(1!)^2}\,\frac{w+1}{2}+\frac{(l-1)l(l+1)(l+2)}{(2!)^2}\,\frac{(w+1)(w+3)}{2\times4}-\frac{(l-2)(l-1)l(l+1)(l+2)(l+3)}{(3!)^2}\,\frac{(w+1)(w+3)(w+5)}{2\times4\times6}+\cdots \quad \text{(B2)}$$

Carlitz[23] has given a simple connection between the Bateman polynomials and some polynomials of Touchard[24] related to the Bernoulli numbers. Next we list a number of series involving F_l which, like (14), can be summed exactly.

Rice[7] gives

$$\left(\frac{2r}{\pi}\right)^{1/2}e^{ir}\,\Phi(-\nu,\,1;\;-2ir)=\sum_{l=0}^{\infty}(2l+1)i^{l}J_{l+1/2}(r)F_l(-2\nu-1), \quad \text{(B3)}$$

where Φ is the confluent hypergeometric function.[9]

Series (B3) appears to hold for all r and ν, although its proof required either $\text{Re}\nu<0$ or $\text{Re}\nu>-1$. Bateman[25] gives

$$\left(\frac{u+v}{u-v}\right)^{\nu}\frac{1}{(u-v)}\,P_\nu\left(\frac{u^2+v^2-2}{u^2-v^2}\right)=\sum_{l=0}^{\infty}(2l+1)Q_l(u)P_l(v)F_l(-2\nu-1). \quad \text{(B4)}$$

Equation (B4) holds for $u>1$, $u>v>-1$, but these are not necessary conditions.[7]

We give only one more series,[6]

$$-\left(\frac{2}{\pi}\right)e^{(2\nu+1)x}\sin(\pi\nu)\cosh x = \sum_{l=0}^{\infty}(2l+1)P_l(\tanh x)F_l(-2\nu-1), \quad \text{(B5)}$$

$$-1<\nu<0, \quad -\infty<x<\infty.$$

Finally, we give one of several recurrence relations,[6]

$$(l+1)^2F_{l+1}(w)=l^2F_{l-1}(w)-(2l+1)wF_l(w). \quad \text{(B6)}$$

APPENDIX C: THE BUCHHOLZ FUNCTIONS

The Buchholz functions[26] $m_\gamma^{(p)}(z)$ and $w_\gamma^{(p)}(z)$ are a system of linearly independent solutions of the differential equation

$$\frac{d^2u}{dz^2}+\frac{1}{z}\frac{du}{dz}+\left(-\frac{1}{4}+\frac{\gamma}{z}-\frac{p^2}{z^2}\right)u=0. \quad \text{(C1)}$$

These functions are simply related to the Whittaker functions $M_{\gamma,p/2}(z)$ and $W_{\gamma,p/2}(z)$ and to the two confluent hypergeometric functions ($\Phi={}_1F_1$) as follows:

$$m_\gamma^{(p)}(z)=\frac{z^{-1/2}M_{\gamma,p/2}(z)}{\Gamma(1+p)} = \frac{e^{-z/2}z^{p/2}}{\Gamma(1+p)}\Phi\left(\frac{p+1}{2}-\gamma,p+1;\ z\right), \quad \text{(C2a)}$$

$$w_\gamma^{(p)}(z)=z^{-1/2}W_{\gamma,p/2}(z) = e^{-z/2}z^{p/2}\Psi\left(\frac{p+1}{2}-\gamma,p+1;\ z\right), \quad \text{(C2b)}$$

and

$$m_\gamma^{(0)}\equiv m_\gamma, \quad w_\gamma^{(0)}\equiv w_\gamma.$$

Equation (C1) results from the separation of the Schrödinger equation for a Coulomb potential in parabolic coordinates.[27]

The asymptotic expansions of the Buchholz functions for $|z|\to\infty$ are

$$m_\gamma^{(p)}(z)\sim\frac{z^{-\gamma-1/2}e^{z/2}}{\Gamma((1+p)/2-\gamma)}\,{}_2F_0\left(\frac{p+1}{2}+\gamma,\ \frac{1-p}{2}+\gamma;\ ;\ \frac{1}{z}\right)+\frac{z^{\gamma-1/2}e^{-z/2}}{\Gamma((1+p)/2+\gamma)}\,e^{\pm i\pi[\gamma-(1+p)/2]}\,{}_2F_0\left(\frac{1+p}{2}-\gamma,\frac{1-p}{2}-\gamma;\ ;-\frac{1}{z}\right), \quad \text{(C3)}$$

where we take the upper sign for $-\frac{3}{2}\pi<\arg z<\frac{1}{2}\pi$ and the lower sign for $-\frac{1}{2}\pi<\arg z<\frac{3}{2}\pi$, and

$$w_\gamma^{(p)}(z)\sim z^{\gamma-1/2}e^{-z/2}\,{}_2F_0\left(\frac{1+p}{2}-\gamma,\ \frac{1-p}{2}-\gamma;\ ;-\frac{1}{z}\right), \quad \text{(C4a)}$$

$$|\arg z|<\tfrac{3}{2}\pi,$$

$$w_\gamma^{(p)}(ze^{\pm i\pi})\sim(ze^{\pm i\pi})^{\gamma-1/2}e^{z/2}\times{}_2F_0\left(\frac{1+p}{2}-\gamma,\ \frac{1-p}{2}-\gamma;\ ;\ \frac{1}{z}\right), \quad \text{(C4b)}$$

$$|\arg z\pm\pi|<\tfrac{3}{2}\pi.$$

We also give two representations for an outgoing spherical wave in terms of Buchholz functions[28]

$$\frac{e^{ikr}}{r}=k\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{w_{-\nu-1/2}(-ik\xi)w_{\nu+1/2}(-ik\eta)d\nu}{\sin(\pi\nu)} \quad \text{(C5)}$$

$$-1<\sigma<0,$$

or

$$\frac{e^{ikr}}{r}=-2ik\sum_{n=0}^{\infty}(-1)^n w_{-n-1/2}(-ik\xi)w_{n+1/2}(-ik\eta), \quad \text{(C6)}$$

$$0\leq\eta<\xi.$$

[1]N. N. Khuri, Phys. Rev. Lett. 10, 420 (1963); Phys. Rev. 132, 914 (1963).

[2]See second paper of Ref. 1, footnote 4 ($z=\cos\theta$).

[3]A brief presentation of some of the results in this paper and related material by the same authors has appeared in Lett. Nuovo Cimento 12, 601 (1975).

[4]S. J. Wallace, Phys. Rev. D 9, 406 (1974); 8, 1846 (1973); E. Predazzi, Ann. Phys. (N.Y.) 36, 228 (1966); 36, 250 (1966); R. Blankenbecler and M. L. Goldberger, Phys. Rev. 126, 766 (1962); M. M. Islam, Saclay Report No. DPh-T/74/97, 1974 (unpublished).

[5]Our normalization is such that $A=\Sigma(2l+1)A_lP_l$, $A_l=\sqrt{s}(e^{2i\delta_l}-1)/(2ik)$, and $d\sigma/d\Omega=|A|^2/s$; k is the c.m. momentum, and $z=\cos\theta$.

[6]H. Bateman, Tôhoku Math. J. 37, 23 (1933); see also Appendixes A and B.

[7]S. O. Rice, Duke Math. J. 6, 108 (1939).

[8]See, for example, W. Rudin, *Real and Complex Analysis* (McGraw-Hill, New York, 1966), theorem 1.38, p. 28.

[9]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1, Sec. 3.8, Eq. (19).

[10]Here and below we are neglecting a $(-1)^l$ factor, since our proofs made use of (2) and hence l integer only.

[11]See, for example, E. Hille, *Analytic Function Theory* (Blaisdell, New York, 1959), Vol. 1, theorem 7.10.3, p. 194.

[12]E. J. Squires, Nuovo Cimento 25, 242 (1962).

[13]See, for example, A. O. Barut, *The Theory of the Scattering Matrix* (Macmillan, New York, 1967), p. 155.

[14]If, for example, the nearest z-plane singularities are poles arising from the exchange of a spinless particle of mass m in the t and u channels, $A_l \propto Q_l$; and using (14), we obtain for large $|\nu|$, $a \sim f_1[(s-3m^2)/m^2]^\nu + f_2[m^2/(s-3m^2)]^\nu$.

[15]See, for example, A. S. B. Holland, *Introduction to the Theory of Entire Functions* (Academic, New York, 1973).

[16]In Eq. (22) $a(s,0)=A(s,0)$, the forward scattering amplitude; this follows from $F_l(-1)=1$ and (2). Likewise $a(s,-1)=A(s,\infty)$, the backward scattering amplitude.

[17]H. Bateman, Ann. Math. 35, 767 (1934), Eq. (6).

[18]S. Pasternack, Philos. Mag. 28, 209 (1939).

[19]Since scattering of a particle by a central potential is axially symmetric, the wave function ψ is independent of the angle ϕ.

[20]Buchholz functions are solutions of the differential equation which results from the separation of the Schrödinger equation for a Coulomb potential in parabolic coordinates.

[21]L. J. Slater, *Generalized Hypergeometric Functions* (Cambridge Univ. Press, Cambridge, England, 1966), Eq. (2.3.3.7), p. 52.

[22]See Ref. 9, Sec. 2.1, Eq. (14).

[23]L. Carlitz, Can. J. Math. 9, 188 (1957).

[24]J. Touchard, Can. J. Math. 8, 305 (1956).

[25]H. Bateman, Duke Math. J. 4, 39 (1938).

[26]H. Buchholz, *The Confluent Hypergeometric Function* (Springer, New York, 1969). The definitions of $m_\gamma^{(p)}$ and $w_\gamma^{(p)}$ in this book differ from the ones in Ref. 9 and also from those in earlier papers of Buchholz.

[27]See, for example, L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1965), 2nd ed., Eq. (37.4), p. 126.

[28]H. Buchholz, Math. Z. 52, 355 (1949).