Dual resonance model incorporating two different parent Regge trajectories in the same channel*

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A four-particle dual model is constructed with two different Regge trajectories in both the s and t channels. The amplitude is crossing-symmetric, has residues that are polynomial of correct order, and has full Regge behavior. In a certain limit it becomes the Euler B function. The multiparticle generalization of the present model will be relevant to construction of hadron models with realistic mass spectra.

I. INTRODUCTION AND MOTIVATION

All attempts to construct dual resonance models have, so far, made the simplifying assumption that there is only one parent Regge trajectory per channel, accompanied by its infinite family of unit-spaced daughter trajectories. In nature, however, the situation is more complicated; for example, in a three-pion channel both the π trajectory [intercept $\alpha(0)\simeq 0$] and the ω trajectory [$\alpha(0)\simeq\frac{1}{2}$] couple. Thus, we need a model that allows for this possibility.

The situation becomes particularly acute when we try to formulate a realistic meson model which includes strangeness, for even if we are prepared to tolerate the π-ω trajectory degeneracy and have only ρ [$\alpha(0)\simeq\frac{1}{2}$] and π [$\alpha(0)\simeq 0$] parent trajectories (in different channels), when we attempt to include K [$\alpha(0)\simeq -\frac{1}{4}$] and K^* [$\alpha(0)\simeq\frac{1}{4}$] then in the channel $K\pi\pi\to K\pi\pi$ both parent trajectories occur in the same channel.

The existing philosophy among dual-model proponents seems to be that one should accommodate two different parent trajectories in the same channel by *summing* different terms. This was suggested, for example, many years ago by Olive and Zakrzewski[1] and by Rittenberg and Rubinstein.[2] The difficulty, known to these authors, of this approach is the proliferation of terms in the N-particle amplitude (the exact number is $[(2N-4)!/(N-1)!(N-2)!]\,2^{N-3}\sim 8^N$ for large N), and the consequent loss of factorization on the daughter trajectories.

An interesting attempt to overcome the ω-π trajectory degeneracy in the Neveu-Schwarz six-pion amplitude was made by Brower and Chu[3]; their modification is made in a phenomenological spirit, without attempting to set up a factorizing system of N-pion amplitudes.

In the present paper we attempt to make the first step toward setting up a dual model with two (or possibly more) different parent trajectories in the same channel. The most interesting case phenomenologically is the six-point function, where the $3\pi\to 3\pi$ and $K\pi\pi\to K\pi\pi$ situations occur, but for the purposes of mathematical simplicity we here concentrate on the four-particle amplitude. The objective is to set up a four-particle amplitude $A(s,t)$ with two different parent trajectories α_s, $\bar\alpha_s$ in the s channel, and correspondingly α_t, $\bar\alpha_t$ in the t channel. The simplest prescription, following the philosophy of summing terms, is to write

$$A(s,t)=B(-\alpha_s,-\alpha_t)+B(-\alpha_s,-\bar\alpha_t)+B(-\bar\alpha_s,-\alpha_t)+B(-\bar\alpha_s,-\bar\alpha_t). \tag{1}$$

But our objective here is to write an integral representation where the different parent trajectories are accommodated in a more essential way; the hope is then that a multiparticle extension will have more favorable factorization properties.

The outline of the present paper is as follows: In Sec. II we make some general considerations about the amplitude; in Sec. III we give our specific proposal. Polynomial residues and Regge behavior are discussed respectively in Secs. IV and V. Finally in Sec. VI there is some discussion.

II. GENERAL CONSIDERATIONS

We want our amplitude to have resonance poles corresponding to α_s, $\bar\alpha_s$, α_t, $\bar\alpha_t$ so it is natural to write a general form

$$A_4=\int_0\int_0\int_0\int_0 dx_1dx_2dx_3dx_4V(x_1x_2x_3x_4)\times x_1^{-\alpha_s-1}x_2^{-\bar\alpha_s-1}x_3^{-\alpha_t-1}x_4^{-\bar\alpha_t-1}\times\prod_{i=1}^{3}\delta(f_i(x_j)), \tag{2}$$

where the upper integration limits are, for the moment, left unspecified (see below) and there are *three* δ functions, since anything less is incompatible with the requirement of polynomial residues. Now the integrand should be crossing-symmetric under the permutation 1234→ 3412 and it follows that, solving in terms of x_1,

$$x_1(x_3(x_1)) = x_1 \tag{3}$$

and hence a possible solution is

$$x_3 = c - x_1, \tag{4}$$

$$x_4 = c - x_2, \tag{5}$$

where c is a constant so that the amplitude simplifies to

$$A_4 = \int\int dx\,dy\, V(x,y) x^{-\alpha_s - 1} y^{-\bar{\alpha}_s - 1} \times (c-x)^{-\alpha_t - 1}(c-y)^{-\bar{\alpha}_t - 1} \times \delta(f(x,y)). \tag{6}$$

The requirement of polynomial residues further imposes the conditions that

$$f\left(0, c - \frac{1}{c}\right) = 0, \tag{7}$$

$$f\left(c, \frac{1}{c}\right) = 0, \tag{8}$$

and symmetry under 1234→ 2143 dictates that

$$f(x,y) = f(y,x). \tag{9}$$

The simplest possibility consistent with these requirements is to inscribe an ellipse inside the square $0 \le x, y \le c$. This gives

$$f(x,y) = (c^2 - 1)(x + y - c)^2 + (y - x)^2 - (c^2 - 1). \tag{10}$$

We shall further require that when $\alpha_s \to \bar{\alpha}_s$, $\alpha_t \to \bar{\alpha}_t$, $c \to 1$ the amplitude reduces precisely to the Euler B function; we see that $f(x,y)$ of Eq. (10) is consistent with this requirement since when $c \to 1$, $f(x,y) = 0 \Rightarrow x = y$.

Our proposed model is not simply the substitution of Eq. (10) into Eq. (6) (this violates Regge behavior) but a modification of this substitution procedure as described in the subsequent section.

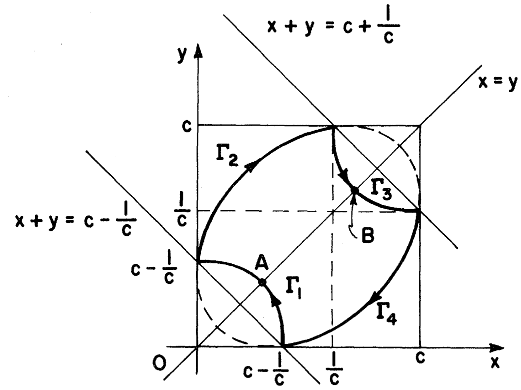

FIG. 1. Integration contour in the x-y plane.

III. THE PROPOSED MODEL

We are now ready to write down our proposed model, which is obviously motivated by the general consideration of the preceding section. The model is

$$A_4 = \int_\Gamma dx (y - x) x^{-\alpha_s - 1} y^{-\bar{\alpha}_s - 1}(c - x)^{-\alpha_t - 1}(c - y)^{-\bar{\alpha}_t - 1}, \tag{11}$$

where the line integral is over the closed contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ depicted in Fig. 1.

The segments Γ_2 and Γ_4 correspond to the ellipse of Eq. (10) and can be solved to give

$$y_{2,4} = \frac{1}{c^2}\{c(c^2 - 1) + x(2 - c^2) \pm 2[x(c - x)(c^2 - 1)]^{1/2}\}, \tag{12}$$

or

$$x_{2,4} = \frac{1}{c^2}\{c(c^2 - 1) + y(2 - c^2) \mp 2[y(c - y)(c^2 - 1)]^{1/2}\}. \tag{13}$$

The segment Γ_1 corresponds to the reflection of the ellipse, Eq. (10), in the line $x + y = c - 1/c$, that is,

$$(x - y)^2 + (c^2 - 1)\left(x + y - c + \frac{2}{c}\right)^2 - (c^2 - 1) = 0, \tag{14}$$

which gives

$$y_1 = \frac{1}{c^2}\left\{(c^2 - 2)\left(c - \frac{1}{c}\right) + x(2 - c^2) + 2\left[(c^2 - 1)\left(x + \frac{1}{c}\right)\left(c - \frac{1}{c} - x\right)\right]^{1/2}\right\}, \tag{15}$$

or

$$x_1 = \frac{1}{c^2}\left\{(c^2 - 2)\left(c - \frac{1}{c}\right) + y(2 - c^2) + 2\left[(c^2 - 1)\left(y + \frac{1}{c}\right)\left(c - \frac{1}{c} - y\right)\right]^{1/2}\right\}. \tag{16}$$

Similarly the segment Γ_3 is the reflection of Eq. (10) in the line $x+y=c+1/c$ giving

$$(x-y)^2+(c^2-1)\left(x+y-c-\frac{2}{c}\right)^2-(c^2-1)=0 \tag{17}$$

corresponding to

$$y_3=\frac{1}{c^2}\left\{(c^2+2)\left(c-\frac{1}{c}\right)+x(2-c^2)-2\left[(c^2-1)\left(c+\frac{1}{c}-x\right)\left(x-\frac{1}{c}\right)\right]^{1/2}\right\}, \tag{18}$$

or

$$x_3=\frac{1}{c^2}\left\{(c^2+2)\left(c-\frac{1}{c}\right)+y(2-c^2)-2\left[(c^2-1)\left(c+\frac{1}{c}-y\right)\left(y-\frac{1}{c}\right)\right]^{1/2}\right\}. \tag{19}$$

This completely specifies the amplitude. After straightforward algebra, one deduces that

$$\begin{aligned}A_4=&\frac{1}{2c^2}\int_0^{1/c}dx\left\{1+\frac{d}{dx}[x(c-x)(c^2-1)]^{1/2}\right\}x^{-\alpha_s-1}(c-x)^{-\alpha_t-1}y_2^{-\bar\alpha_s-1}(c-y_2)^{-\bar\alpha_t-1}\\&-\frac{1}{2c^2}\int_0^{A}dx\left\{1+\frac{d}{dx}\left[\left(x+\frac{1}{c}\right)\left(c-\frac{1}{c}-x\right)(c^2-1)\right]^{1/2}\right\}x^{-\alpha_s-1}(c-x)^{-\alpha_t-1}y_1^{-\bar\alpha_s-1}(c-y_1)^{-\bar\alpha_t-1}\\&-\frac{1}{2c^2}\int_{1/c}^{B}dx\left\{1-\frac{d}{dx}\left[\left(x-\frac{1}{c}\right)\left(c+\frac{1}{c}-x\right)(c^2-1)\right]^{1/2}\right\}x^{-\alpha_s-1}(c-x)^{-\alpha_t-1}y_3^{-\bar\alpha_s-1}(c-y_3)^{-\bar\alpha_t-1}\\&+(\alpha_s\leftrightarrow\bar\alpha_s;\ \alpha_t\leftrightarrow\bar\alpha_t).\end{aligned} \tag{20}$$

Here we have defined

$$A=\frac{c}{2}+\frac{1}{2}-\frac{1}{c}, \tag{21}$$

$$B=\frac{c}{2}-\frac{1}{2}+\frac{1}{c} \tag{22}$$

as the coordinates of the points A, B in Fig. 1.

Note that in the limit $c\to 1$, then $A\to 0$, $B\to 1$, and $y_i\to x$ ($i=1, 2, 3, 4$) so we obtain

$$\lim_{c\to 1}A_4=\int_0^1 dx\, x^{-\alpha_s-\bar\alpha_s-2}(1-x)^{-\alpha_t-\bar\alpha_t-2} \tag{23}$$

corresponding to an Euler B function with effective trajectories

$$\alpha_s^{\text{eff}}=\alpha_s+\bar\alpha_s+1, \tag{24}$$

$$\alpha_t^{\text{eff}}=\alpha_t+\bar\alpha_t+1. \tag{25}$$

Note, in particular, that the (equal) Regge slopes of $\alpha_s, \bar\alpha_s$ must be equal to $\frac{1}{2}\alpha^{\text{eff}\,\prime}$ for consistency; this will be important in the later development.

IV. POLYNOMIAL RESIDUES

Because of the symmetry properties built into the amplitude we need study in detail only one of the four singular points. We choose to study the poles in α_s which arise for $x\sim 0$ ($y\sim c-1/c$). The contributions are from two terms in Eq. (20), namely

$$\begin{aligned}&\frac{1}{2c^2}\int_0^{1/c}dx\left\{1+\frac{d}{dx}[x(c-x)(c^2-1)]^{1/2}\right\}x^{-\alpha_s-1}(c-x)^{-\alpha_t-1}y_2^{-\bar\alpha_s-1}(c-y_2)^{-\alpha_t-1}\\&\qquad-\frac{1}{2c^2}\int_0^{A}dx\left\{1+\frac{d}{dx}\left[\left(x+\frac{1}{c}\right)\left(c-\frac{1}{c}-x\right)(c^2-1)\right]^{1/2}\right\}x^{-\alpha_s-1}(c-x)^{-\alpha_t-1}y_1^{-\bar\alpha_s-1}(c-y_1)^{-\bar\alpha_t-1}.\end{aligned} \tag{26}$$

We deal with the first term first. Using the formula for y_2 in Eq. (12) we find that

$$y_2^{-\bar{\alpha}_s-1}(c-y_2)^{-\bar{\alpha}_t-1}=\left(c-\frac{1}{c}\right)^{-\bar{\alpha}_s-1}c^{-\bar{\alpha}_t+1}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}P_a^{(1)}(-\bar{\alpha}_s-1)P_b^{(2)}(-\bar{\alpha}_t-1)x^{(a+b)/2} \tag{27}$$

$$=\left(c-\frac{1}{c}\right)^{-\bar{\alpha}_s-1}c^{-\bar{\alpha}_t+1}\sum_{n=0}^{\infty}P_n(-\bar{\alpha}_t-1)x^{n/2}, \tag{28}$$

where

$$P_a^{(1)}(\alpha)=\left(\frac{c}{2}\right)^a\left(c-\frac{1}{c}\right)^{-a/2}\left(\frac{2}{c^2}-1\right)^a\sum_{i=0}^{a}\binom{\alpha}{i}2^{2i}(2-c^2)^{-i}f(a,i), \tag{29}$$

$$P_b^{(2)}(\alpha)=\left(\frac{c}{2}\right)^b\left(c-\frac{1}{c}\right)^{b/2}\left(\frac{2}{c^2}-1\right)^b\sum_{i=0}^{b}(-1)^i\binom{\alpha}{i}2^{2i}(2-c^2)^{-i}(c^2-1)^if(b,i), \tag{30}$$

$$P_n(-\bar{\alpha}_t-1)=\sum_{r=0}^{n}P_{n-r}^{(1)}(-\bar{\alpha}_s-1)P_r^{(2)}(-\bar{\alpha}_t-1), \tag{31}$$

$$f(a,i)=\sum_r(-1)^r\binom{i}{a-i-2r}\binom{r+i-a/2}{r}2^{2r}(2-c^2)^{-2r}(c^2-1)^r. \tag{32}$$

Further, we need the power expansions

$$(c-x)^{-\alpha_t-1}=\sum_{n=0}^{\infty}\binom{-\alpha_t-1}{n}c^{-\alpha_t-1-n}(-1)^nx^n \tag{33}$$

and

$$\frac{d}{dx}[x(c-x)(c^2-1)]^{1/2}=[(c^2-1)]^{1/2}\sum_{m=0}^{\infty}\binom{1/2}{m}(-1)^m(m+\tfrac{1}{2})c^{1/2-m}x^{m-1/2}. \tag{34}$$

Combining these results, we find that if we define

$$R_k(t)=\sum_{m=0}^{[k/2]}(-1)^m\binom{-\bar{\alpha}_t-1}{m}P_{k-2m}(-\bar{\alpha}_t-1), \tag{35}$$

$$R_n'(t)=\sum_{m=0}^{[n/2]}R_{n-2m}(t)\binom{1/2}{m}(-1)^m(m+\tfrac{1}{2})c^{1/2-m}[(c^2-1)]^{1/2}, \tag{36}$$

then the first term in expression (26) is equal to

$$\frac{1}{2c^2}\int_0^{1/c}dx\,x^{-\alpha_s-1}\left(c-\frac{1}{c}\right)^{-\bar{\alpha}_s-1}c^{\Delta}\sum_{n=0}^{\infty}S_n(t)x^{n/2-1/2}, \tag{37}$$

where

$$S_0(t)=R_0'(t)=\tfrac{1}{2}[c(c^2-1)]^{1/2}, \tag{38}$$

$$S_n(t)=R_n'(t)+R_{n-1}(t),\quad n\geq 1 \tag{39}$$

and

$$\Delta=\bar{\alpha}_s-\alpha_s.$$

In particular, $S_n(t)$ is a polynomial of degree n in t. Therefore, if we define as the actual trajectory

$$\alpha(s)=2\alpha_s+1, \tag{40}$$

then we can rewrite this term as

$$c^{\Delta-2}\left(c-\frac{1}{c}\right)^{-[\bar{\alpha}(s)+1]/2}\sum_{n=0}^{\infty}\frac{S_n(t)}{n-\alpha(s)}. \tag{41}$$

Concerning the second term in (26), it is straightforward to show that it may be rewritten as

$$-c^{\Delta-2}\left(c-\frac{1}{c}\right)^{-[\bar{\alpha}(s)+1]/2}\sum_{n=0}^{\infty}\frac{Q_n(t)}{2n+1-\alpha(s)}, \tag{42}$$

where $Q_n(t)$ is a polynomial of degree n in t.

Thus this second term contributes residues of lower degree than the first, and does not contribute to the parent resonances, only to low-lying daughters.

What is worth emphasizing is the identification, Eq. (40), which is precisely that implied by the limit $c\to 1$ discussed in the preceding section [see, in particular, Eq. (24)]. To see this, note that if on taking the limit $c\to 1$ we also take $\Delta\to 0$ ($\Rightarrow \alpha_s\to\bar{\alpha}_s$) then Eq. (24) coincides with Eq. (40). This is a happy event since it ties together the original choice of the *quadratic* form in Eq. (10) with the existence of the Euler-B-function limit; thus the insistence on this limit removes any arbitrariness in the choice of the integration contour.

V. REGGE BEHAVIOR

Our proof that the residues are polynomials of the correct degree is a necessary, but *not* sufficient, condition for the amplitude to possess correct Regge behavior. This is because of the extra s dependence implied by the factor $y^{-\alpha s-1}$ in the expansions made in the preceding section. Because of this there is the danger that the amplitude, although meromorphic, may not be completely expressed in terms of its poles (with energy-independent residues at constant momentum transfer) and that there could be, in addition, an entire function. Such an entire function would, in general, lead to violation of Regge behavior because it is dictated by the Phragmén-Lindelöf theorem that an entire function must increase in at least some directions of complex plane.[4,5]

In order to investigate this question we use the Laplace-Watson approach to the asymptotic expansion of integrals.[6] We may rewrite Eq. (20) in the form

$$A_4=-\frac{1}{2c^2}\int_0^A dx(xy_1)^{-\bar\alpha_s-1}[(c-x)(c-y_1)]^{-\bar\alpha_t-1}[x(c-x)]^{\Delta}\frac{d}{dx}\left\{x+\left[\left(x+\frac{1}{c}\right)\left(c-\frac{1}{c}-x\right)(c^2-1)\right]^{1/2}\right\}$$
$$+\frac{1}{2c^2}\int_0^{1/c} dx(xy_2)^{-\bar\alpha_s-1}[(c-x)(c-y_2)]^{-\bar\alpha_t-1}[x(c-x)]^{\Delta}\frac{d}{dx}\{x+[x(c-x)(c^2-1)]^{1/2}\}$$
$$-\frac{1}{2c^2}\int_{1/c}^B dx(xy_3)^{-\bar\alpha_s-1}[(c-x)(c-y_3)]^{-\bar\alpha_t-1}[x(c-x)]^{\Delta}\frac{d}{dx}\left\{x-\left[\left(x-\frac{1}{c}\right)\left(c+\frac{1}{c}-x\right)(c^2-1)\right]^{1/2}\right\}$$
$$+(\alpha_s\to\bar\alpha_s;\ \alpha_t\to\bar\alpha_t). \tag{43}$$

The Regge asymptotic behavior of each of the three terms in (43) will now be examined. Let us define

$$v_1=\frac{1}{c}\left\{x+\left[\left(x+\frac{1}{c}\right)\left(c-\frac{1}{c}-x\right)(c^2-1)\right]^{1/2}\right\}, \tag{44}$$

$$z_1=v_1^2-\left(1-\frac{1}{c^2}\right)^2; \tag{45}$$

then the first term in (43) is equal, without approximation, to

$$-\frac{1}{4c}(4c^2)^{-\Delta}\int_0^{A^2}\frac{dz_1}{[z_1+(1-1/c^2)]^{1/2}}z_1^{-\bar\alpha_s-1}\left\{3-\frac{2}{c^2}+z_1-2\left[z_1+\left(1-\frac{1}{c^2}\right)^2\right]^{1/2}\right\}^{-\bar\alpha_t-1}$$
$$\times\left(c^4-\left\{3-\frac{2}{c^2}-2\left[z_1+\left(1-\frac{1}{c^2}\right)^2\right]^{1/2}+\{(c^2-1)[c^2-[1-2/c^2-2(z_1+(1-1/c^2)^2)^{1/2}]]^2\}^{1/2}\right\}^2\right)^{\Delta}. \tag{46}$$

Now, since $A^2<1$ (provided $c<2$), this term vanishes exponentially as $\mathrm{Re}\,\alpha_s\to-\infty$.

Concerning the second term in (43), define

$$v_2^2=xy_2, \tag{47}$$

whereupon this term becomes

$$\frac{1}{2c}\int_0^1 dv_2 v_2^{-2\bar\alpha_s-2}(1-v_2)^{-2\bar\alpha_t-2}(4c^2)^{-\Delta}[c^4-(1-2v_2+\{(c^2-1)[c^2-(1-2v_2)^2]\}^{1/2})^2]^{\Delta}, \tag{48}$$

and when $\mathrm{Re}\,\alpha_s\to-\infty$ the dominant contribution is from $v_2\sim 1$. The final factor has the Taylor expansion

$$[c^4-(1-2v_2+\{(c^2-1)[c^2-(1-2v_2)^2]\}^{1/2})^2]^{\Delta}=\sum_{n=0}^{\infty}\alpha_n(1-v_2)^n, \tag{49}$$

where, for example,

$$\alpha_0 = [4(c^2-1)]^{\Delta}, \tag{50}$$

$$\alpha_1 = -8\Delta(c^2-1)[4(c^2-1)]^{\Delta-1}, \tag{51}$$

$$\alpha_2 = 16\Delta[2(\Delta-1)(c^2-2)^2 + c^4 - 6c^2 + 4][4(c^2-1)]^{\Delta-2}. \tag{52}$$

Making the change of variables

$$v_2 = e^{-w} \tag{53}$$

we obtain the Laplace transform

$$\sum_{n=0}^{\infty} \alpha_n \frac{1}{2c}(4c^2)^{-\Delta} \int_0^{\infty} dw\, e^{w(2\overline{\alpha}_s-1)}(1-e^{-w})^{-2\overline{\alpha}_t-2+n}$$

$$= \frac{1}{2c}\left[\frac{1}{c^2}(c^2-1)\right]^{\Delta}(1-2\overline{\alpha}_s)^{2\overline{\alpha}_t+1}\Gamma(-2\overline{\alpha}_t-1) + (\text{correction terms} \sim 1/s). \tag{54}$$

For the third term we change variables to

$$v_3 = \frac{1}{c}\left\{x - \left[\left(x-\frac{1}{c}\right)\left(c+\frac{1}{c}-x\right)(c^2-1)\right]^{1/2}\right\}, \tag{55}$$

$$z_3 = v_3^{\,2} + 1 - \frac{1}{c^4}, \tag{56}$$

and rewrite this term as

$$\frac{1}{4c}(4c^2)^{-\Delta}\int_{B^2}^{1} dz_3 \frac{1}{(z_3-1+1/c^4)^{1/2}} z_3^{-\overline{\alpha}_s-1}\left[\frac{2}{c^2}-1+z_3-2\left(z_3-1+\frac{1}{c^4}\right)^{1/2}\right]^{-\overline{\alpha}_t-1}$$

$$\times\left\{c^4 - \left[1-\frac{2}{c^2}+2\left(z_3-1+\frac{1}{c^4}\right)^{1/2} - \{(c^2-1)(c^2-[1+2/c^2-2(z_3-1+1/c^4)^{1/2}]^2)\}^{1/2}\right]^2\right\}^{\Delta}. \tag{57}$$

The dominant contribution in the limit that we are considering is from $z_3 \sim 1$, so we write

$$z_3 = e^{-w} \tag{58}$$

and find that the leading term as $\text{Re}\,\alpha_s \to -\infty$ is $(B = c/2 - \frac{1}{2} + 1/c < 1)$,

$$\frac{c}{4}\left[\frac{1}{c^2}(c^2-1)\right]^{\Delta}\int_0^{-\ln B^2} dw\, e^{w\overline{\alpha}_s}[(c-1)w]^{-\overline{\alpha}_t-1} = \frac{c}{4}\left[\frac{1}{c^2}(c^2-1)\right]^{\Delta}(c-1)^{-\overline{\alpha}_t-1}(-\overline{\alpha}_s)^{-\overline{\alpha}_t}\int_0^{(-\overline{\alpha}_s)\ln(1/B^2)} dz\, e^{-z} z^{-\overline{\alpha}_t-1}. \tag{59}$$

To summarize and collect together the results of this section, we have shown that for $\text{Re}\,\alpha_s \to -\infty$ at fixed t

$$A_4 \sim \frac{1}{2c}\left[\frac{1}{c^2}(c^2-1)\right]^{\Delta}(1-2\overline{\alpha}_s)^{2\overline{\alpha}_t+1}\Gamma(-2\overline{\alpha}_t-1)$$

$$+\frac{c}{4}\left[\frac{1}{c^2}(c^2-1)\right]^{\Delta}(c-1)^{-\overline{\alpha}_t-1}(-\overline{\alpha}_s)^{-\overline{\alpha}_t}\Gamma(-\overline{\alpha}_t) + (\alpha_s \leftrightarrow \overline{\alpha}_s;\ \alpha_t \leftrightarrow \overline{\alpha}_t;\ \Delta \leftrightarrow -\Delta). \tag{60}$$

This result agrees precisely with our expectations from the discussions of the preceding sections; since if we define the actual trajectory again as $\overline{\alpha}(t) = 2\overline{\alpha}_t + 1$ then the leading Regge behavior in

$$A_4 \sim s^{\alpha(t)} + s^{\overline{\alpha}(t)} \tag{61}$$

as required.

Of course, we have derived Eq. (60) only in the limit $\text{Re}\,\alpha_s \to -\infty$ and we now must discuss the Regge behavior in the right-half s plane $(\text{Re}\,\alpha_s \to +\infty)$. To do this we exploit Watson's lemma[6] which, in the present context, may most usefully be stated as follows: Given that

$$A(\alpha_s, \alpha_t) = \int_0^{\infty} dy\, e^{y\alpha_s} g(e^{-y}, \alpha_t) \tag{62}$$

and that $g(e^{-y}, \alpha_t)$ has the asymptotic expansion

$$g(e^{-y}, \alpha_t) \sim \sum_{n=0}^{\infty} b_n(\alpha_t) y^{n-\alpha_t-1}, \quad (63)$$

then for $A(\alpha_s, \alpha_t)$ the asymptotic expansion

$$A(\alpha_s, \alpha_t) \sim \sum_{n=0}^{\infty} b_n(\alpha_t)(-\alpha_s)^{\alpha_t - n}\Gamma(n-\alpha_t) \quad (64)$$

is valid for *all* $|\arg(-\alpha_s)| < \pi$ *provided* that

(i) $g(e^{-y})$ is analytic for $\mathrm{Re}\, y > 0$, and

(ii) for every value of $\arg(y)$ between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ there exists a positive M such that for $|y| > 0$, $|g(e^{-y})| < M$.

In the present case, we can confirm that for each of the terms in Eq. (43) the integrand has the required analyticity properties and that rotation of the contour is allowed.

VI. DISCUSSION

It has been shown here how to construct a four-particle dual model with two different parent trajectories per channel; the amplitude has the correct polynomial residues and Regge behavior for $|\alpha_s| \to \infty$ at fixed t, $|\alpha_t| \to \infty$ at fixed s. Its construction is strongly constrained by the requirement that when the parameter $c \to 1$ it becomes the Euler B function.

The parameters Δ and c are free and independent in all considerations we have made. We note only that if $c = \sqrt{2}$ the integration contour becomes especially symmetric; and if $\Delta = \frac{1}{2}$ there is a square-root branch point rather than a more complicated sheet structure. Neither choice is yet compelling, and the absolute value of the intercepts is still unconstrained.

The present study leads naturally to several further consistency questions:

(i) What is the behavior of this amplitude when $s \to \infty$, $t \to -\infty$ at fixed u?

(ii) What is the fixed-angle behavior?

(iii) Are the partial-wave projections of the residues positive?

Concerning question (iii) we expect that for some finite range of c and Δ, in the vicinity of $c = 1$ and $\Delta = 0$, and for intercept values near one, there must be positivity since for these precise values it is known to be true.

In addition we may ask:

(iv) How does the answer to question (iii) depend not only on c, Δ, and the absolute intercept values but also on the space-time dimensionality?

If acceptable answers to these questions can be proved, then the next step is to generalize to a many-particle amplitude. After all, the original motivation as discussed in the Introduction was to study the channels $3\pi \to 3\pi$ and $K\pi\pi \to K\pi\pi$ occurring in the six-pseudoscalar amplitude.

These questions are under investigation, and we hope to report on their answers on a future occasion.

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[1]D. I. Olive and W. J. Zakrzewski, Nucl. Phys. B21, 303 (1970).

[2]J. D. Dorren, V. Rittenberg, and H. R. Rubinstein, Nuovo Cimento 3A, 385 (1971).

[3]R. C. Brower and G. Chu, Phys. Rev. D 7, 56 (1973).

[4]R. P. Boas, *Entire Functions* (Academic, New York, 1954).

[5]R. Oehme, Lett. Nuovo Cimento 2, 53 (1969); Nucl. Phys. B16, 161 (1970).

[6]J. D. Murray, *Asymptotic Analysis* (Clarendon, Oxford, 1974).