Background-field method versus normal field theory in explicit examples: One-loop divergences in the S matrix and Green's functions for Yang-Mills and gravitational fields

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The gauge dependence in the background-field method of counterterms on and off the mass shell is investigated in several gauges for Yang-Mills and gravitational fields, including the axial gauge. The results agree with theorems of DeWitt and Kallosh. The relation between the quantum and classical gauge-fixing terms is discussed. It is shown how to determine specific S-matrix elements in the background-field method. A complete calculation using normal field theory of the one-loop divergences of scalar-scalar scattering through graviton exchange is presented. The results agree with those obtained in the background-field method by 't Hooft and Veltman.

I. INTRODUCTION

Whereas on the theoretical side several excellent articles have been written on the relation between the background-field method[1-3] and normal field theory, there is a conspicuous absence of explicit calculations verifying these theoretical results. Now that the background-field method has been proven useful in calculations[4-8] which would have been extremely difficult in normal field theory, it is desirable to check and elucidate in explicit examples the relation between the two methods.

In this paper we exhibit in several examples the gauge dependence off and on shell in the background-field method of counterterms for pure gauge fields and for gauge fields interacting with matter fields, thus verifying theorems of DeWitt[1] and Kallosh.[3] We also present a complete calculation of the one-loop divergences of scalar-scalar scattering through graviton exchange using normal field theory, thus verifying the background-field results of 't Hooft and Veltman[4] and the equality of the S matrices in both formalisms.

Why does one use the background-field method, for example in the calculations which dealt with the renormalizability of quantum gravity?[4-8] In this method one considers both classical and quantum fields and as dictated by covariant quantization, one breaks only the invariance of the theory under quantum gauge transformations. There remains then the gauge invariance of the classical fields, and it is this residual gauge invariance which serves as a useful bookkeeping device, thereby rendering the algebra manageable. An even greater advantage of the background-field method is the existence of a certain local gauge invariance, even present for nongauge theories, which enabled 't Hooft[5] to derive a simple algorithm for the one-loop divergences of most Lagrangian field theories.[9]

However, several questions arise. For one-particle-irreducible diagrams, there is a difference between normal field theory and the background-field method insofar as the gauge-fixing term (and hence the ghost term) may introduce additional vertices in the latter method. Do these additional vertices affect the S matrix and the counterterms? DeWitt has proved that this is not the case for the S matrix[1] (and hence for the counterterms on the mass shell), whereas Kallosh has shown[3] that for pure Yang-Mills fields the counterterms are independent of the quantum gauge-fixing term even off the mass shell—a result which is well known not to be true in normal field theory. We will investigate whether her theorem can be extended to pure gravitation, and check these statements in explicit calculations involving both Yang-Mills and gravitational fields.

In the background-field method one obtains the Green's functions from the one-particle-irreducible diagrams by substituting for the background fields suitable solutions of the classical field equations. How does one extract from this result the S matrix for a given process and does one recover the results of normal field theory? The answer to the latter question has been given by Kallosh, who stressed the pivotal role of the axial gauge.[3] Not only can the equivalence of canonical and covariant quantization be proved in this gauge,[10] but the axial gauge is also the only gauge in the background-field method which does not introduce additional vertices. From this latter property equality of the S matrices can be proved. We will check these theorems by comparing the counterterms for a given physical process calculated both in the background-field method and in normal field theory and establishing their equality when the external lines are on the mass shell.

The paper is organized as follows. In the next section we summarize some aspects of background-field theory and extract the divergences in the S matrix for gravitational scalar-scalar scattering from the one-loop divergences of the scalar-graviton system, as calculated in the background-field formalism by 't Hooft and Veltman.[4] In Sec. III we investigate whether Kallosh's theorem on the gauge independence of the counterterms off the mass shell can be extended to pure gravitation. We then consider four different gauges in the background-field method, including the axial gauge, and calculate the corresponding counterterms for Yang-Mills and gravitational fields. The calculations are very elaborate (which might explain their absence in the literature), but since they proceed along the same lines as in Refs. 4, 6–8, we give only a few intermediate results. In Sec. IV the complete calculation of the one-loop counterterms in normal field theory for scalar-scalar scattering through graviton exchange is presented. The calculations are performed using a symbolic manipulation program.[11] Several Ward identities, derived in Appendix B, are used to check the two- and three-point functions on and off the mass shell, and the final result is compared with the result for the same process in the background-field method as derived in Sec. II.

We use the notation $\eta_{\mu\nu}$ for the flat-space metric, which is (1, −1, −1, −1) throughout the paper.

II. COUNTERTERMS FOR A GIVEN PHYSICAL PROCESS IN THE BACKGROUND-FIELD METHOD

For a nongauge theory the background-field method is defined by the following steps. First, replace in the original classical Lagrangian any field ϕ by the sum $V+\phi$, where V is called the background (or classical or external) field and ϕ the quantum (or internal) field. Expanding the Lagrangian in ϕ and discarding terms independent of ϕ and linear in ϕ, the effective Lagrangian for loop diagrams is given by

$$\mathcal{L}^{\text{eff}}(V,\phi)=\mathcal{L}(V+\phi)-\mathcal{L}(V)-\phi\frac{\delta\mathcal{L}(V)}{\delta V}. \tag{1}$$

One-particle-irreducible loop diagrams are now calculated by using the quantum fields ϕ inside loops, while the classical fields V appear at the external vertices (see Fig. 1). It follows that one needs only the terms quadratic in ϕ in Eq. (1) for one-loop calculations. Any diagram with N external V lines thus calculated is identical to the corresponding diagram in normal field theory (where internal and external fields are both denoted by ϕ) because the propagators and the vertices are the same in both cases.

The second step in the background-field method is to require that the classical fields V satisfy the classical equation of motion

$$\frac{\delta\mathcal{L}(V)}{\delta V}=0. \tag{2}$$

If this is the case, then each single V line in the background-field method corresponds in normal field theory to the sum of all tree graphs which have at the end points all possible in fields on the mass shell and with physical polarizations. This correspondence is sketched in Fig. 2.

For gauge theories some new features are present. For clarity we consider the case of Yang-Mills fields. The classical Yang-Mills action $\mathcal{L}(V_\mu^a+\phi_\mu^a)$ is invariant under the gauge transformation

$$\delta(V_\mu^a+\phi_\mu^a)=\partial_\mu\Lambda^a+g\epsilon^{abc}(V_\mu^b+\phi_\mu^b)\Lambda^c. \tag{3}$$

This transformation can be realized in the following two ways:

$$\delta V_\mu^a=0;\quad \delta\phi_\mu^a=(D_\mu\Lambda)^a+g\epsilon^{abc}\phi_\mu^b\Lambda^c, \tag{4}$$

$$\delta V_\mu^a=(D_\mu\Lambda)^a;\quad \delta\phi_\mu^a=g\epsilon^{abc}\phi_\mu^b\Lambda^c, \tag{5}$$

where the covariant derivative D is here (and also in the gravitational case) always with respect to the classical field V: $(D_\mu\Lambda)^a=\partial_\mu\Lambda^a+g\epsilon^{abc}V_\mu^b\Lambda^c$. Equation (4) describes the so-called quantum gauge transformations. If Eq. (2) is satisfied, then also $\mathcal{L}^{\text{eff}}(V,\phi)$ in Eq. (1) is invariant under these quantum gauge transformations. We break this invariance by adding, as usual, a gauge-fixing term $\mathcal{L}^B(V,\phi)$. Many terms will do, e.g.,

$$\mathcal{L}^B(V,\phi)=-\tfrac{1}{2}(\partial_\mu\phi_\mu^a)^2,\ -\tfrac{1}{2}(D_\mu\phi_\mu^a)^2,\ \text{or}\ -\tfrac{1}{2}(\phi_3^a)^2. \tag{6}$$

The Lagrangian in Eq. (1) is invariant under the background gauge transformations defined by Eq. (5), whether or not Eq. (2) is satisfied. Under these transformations V transforms as a Yang-Mills field (or tensor in the gravitational case)

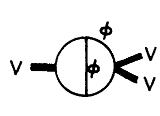

FIG. 1. A typical diagram in the background-field method. Heavy lines denote external fields V, and regular lines the quantum field ϕ.

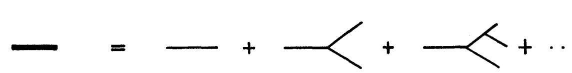

FIG. 2. Tree-graph expansion corresponding to a single external field V.

and ϕ as an ordinary isovector (or again tensor in the gravitational case). If we choose the gauge-fixing term to be background gauge invariant, then in all known cases the ghost Lagrangian can be made background gauge invariant also by defining suitable background gauge transformations for the ghost fields. It follows that the Green's functions and the counterterms will be invariant under Eq. (5) as well. In particular, if one calculates one-loop diagrams with the Lagrangian $\mathcal{L}^{\text{eff}}(V,\phi)+\mathcal{L}^B+\mathcal{L}^G$ (where $\mathcal{L}^G$ is the ghost term), then one obtains for the divergent parts of the Green's functions, when regularized by the dimensional regularization scheme,[12] $(n-4)^{-1}$ times local functions of V. These objects are the counterterms, and they are invariant when the fields V transform as in Eq. (5).

We note that in the examples of Eq. (6) the gauge-fixing terms are not functions of $V+\phi$ alone. In general $\mathcal{L}^B(V,\phi)$ introduces new vertices into the background field which are absent in normal field theory. For example, the second term in Eq. (6) contains a new vertex of the form $V\phi\phi$. Consequently, the one-particle-irreducible graphs in the background-field method no longer agree *a priori* with those in normal field theory. However, DeWitt[1] and Kallosh[3] have shown that the S matrix is independent of these new vertices and equal to the S matrix of normal field theory, provided the gauge-fixing term is invariant under the background gauge transformations.

Let us now turn to the example of massless scalar-scalar scattering through two-graviton exchange and discuss in detail how to obtain the counterterms for this given physical process in the background-field method. The classical Lagrangian is in this case[13]

$$\mathcal{L}=\sqrt{-g}\left[\frac{2}{\kappa^2}R(g)+\tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}\right]. \tag{7}$$

We introduce the background fields $g_{\mu\nu}$ and V by the replacements

$$g_{\mu\nu}\to g_{\mu\nu}+\kappa h_{\mu\nu}, \tag{8}$$

$$\phi\to V+\phi, \tag{9}$$

and add the gauge-fixing term[13] (indices are contracted by means of $g_{\mu\nu}$)

$$\mathcal{L}^B(g,V;h,\phi)=\sqrt{-g}\,(-D^\mu h_{\mu\nu}+\tfrac{1}{2}D_\nu h^\mu{}_\mu-\tfrac{1}{2}\kappa\phi\partial_\nu V)^2, \tag{10}$$

which leads in the well-known way to the ghost Lagrangian. Using this action 't Hooft and Veltman[4] have calculated the divergences of the sum of all one-particle-irreducible one-loop diagrams and found for the counterterms[13]

$$\Delta\mathcal{L}=\frac{\sqrt{-g}}{8\pi^2(n-4)}\left[\frac{9}{720}R^2+\frac{43}{120}R_{\alpha\beta}R^{\alpha\beta}+\frac{\kappa^4}{8}(\partial_\mu V\partial_\nu Vg^{\mu\nu})^2-\frac{\kappa^2}{24}R(\partial_\mu V\partial_\nu Vg^{\mu\nu})+\kappa^2(D_\mu D^\mu V)^2\right]. \tag{11}$$

Inserting, as discussed earlier, the classical field equations

$$D_\mu D^\mu V=0,\qquad R_{\mu\nu}=-\frac{\kappa^2}{4}(D_\mu V)(D_\nu V), \tag{12}$$

they found for the on-shell counterterms

$$\Delta\mathcal{L}=\frac{\sqrt{-g}}{8\pi^2(n-4)}\,\frac{203}{80}R^2. \tag{13}$$

We will now discuss how to extract from this result the counterterm for gravitational scattering of massless scalars.

The classical fields $g_{\mu\nu}$ and V are the solutions of Eq. (12) with Feynman boundary conditions, as discussed in Ref. 1. Solving these equations iteratively we get the perturbation expansion

$$g_{\mu\nu}(x)=g^{\text{in}}_{\mu\nu}(x)+\kappa\int D^F_{\mu\nu,\rho\sigma}(x-x')J_{\rho\sigma}(g^{\text{in}}_{\mu\nu},V^{\text{in}},x')d^4x'+\cdots, \tag{14}$$

$$V(x)=V^{\text{in}}(x)+\kappa\int D^F(x-x')J(g^{\text{in}}_{\mu\nu},V^{\text{in}},x')d^4x'+\cdots, \tag{15}$$

where $g^{\text{in}}_{\mu\nu}$ and V^{in} are in fields on their mass shell and with physical polarizations. If one wants to obtain the counterterms for a given process with, say, n external gravitons and m external scalars, then one must insert Eqs. (14) and (15) into $\Delta\mathcal{L}(g_{\mu\nu},V)$ and collect from the resulting infinite series all terms with the correct number of in fields. In order to define the propagators D in Eqs. (14) and (15), one must also introduce gauge-fixing terms for the classical fields. The insertion of the perturbation solution of Eqs. (14) and (15) into the counterterms produces genuine tree diagrams. That the resulting S-matrix elements are also independent of the gauge-fixing terms for the classical fields can be understood in the following way. One can consider $\Delta\mathcal{L}(V)$ as a new local interaction in a new Lagrangian $\mathcal{L}'(V)$ which is equal to the original unbroken Lagrangian $\mathcal{L}(V)$ to which we add a classical gauge-fixing term $\mathcal{L}^B(V)$ and this new interaction $\alpha\Delta\mathcal{L}(V)$. Since we know that the tree graphs corresponding to $\mathcal{L}' = \mathcal{L} + \mathcal{L}^B + \alpha\Delta\mathcal{L}$ are independent of $\mathcal{L}^B$, the same holds for all tree graphs linear in α, i.e., for the counterterms into which the classical tree graphs are inserted. The on-shell counterterms for a given physical process are thus independent both of the quantum and of the classical gauge-fixing terms, and these two terms may be chosen differently. For scalar-scalar scattering we should therefore replace each field $g_{\mu\nu}(x)$ in R in Eq. (13) by the expansion of Eq. (14), and collect all terms with four V^{in} fields and no $g^{\text{in}}_{\mu\nu}$ fields. However, in this particular example we can simplify the algebra somewhat. We first express R in terms of V by means of Eq. (12) and thus find the equivalent result

$$\Delta\mathcal{L} = \frac{\sqrt{-g}}{8\pi^2(n-4)}\,\frac{203}{1280}\,\kappa^4(\partial_\mu V)^4. \qquad (16)$$

That all expressions for $\Delta\mathcal{L}$ which are equal modulo the classical field equations give the same S matrix contributions after insertion of the iterative solution of Eqs. (14) and (15) follows from the fact that the classical field equations themselves can be viewed as relations between infinite series of tree graphs. In Eq. (16) only the first term in the expansion for V in Eq. (15) contributes to scalar-scalar scattering. Using the kinematics of Fig. 3, we conclude that the sum of the one-loop divergences in the S matrix for gravitational massless scalar-scalar scattering is given by

$$S^{\text{div}} = \frac{-i\kappa^4}{8\pi^2(n-4)}\,\frac{203}{640}\,(s^2+t^2+u^2). \qquad (17)$$

In Sec. IV we will recalculate this same quantity using normal field theory.

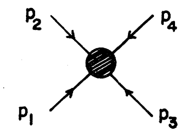

FIG. 3. Kinematics for scalar-scalar scattering. Mandelstam variables are $s=(p_1+p_2)^2$, $t=(p_1+p_3)^2$, and $u=(p_1+p_4)^2$.

III. GAUGE DEPENDENCE OF COUNTERTERMS IN THE BACKGROUND-FIELD METHOD

DeWitt[1] has shown that the generating functional for loop diagrams,

$$Z(V) = \int d\phi\, d\chi\, d\chi^* \exp\left\{ i\int d^4x\left[\mathcal{L}(V+\phi) - \mathcal{L}(V) - \phi\frac{\delta\mathcal{L}}{\delta V} + \mathcal{L}^B + \mathcal{L}^G\right]\right\}, \qquad (18)$$

is independent of $\mathcal{L}^B$ provided $\delta\mathcal{L}/\delta V = 0$ and provided $\mathcal{L}^B$ is background gauge invariant. The proof proceeds as follows. If one replaces $\mathcal{L}^B$ by $\mathcal{L}^B + \delta\mathcal{L}^B$ in Eq. (18), then, introducing new integration variables ϕ', χ', and $\chi^{*\prime}$ related to the old ones by a quantum gauge transformation where the gauge parameters are determined by $\delta\mathcal{L}^B$, one recovers the original expression in Eq. (18), provided that $\delta\mathcal{L}/\delta V = 0$ and $\mathcal{L}^B$ is background gauge invariant.

It is, however, interesting to ask what can be said about the gauge dependence of $Z(V)$ if these requirements are not fulfilled. Kallosh has observed that for pure Yang-Mills fields the counterterms are actually independent of the quantum gauge-fixing term even off the mass shell, i.e., when $\delta\mathcal{L}/\delta V$ is not equal to zero. Her proof proceeds as follows.[3,14] Under the same change of variables $(\phi^a_\mu)' = \phi^a_\mu + \delta\phi^a_\mu$ that DeWitt uses, it follows from Eq. (18) that, in an obvious notation,

$$Z_{\mathcal{L}^B+\delta\mathcal{L}^B}(V) - Z_{\mathcal{L}^B}(V) = -i\int\left[\frac{\delta\mathcal{L}(V)}{\delta V^a_\mu}\right]\langle\delta\phi^a_\mu\rangle d^4x, \qquad (19)$$

where $\langle\delta\phi^a_\mu\rangle$ is the expectation value of the variation $\delta\phi^a_\mu$ in the path-integral sense. For Yang-Mills fields $\delta\mathcal{L}/\delta V^a_\mu = D_\nu G^a_{\mu\nu}$ and, since both Z functionals in Eq. (19) are background gauge-invariant, so is the right-hand side of this equation. In particular, since the counterterms $\Delta\mathcal{L}$ are local and background gauge-invariant functions of dimension four, it follows that

$$\Delta\mathcal{L}_{\mathcal{L}^B+\delta\mathcal{L}^B} - \Delta\mathcal{L}_{\mathcal{L}^B} = \alpha(D_\mu G^a_{\mu\nu})^2. \qquad (20)$$

However, the right-hand side of this equation contains terms with six external Yang-Mills fields, and since one-particle-irreducible diagrams with six external Yang-Mills fields are convergent, it follows that α must be equal to zero. We note that in the presence of matter fields this conclusion is not valid, although Eq. (19) still holds. One obtains terms proportional to the matter-field classical equations, and the change in the counterterms need not be zero in this case.

We will now investigate whether a similar theorem holds for gravitation without matter fields. For pure gravitation we have

$$\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} = \sqrt{-g}\,(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R), \qquad (21)$$

where $g_{\mu\nu}$ is the classical gravitational field, and the same arguments as before lead to the following expression for the divergent part of the expectation value of the variation of the quantum gravitational field in the one-loop approximation:

$$\langle \delta h_{\mu\nu} \rangle = \frac{\kappa}{n-4}(\alpha R_{\mu\nu} + \beta g_{\mu\nu} R). \tag{22}$$

However, unlike the case of Yang-Mills theory, all one-particle-irreducible loop diagrams with arbitrarily many external gravitons are equally divergent; this is caused by the double-derivative couplings of gravitons to each other. It follows that the coefficients α and β in Eq. (22) need not be zero, and the counterterms in pure gravitation will, in general, be gauge dependent off the mass shell. We have not verified this explicitly in an example, because any gauge other than the harmonic gauge in Eq. (10) leads to a complicated propagator, involving momentum factors in the numerator, thus excluding the applicability of the algorithm of Ref. 5. However, in example 4 below we show that for matter-gravity interactions the counterterms are gauge dependent off shell.

We will now verify DeWitt's and Kallosh's theorems in a series of four examples. The calculations are quite elaborate, but since there are no difficulties of principle we merely record below the answers.

Example 1: Pure Yang-Mills theory in a covariant parameter-dependent gauge. The Lagrangian is given in the Lorentz-type gauge by

$$\mathcal{L} = -\tfrac{1}{4}[G^a_{\alpha\beta}(V^a_\mu + \phi^a_\mu)]^2 - \tfrac{1}{4}[G^a_{\alpha\beta}(V)]^2 - \phi^a_\nu[D_\mu G^a_{\mu\nu}(V)] - \tfrac{1}{2}\alpha(D_\mu \phi^a_\mu)^2. \tag{23}$$

For $\alpha = 1$, 't Hooft[5] has calculated the counterterms and found

$$\Delta\mathcal{L} = \frac{g^2}{8\pi^2(n-4)}\frac{11}{6}[G^a_{\alpha\beta}(V)]^2, \tag{24}$$

which result incidentally proves the asymptotic freedom of Yang-Mills fields. For $\alpha \neq 1$ we cannot use his algorithm since the propagator is not of the Feynman form

$$P^{ab}_{\mu\nu}(k) = -i[\eta_{\mu\nu} - (\alpha - 1)k_\mu k_\nu/k^2]\delta^{ab}/k^2. \tag{25}$$

However, the only background gauge-invariant counterterm is the one in Eq. (24). In order to check that for $\alpha \neq 1$ the same factor $\frac{11}{6}$ is found also, we calculate the two-point function. The $V\phi\phi$ vertex is given by

$$\mathcal{L}(V,\phi,\phi) = g\epsilon^{abc}V^b_\nu[-(\partial_\nu\phi^a_\mu)\phi^c_\mu + 2(\partial_\mu\phi^a_\nu)\phi^c_\mu - (1+\alpha)(\partial_\mu\phi^a_\mu)\phi^c_\nu], \tag{26}$$

while the ghost-ghost-V vertex is the usual expression, independent of α. Straightforward calculation indeed also reproduces for arbitrary α the same coefficient $\frac{11}{6}$, confirming Kallosh's theorem.[3] This result can be understood in a more physical way. Absence of a counterterm $(D_\mu V^a_\mu)^2$ implies that the Z factors for α and V^a_μ are equal, $Z_\alpha = Z$, while renormalizability of massless Yang-Mills fields implies that $Z = Z_g$. Since, however, the coupling-constant renormalization is given by

$$g^R = \frac{Z^{3/2}}{Z_g} g = Z^{1/2} g, \tag{27}$$

we see that in this example gauge independence of the counterterms off the mass shell is equivalent to the gauge independence of the coupling-constant renormalization.

Example 2: Pure Yang-Mills fields in a noncovariant gauge. In order to extend the results of DeWitt to gauge-fixing terms which are not background gauge independent, we consider the quantum gauge-fixing term

$$\mathcal{L}^B = -\tfrac{1}{2}(\partial_\mu \phi^a_\mu)^2. \tag{28}$$

Global isospin invariance still holds, hence *a priori*

$$\Delta\mathcal{L} \sim a_1(G^a_{\mu\nu})^2 + a_2(G^a_{\mu\nu}\epsilon^{abc}V^b_\mu V^c_\nu) + a_3(\vec{V}_\mu \times \vec{V}_\nu)^2 + a_4(\partial_\mu V^a_\mu)^2 + a_5(V^a_\mu V^a_\mu)^2. \tag{29}$$

Since the propagators are still of the Feynman form, we can apply 't Hooft's algorithm[5] and find, after a tedious calculation,

$$\Delta\mathcal{L} = \frac{g^2}{8\pi^2(n-4)}[\tfrac{5}{6}(G^a_{\mu\nu})^5 - g\epsilon^{abc}G^a_{\mu\nu}V^b_\mu V^c_\nu]. \tag{30}$$

Off the mass shell this is no longer of the form of Eq. (24), hence Kallosh's theorem does not hold for noncovariant gauges. On shell, however, $D_\mu G^a_{\mu\nu} = 0$, from which one readily derives

$$(G^a_{\mu\nu})^2 = -g\epsilon^{abc}G^a_{\mu\nu}V^b_\mu V^c_\nu \quad \text{(on shell)}. \tag{31}$$

Inserting Eq. (31) into Eq. (30) and comparing with Eq. (24), we see that DeWitt's theorem still holds for this particular noncovariant gauge. One might have chosen, instead of Eq. (28),

$$\mathcal{L}^B = -\tfrac{1}{2}(\partial_\mu\phi^a_\mu + \partial_\mu V^a_\mu)^2. \tag{32}$$

We still get the same off-shell counterterm of Eq. (30), since the terms of $\mathcal{L}^B$ quadratic in ϕ are the same. The classical field equations still are $D_\mu G^a_{\mu\nu} = 0$; we stress that one should not simply define them by requiring that all terms linear in ϕ vanish, but by $\delta\mathcal{L}/\delta V = 0$ instead of $\delta(\mathcal{L} + \mathcal{L}^B)/\delta V = 0$. We conclude that in this particular example even background noncovariant gauge-fixing terms give a correct result. It is, however, more in the spirit of the background-field method to use background gauge-invariant quantum gauge-fixing terms.

Example 3: Yang-Mills fields in the axial gauge. In order to verify that the counterterms are not only parameter independent but even independent of the functional form of the gauge-fixing term, we replace the Lorentz-type gauge-fixing term in example 1 by the axial gauge-fixing term

$$\mathcal{L}^B = -\tfrac{1}{2}\beta(\phi_3^a)^2. \tag{33}$$

The Lagrangian is still background gauge invariant, but no longer Lorentz invariant. (This gauge-fixing term is not a mass term since only the component $\mu = 3$ of ϕ_μ^a appears.) It follows that

$$\Delta\mathcal{L} = a[G^a_{\mu\nu}(V)]^2 + b[G^a_{\mu 3}(V)]^2. \tag{34}$$

However, Eq. (34) can be absorbed into the original action by rescaling of V_3^a, V_μ^a, and g only if $b = 0$. Hence, invoking renormalizability of Yang-Mills theory, we conclude that $b = 0$, and proceed to calculate a in Eq. (34). Calculating the two-point function $\langle V_3^a V_3^a\rangle$, we find indeed the same coefficient $\frac{11}{6}$, thus verifying DeWitt's and Kallosh's theorems. Some details are given in Appendix A.

Example 4: Scalar-graviton interactions with a parameter-dependent gauge. We return to the Lagrangian in Eq. (7), but supplement it this time with the quantum gauge-fixing term

$$\mathcal{L}^B = \sqrt{-g}\,(-D^\mu h_{\mu\nu} + \tfrac{1}{2}D_\nu h_\mu^\mu - \tfrac{1}{2}\alpha\phi\partial_\nu V)^2, \tag{35}$$

where all operations such as covariant differentiation and squaring are again taken with respect to $g_{\mu\nu}$. The propagators are still of Feynman form; hence we apply the algorithm of Ref. 5. After a lengthy calculation we find

$$\Delta\mathcal{L} = \Delta\mathcal{L}(\alpha = 1) + \frac{\sqrt{-g}\,\kappa^2}{8\pi^2(n-4)}\left\{\frac{(\alpha-1)}{4}(\Box V)^2 + \frac{(\alpha-1)^2}{8}[R(\partial_\mu V)^2 + \tfrac{1}{4}\kappa^2(\partial_\mu V)^4]\right\}. \tag{36}$$

Inserting the classical field equations of Eq. (12) we see that $\Delta\mathcal{L}$ is indeed α independent on shell, in agreement with DeWitt's theorem, but α dependent off shell, in agreement with the conclusions following Eq. (20).

IV. ONE-LOOP DIVERGENCES OF GRAVITATIONAL SCALAR-SCALAR SCATTERING IN NORMAL FIELD THEORY

In this section we give the results of our calculations for the ultraviolet divergences in gravitational scattering of two identical massless scalar particles in the one-loop approximation using normal field theory. The divergences show up as poles at $n = 4$ in the dimensional regularization scheme[12] which we use throughout. The calculation is carried out in several steps. We first calculate the two-point functions, i.e., self-energy diagrams for graviton and scalar particles, then gravitational corrections to the graviton-scalar-scalar vertex, and finally the divergences in scalar-scalar scattering.

At various steps we show that the results satisfy Ward identities. This serves two purposes: (1) to show that the dimensional regularization scheme indeed preserves the Ward identities and (2) to use the Ward identities in checking our calculations.

The Lagrangian of the system considered is given by

$$\mathcal{L} = \mathcal{L}^E + \mathcal{L}^S + \mathcal{L}^B + \mathcal{L}^G, \tag{37}$$

where $\mathcal{L}^E + \mathcal{L}^S$ (Einstein plus scalar) is given in Eq. (7). $\mathcal{L}^B$ and $\mathcal{L}^G$ are, respectively, the gauge-fixing term and the ghost Lagrangian derived from it. In the harmonic gauge

$$\mathcal{L}^B = (h_{\alpha\mu,\alpha} - \tfrac{1}{2}h_{\alpha\alpha,\mu})^2, \tag{38}$$

and $\mathcal{L}^G$ becomes, for a complex vector ghost field χ_μ,

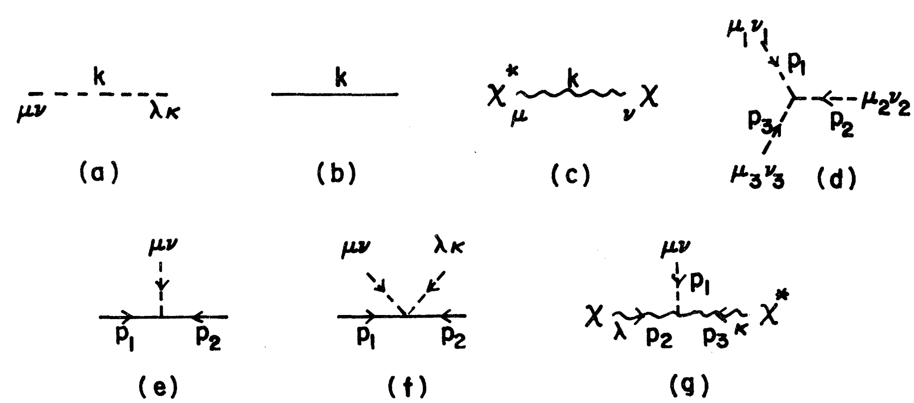

FIG. 4. Feynman rules for the graviton-scalar system. Broken lines, solid lines, and wavy lines represent graviton, scalar, and ghost fields, respectively.

$$\mathcal{L}^G = \kappa\chi_\mu^*\left(\frac{1}{\kappa}\eta_{\mu\beta}\Box + h_{\beta\alpha,\alpha}\partial_\mu + h_{\mu\beta,\alpha}\partial_\alpha + h_{\mu\beta}\Box + h_{\mu\alpha,\beta}\partial_\alpha + h_{\mu\alpha,\beta\alpha} - h_{\beta\alpha,\mu}\partial_\alpha - \tfrac{1}{2}h_{\alpha\alpha,\beta}\partial_\mu - \tfrac{1}{2}h_{\alpha\alpha,\beta\mu}\right)\chi_\beta \tag{39}$$

The relevant Feynman rules can therefore be summarized as follows (see Fig. 4):
(a) graviton propagator,

$$P_{\mu\nu,\lambda\kappa}(k) = (i/2k^2)(\eta_{\mu\lambda}\eta_{\nu\kappa} + \eta_{\mu\kappa}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\kappa}); \tag{40}$$

(b) scalar propagator,

$$P(k) = i/k^2; \tag{41}$$

(c) ghost propagator,

$$P_{\mu\nu}(k) = -i\eta_{\mu\nu}/k^2; \tag{42}$$

(d) three-graviton vertex,

$$\begin{aligned} V(\mu_1\nu_1 p_1; \mu_2\nu_2 p_2; \mu_3\nu_3 p_3) &= -2i\kappa(p_{1\nu_3}p_{3\mu_2}\eta_{\mu_1\nu_2}\eta_{\nu_1\mu_3} - \tfrac{1}{2}p_2\cdot p_3\eta_{\mu_1\nu_3}\eta_{\nu_1\mu_2}\eta_{\nu_2\mu_3} + \tfrac{1}{2}p_{3\mu_2}p_{2\nu_3}\eta_{\mu_1\nu_2}\eta_{\nu_1\mu_3} \\ &\quad + \tfrac{1}{2}p_2\cdot p_3\eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2}\eta_{\mu_3\nu_3} + \tfrac{1}{4}p_{1\mu_3}p_{2\nu_3}\eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2} + \tfrac{1}{2}p_{1\nu_2}p_{1\mu_3}\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_3} \\ &\quad - \tfrac{1}{4}p_{2\mu_3}p_{3\nu_2}\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_3} + \tfrac{1}{8}p_2\cdot p_3\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_3}\eta_{\nu_2\mu_3} - \tfrac{1}{8}p_2\cdot p_3\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3} \\ &\quad - \tfrac{1}{4}p_{2\nu_3}p_{2\mu_3}\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} + \tfrac{1}{2}p_{2\nu_3}p_{2\mu_3}\eta_{\mu_1\nu_2}\eta_{\mu_2\nu_1}) \\ &\quad + \text{permutations among indices 1, 2, and 3 and over-all symmetrization in } \mu \text{ and } \nu;^{15} \end{aligned} \tag{43}$$

(e) graviton-scalar-scalar vertex,

$$V_{\mu\nu}(p_1, p_2) = -\tfrac{1}{2}i\kappa(\eta_{\mu\nu}p_1\cdot p_2 - p_{1\mu}p_{2\nu} - p_{1\nu}p_{2\mu}); \tag{44}$$

(f) two-graviton-two-scalar vertex,

$$\begin{aligned} V_{\mu\nu,\lambda\kappa}(p_1, p_2) = -\tfrac{1}{4}i\kappa^2[&(\eta_{\mu\nu}\eta_{\lambda\kappa} - \eta_{\mu\lambda}\eta_{\nu\kappa} - \eta_{\mu\kappa}\eta_{\nu\lambda})(p_1\cdot p_2) + \eta_{\nu\lambda}(p_{1\mu}p_{2\kappa} + p_{1\kappa}p_{2\mu}) + \eta_{\mu\kappa}(p_{1\lambda}p_{2\nu} + p_{1\nu}p_{2\lambda}) \\ &+ \eta_{\mu\lambda}(p_{1\nu}p_{2\kappa} + p_{1\kappa}p_{2\nu}) + \eta_{\nu\kappa}(p_{1\lambda}p_{2\mu} + p_{1\mu}p_{2\lambda}) - \eta_{\mu\nu}(p_{1\lambda}p_{2\kappa} + p_{1\kappa}p_{2\lambda}) - \eta_{\lambda\kappa}(p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu})]; \end{aligned} \tag{45}$$

(g) graviton-ghost vertex,

$$\begin{aligned} V_{\mu\nu,\lambda,\kappa}(p_1, p_2, p_3) = -i\kappa[&\eta_{\mu\lambda}(p_{3\kappa}p_{2\nu} - p_{3\nu}p_{2\kappa}) - p_2\cdot p_3\eta_{\mu\kappa}\eta_{\nu\lambda} - p_{1\lambda}p_{3\nu}\eta_{\mu\kappa} + \tfrac{1}{2}p_{1\lambda}p_{3\kappa}\eta_{\mu\nu}] \\ &+ \text{symmetrization in } \mu \text{ and } \nu.^{15} \end{aligned} \tag{46}$$

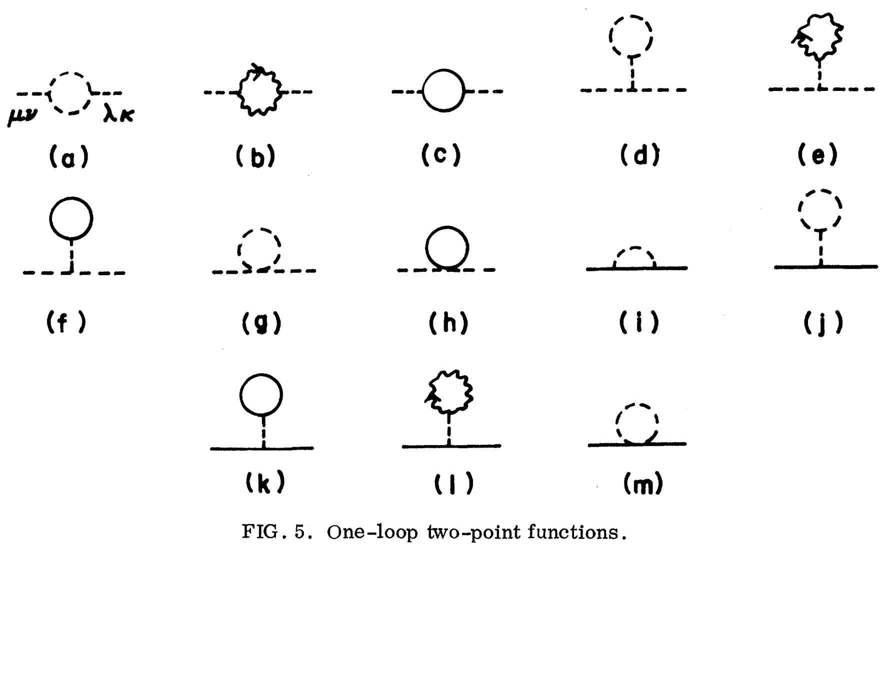

FIG. 5. One-loop two-point functions.

We have checked that the three-graviton vertex satisfies the two Ward identities

$$p_{1\mu_1}p_{2\mu_2}p_{3\mu_3}V(\mu_1\nu_1p_1;\mu_2\nu_2p_2;\mu_3\nu_3p_3)=0 \tag{47}$$

and

$$p_{1\mu_1}V(\mu_1\nu_1p_1;\mu_2\nu_2p_2;\mu_3\nu_3p_3)=0, \tag{48}$$

where in Eq. (48) the gravitons with indices 2 and 3 are on the mass shell. These identities follow from

$$\langle T[(h_{\mu\nu,\nu}(x)-\tfrac{1}{2}h_{\nu\nu,\mu}(x))(h_{\alpha\gamma,\gamma}(y)-\tfrac{1}{2}h_{\gamma\gamma,\alpha}(y))(h_{\beta\kappa,\kappa}(z)-\tfrac{1}{2}h_{\kappa\kappa,\beta}(z))]\rangle=0 \tag{49}$$

and

$$\langle T[(h_{\mu\nu,\nu}(x)-\tfrac{1}{2}h_{\nu\nu,\mu}(x))h_{\alpha\beta}(y)h_{\gamma\delta}(z)]\rangle=0, \tag{50}$$

where the second and the third gravitons are on the mass shell in Eq. (50). One may easily derive these Ward identities using the techniques of Refs. 16–18 (see Appendix B).

We first calculate the divergences for the graviton self-energy diagrams (Fig. 5). In terms of the divergent integral D,

$$D=\frac{1}{(2\pi)^4}\int\frac{d^nk}{(k^2+M^2)^2}=\frac{i\pi^{n/2}}{(2\pi)^4}\Gamma(2-\tfrac{1}{2}n)=\frac{-i}{8\pi^2(n-4)}+\text{finite terms}, \tag{51}$$

we find the contribution from the graviton loop

$$S(5(\text{a}))=\kappa^2[\tfrac{13}{15}p_\mu p_\nu p_\lambda p_\kappa+\tfrac{27}{80}(p\cdot p)^2\eta_{\mu\nu}\eta_{\lambda\kappa}+\tfrac{71}{240}(p\cdot p)^2(\eta_{\kappa\mu}\eta_{\lambda\nu}+\eta_{\lambda\mu}\eta_{\kappa\nu})$$
$$-\tfrac{19}{60}(p\cdot p)(p_\mu p_\nu\eta_{\lambda\kappa}+p_\lambda p_\kappa\eta_{\mu\nu})-\tfrac{11}{40}(p\cdot p)(p_\mu p_\kappa\eta_{\lambda\nu}+p_\mu p_\lambda\eta_{\kappa\nu}+p_\nu p_\kappa\eta_{\lambda\mu}+p_\nu p_\lambda\eta_{\kappa\mu})]D \tag{52}$$

and the contribution from the ghost loop

$$S(5(\text{b}))=\kappa^2[-\tfrac{1}{6}p_\mu p_\nu p_\lambda p_\kappa-\tfrac{7}{48}(p\cdot p)^2\eta_{\mu\nu}\eta_{\lambda\kappa}-\tfrac{1}{24}(p\cdot p)^2(\eta_{\kappa\mu}\eta_{\lambda\nu}+\eta_{\lambda\mu}\eta_{\kappa\nu})$$
$$+\tfrac{1}{8}(p\cdot p)(p_\mu p_\nu\eta_{\lambda\kappa}+p_\lambda p_\kappa\eta_{\mu\nu})+\tfrac{1}{48}(p\cdot p)(p_\mu p_\kappa\eta_{\lambda\nu}+p_\mu p_\lambda\eta_{\kappa\nu}+p_\nu p_\kappa\eta_{\lambda\mu}+p_\nu p_\lambda\eta_{\kappa\mu})]D. \tag{53}$$

The contribution from the scalar loop gives

$$S(5(\text{c}))=\kappa^2[\tfrac{1}{60}p_\mu p_\nu p_\lambda p_\kappa+\tfrac{1}{80}(p\cdot p)^2\eta_{\mu\nu}\eta_{\lambda\kappa}+\tfrac{1}{480}(p\cdot p)^2(\eta_{\mu\lambda}\eta_{\nu\kappa}+\eta_{\mu\kappa}\eta_{\nu\lambda})$$
$$-\tfrac{1}{80}(p\cdot p)(p_\mu p_\nu\eta_{\lambda\kappa}+p_\lambda p_\kappa\eta_{\mu\nu})-\tfrac{1}{480}(p\cdot p)(p_\mu p_\kappa\eta_{\lambda\nu}+p_\mu p_\lambda\eta_{\kappa\nu}+p_\nu p_\kappa\eta_{\lambda\mu}+p_\nu p_\lambda\eta_{\kappa\mu})]D. \tag{54}$$

Tadpole diagrams [i.e., Figs. 5(d), 5(e), 5(f), 5(g), and 5(h)] vanish in the dimensional regularization scheme.

According to the Ward identity[16]

$$\langle T[(h_{\mu\nu,\nu}(x)-\tfrac{1}{2}h_{\nu\nu,\mu}(x))(h_{\lambda\kappa,\kappa}(y)-\tfrac{1}{2}h_{\kappa\kappa,\lambda}(y))]\rangle=-\delta_{\mu\lambda}\delta^4(x-y) \tag{55}$$

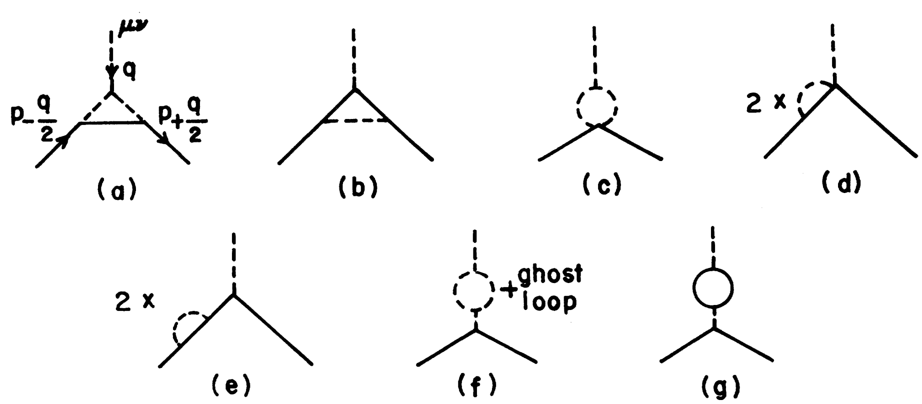

FIG. 6. Gravitational corrections to the graviton-scalar-scalar vertex.

the contribution $T_{\mu\nu,\lambda\kappa}(p) = S(5(\text{a})) + S(5(\text{b}))$ to the graviton self-energy diagrams should satisfy

$$p_\nu p_\kappa T_{\mu\nu,\lambda\kappa}(p) = 0. \tag{56}$$

This holds, as may be easily verified from Eqs. (52) and (53). Similarly, $S(5(\text{c}))$ satisfies the same identity. The divergence for the massless scalar self-energy diagram, Fig. 5(i), is found to be zero zero. Tadpole diagrams [i.e., Figs. 5(j), 5(k), 5(l), and 5(m)] again vanish.

There are seven diagrams, as shown in Fig. 6, for the gravitational corrections to the graviton-scalar-scalar vertex. We list the results for the divergences of each diagram:

$$\begin{aligned}
&S(6(\text{a})) = \kappa^3(\tfrac{1}{2}q^2 p_\mu p_\nu - \tfrac{1}{8}q^2 q_\mu q_\nu + \tfrac{1}{8}q^4\eta_{\mu\nu})D,\\
&S(6(\text{b})) = \kappa^3(-\tfrac{1}{6}q^2 p_\mu p_\nu + \tfrac{1}{24}q^2 q_\mu q_\nu - \tfrac{1}{12}q^4\eta_{\mu\nu})D,\\
&S(6(\text{c})) = \kappa^3(-\tfrac{2}{3}q^2 p_\mu p_\nu - \tfrac{1}{24}q^2 q_\mu q_\nu + \tfrac{1}{24}q^4\eta_{\mu\nu})D,\\
&S(6(\text{d})) = \kappa^3(\tfrac{1}{6}q^2 p_\mu p_\nu + \tfrac{1}{24}q^2 q_\mu q_\nu)D,\\
&S(6(\text{e})) = 0,\\
&S(6(\text{f})) = \kappa^3(\tfrac{61}{120}q^2 p_\mu p_\nu + \tfrac{23}{480}q^2 q_\mu q_\nu - \tfrac{23}{480}q^4\eta_{\mu\nu})D,\\
&S(6(\text{g})) = \kappa^3(\tfrac{1}{240}q^2 p_\mu p_\nu + \tfrac{1}{320}q^2 q_\mu q_\nu - \tfrac{1}{320}q^4\eta_{\mu\nu})D.
\end{aligned} \tag{57}$$

Here we have a Ward identity (for a derivation see Appendix B).

$$\langle T[(h_{\mu\nu,\nu}(x) - \tfrac{1}{2}h_{\nu\nu,\mu}(x))\phi(y)\phi(z)]\rangle = 0, \tag{58}$$

where the scalars are on the mass shell. It follows that multiplying the sum of $S(6(\text{a})) + \cdots + S(6(\text{g}))$ by q_ν yields zero [see Eq. (B8)]. Using the above results, we can show that this identity is valid.

Finally we calculate the divergences for scalar-scalar scattering (Fig. 7) and find, using $s = 2p_1 \cdot p_2$, $t = 2p_1 \cdot p_3$, and $u = 2p_1 \cdot p_4$,

$$\begin{aligned}
&S(7(\text{a})) = \tfrac{1}{16}\kappa^4 s^2 D,\\
&S(7(\text{b})) = \tfrac{1}{16}\kappa^4 u^2 D,\\
&S(7(\text{c})) = \tfrac{1}{8}\kappa^4(2s^2 + 2u^2 - t^2)D,\\
&S(7(\text{d})) = \tfrac{1}{16}\kappa^4(-2s^2 - 2u^2 + t^2)D,\\
&S(7(\text{e})) = -\tfrac{1}{48}\kappa^4(2s^2 + 2u^2 + t^2)D,\\
&S(7(\text{f})) = \tfrac{1}{1920}\kappa^4(123s^2 + 123u^2 - 37t^2)D.
\end{aligned} \tag{59}$$

Summing these contributions we get the divergence for the t-channel diagrams

$$\frac{\kappa^4}{1920}(403s^2 + 403u^2 - 197t^2)D. \tag{60}$$

By interchanging t and s and t and u in Eq. (60) we get the contributions for the s channel and u channel, respectively. Adding the s-, t-, and u-channel contributions we obtain the one-loop ultraviolet divergence for gravitational scalar-scalar scattering:

$$\frac{203}{640}\kappa^4(s^2 + t^2 + u^2)D, \tag{61}$$

which agrees with the result obtained in Sec. II.

V. CONCLUSIONS

In this article we have verified in several examples involving Yang-Mills and gravitational fields that in the background-field method counterterms are independent of the choice of quantum gauge, provided the gauge-fixing term is background gauge invariant and provided the classical fields satisfy the classical field equations. The classical field equations are solved by adding a classical

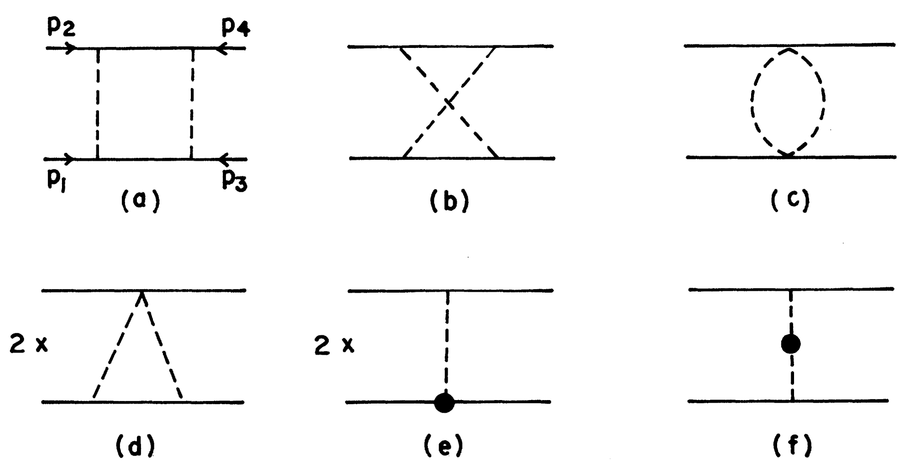

FIG. 7. Gravitational scalar-scalar scattering. ● in (e) denotes the sum of Figs. (6a), (6b), (6c), (6d), and (6e). ● in (f) denotes the sum of graviton, scalar, and ghost one-loop contributions.

gauge-fixing term to the classical action. For the counterterms (but presumably not for the whole S matrix) the classical and quantum gauge-fixing terms may differ. In particular, we have checked the on-shell gauge independence and off-shell gauge dependence of the graviton-scalar system. These results are in agreement with theorems by DeWitt[1] and Kallosh.[3]

The counterterms of pure Yang-Mills theory are found to be gauge independent even off shell, in agreement with a theorem by Kallosh.[3] We have shown that the proof of this theorem is not valid for pure gravitation, and verified that it does not hold for gauge fields coupled to matter either.

Finally, we have calculated using normal field theory the one-loop counterterms for gravitational scalar-scalar scattering. The results agree with the corresponding counterterms in the background-field method, which we have extracted from the results of 't Hooft and Veltman.[4]

We conclude that there is complete agreement between the background-field method and normal field theory and note that for certain calculations the background-field method is vastly superior.

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APPENDIX A

The propagator for Yang-Mills bosons in the axial gauge $-\frac{1}{2}\beta(\phi_3^a)^2$ for $\beta\to 0$ is given by[19,20]

$$\Pi_{jl}^{ab}(k) = -i\delta^{ab}(\eta_{jl} - k_j k_l/k_3^2)/k^2, \qquad \Pi_{33}^{ab}(k) = \Pi_{3j}^{ab} = 0, \tag{A1}$$

while only the vertex $-(\partial_\nu\phi_\mu^a)\phi_\mu^c g\epsilon^{abc}V_\nu^b$ in Eq. (26) with $\alpha\to 0$ can contribute to $\langle V_3^a V_3^a\rangle$. Consider the contribution to $\Delta\mathcal{L}$ of the form $2\alpha(\partial_1 V_3^a)^2$; consequently

$$\alpha = \frac{i}{4(V_3^a)^2}\frac{\partial^2}{\partial k^2}M\bigg|_{k=0}, \tag{A2}$$

where

$$M = \frac{6i(V_3^a)^2}{(2\pi)^4}\int d^4p\,\frac{2p_3^{\,2}}{(p+k)^2p^2}\left[\eta_{ij} - \frac{(p+k)_i(p+k)_j}{(p_3+k_3)^2}\right]\left[\eta_{ij} - \frac{p_ip_j}{p_3^{\,2}}\right]. \tag{A3}$$

Integrals of this type have been discussed in Ref. 19, and following the methods of that reference we find, defining

$$I(X) \equiv \frac{1}{(2\pi)^4}\int\frac{d^4pX}{(p+k)^2p^2}, \tag{A4}$$

that for

$$X = \left\{\frac{(p_1p_3)^2}{(p+k)^4}, \frac{p_1^{\,2}p^4}{(p+k)^4p_3^{\,2}}, \frac{(p_1p)^2}{(p+k)^2p_3^{\,2}}, \frac{p_3^{\,2}}{(p+k)^2}, \frac{p^4}{(p+k)^2p_3^{\,2}}, \frac{p_1^{\,2}}{p_3^{\,2}}\right\} \tag{A5}$$

the results are

$$I(X) = \{\tfrac{1}{24}, -1, -1, \tfrac{1}{4}, -2, -1\}\times\frac{1}{8\pi^2(n-4)}. \tag{A6}$$

With Eqs. (A2)–(A6) the results of example 3 in Sec. III can be reproduced.

APPENDIX B

Ward identities. The generating functional for the graviton-scalar system is given by

$$Z(j_{\mu\nu}, j) = \int dh_{\mu\nu}\,d\phi\,d\chi\,d\chi^*\exp\{i\int d^4x[\mathcal{L}^E + \mathcal{L}^S + (\partial_\mu\tilde{h}_{\mu\nu})^2 + \chi_\alpha^* M_{\alpha\beta}\chi_\beta + j_{\mu\nu}h_{\mu\nu} + j\phi]\}, \tag{B1}$$

where $\mathcal{L}^E$ and $\mathcal{L}^S$ are the Einstein and scalar Lagrangians, $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h_{\lambda\lambda}$ with $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, and $M_{\alpha\beta} = \kappa\delta(\partial_\mu\tilde{h}_{\mu\alpha})/\delta\xi^\beta$ where $\kappa^{-1}\xi$ are the dimensionless gauge parameters. The classical actions are invariant under the gauge transformations

$$\kappa\delta h_{\mu\nu} = \xi^\alpha{}_{,\mu}g_{\alpha\nu} + \xi^\alpha{}_{,\nu}g_{\mu\alpha} + \xi^\alpha g_{\mu\nu,\alpha}, \tag{B2}$$

$$\delta\phi = \xi^\alpha\phi_{,\alpha}. \tag{B3}$$

Only the gauge-fixing term and the two source terms in Eq. (B1) are not invariant under Eqs. (B2) and

(B3) when ξ^α is restricted by

$$\xi^\alpha(x) = \int M^{-1}(x,y)^{\alpha\beta}\lambda_\beta(y)d^4y, \quad \lambda \text{ arbitrary,} \tag{B4}$$

where M^{-1} is defined by $M_{\alpha\beta}(x)M^{-1}(x,y)^{\beta\gamma} = \delta^\gamma_\alpha\delta^4(x-y)$ [but $M^{-1}M \neq MM^{-1}$, hence Eq. (B4) gives a true restriction]. The generator of all Ward identities is obtained as usual[16-18] by making the change of variables of Eqs. (B2) and (B3) in Eq. (B1). One finds for the coefficient of $\lambda_\sigma(y)$

$$\int d^4x\{2\partial_\mu \tilde{h}_{\mu\sigma}(x)\delta^4(x-y) + j_{\mu\nu}(x)[g_{\alpha\nu}(x)\partial x_\mu M^{-1}(x,y)^{\alpha\sigma} + g_{\mu\alpha}(x)\partial x_\nu M^{-1}(x,y)^{\alpha\sigma} + \partial x_\alpha g_{\mu\nu}(x)M^{-1}(x,y)^{\alpha\sigma}]$$
$$+\kappa j(x)\partial x_\alpha \phi(z)M^{-1}(x,y)^{\alpha\sigma}\}Z(j_{\mu\nu},j)=0, \tag{B5}$$

where one should replace $h_{\mu\nu}(x)$ by $-i\delta/\delta j_{\mu\nu}(x)$ and $\phi(x)$ by $-i\delta/\delta j(x)$ and substitute

$$M^{-1}(x,y)_{\alpha\beta} = -i\chi^*_\alpha(x)\chi_\beta(y). \tag{B6}$$

Differentiating Eq. (B5) once with respect to $j(z_1)$ and once with respect to $j(z_2)$ we get a relation between the graviton-scalar-scalar three-point function and diagrams involving ghost fields. Multiplying this relation by the three factors k^2, $p_1^2 - m^2$, and $p_2^2 - m^2$, where k and p_i are the graviton and the two scalar momenta, and putting the two scalar fields on their mass shell, we obtain the following result for the truncated graviton-scalar three-point function:

$$\langle h_{\mu\nu,\nu}(x)\phi(z_1)\phi(z_2)\rangle^{\text{truncated}} = 0. \tag{B7}$$

We have used the relation

$$\int \partial x_\mu P_{\mu\nu,\rho\sigma}(x,x')\tilde{h}_{\rho\sigma}(x')d^4x' = \partial x_\mu h_{\mu\nu}(x), \tag{B8}$$

where P is the graviton propagator, and the fact that the diagrams with ghost fields do not have poles both at $p_1^2 - m^2 = 0$ and $p_2^2 - m^2 = 0$.

The Ward identities in Eqs. (49) and (56) are derived using the techniques of Ref. 16. The Ward identity in Eq. (50) was derived in Ref. 21 and means that gravitons couple also to gravitons through a conserved energy-momentum tensor.

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[14]P. van Nieuwenhuizen thanks R. Kallosh for a correspondence on this point.

[15]By symmetrization we mean $A_{\mu\nu} \to \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$.

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