Solutions for the motion of an electron in electromagnetic fields

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(Received 25 September 1974)

New exact solutions of the Lorentz, Hamilton-Jacobi, Klein-Gordon, and Dirac equations for an electron moving in the field of a plane wave and in electric and magnetic fields have been found. The electric and magnetic fields are parallel to the direction of propagation of the plane wave. The magnetic field is constant and the electric field is an arbitrary function of the combination $ct-z$.

New exact solutions of the Lorentz, Hamilton-Jacobi, Klein-Gordon, and Dirac equations in the electromagnetic field configurations, including a plane wave, and uniform magnetic and electric fields, are presented in this paper. The directions of magnetic and electric fields are chosen to be parallel to the direction of the plane-wave propagation. The solutions are possible for the case when the electric field is constant as well as when it is an arbitrary function of the combination $ct-z$. However, the case of a constant electric field is of special interest as the fields satisfy the Maxwell equations without charges and currents. If one excludes the electric field from the solutions obtained, one has the Redmond solutions.[1]

The configuration of the electromagnetic field under consideration will be defined by means of the potentials A_μ as

$$\begin{aligned} A_0 &= A_3 = c\hbar e^{-1} f_0(\xi)\,, \\ A_1 &= Hy + c\hbar e^{-1} f_1(\xi)\,, \\ A_2 &= -c\hbar e^{-1} f_2(\xi)\,. \end{aligned} \tag{1}$$

Such a choice of potentials corresponds to the above-mentioned fields

$$\begin{aligned} E_x &= H_y = c\hbar e^{-1} f_1{}'(\xi), \\ E_y &= -H_x = -c\hbar e^{-1} f_2{}'(\xi), \\ E_z &= 2c\hbar e^{-1} f_0{}'(\xi), \\ H_z &= H\,. \end{aligned}$$

I. CLASSICAL EQUATIONS OF MOTION

Let the Lorentz equations be written in the standard form[2]

$$\begin{aligned} mc^2 \ddot{x}^i &= e g^{ij} F_{jk} \dot{x}^k\,, \quad \dot{x}^k = dx^k/ds\,, \\ -g^{00} &= g^{11} = g^{22} = g^{33} = -1, \quad g^{\alpha\beta} = 0, \quad \alpha \neq \beta\,. \end{aligned} \tag{2}$$

One can easily find from Eqs. (2) the three first obvious integrals of motion and the general solution of the classical problem in quadratures. However, the system of classical first integrals may be chosen in different ways. Here they are selected so that classical first integrals should have analogs in the quantum theory.

After subtraction of Eq. (2) for $i=0$ from the equation for $i=3$ one can find the first integral of motion λ,

$$K_0 \dot{\xi} + 2 f_0 = 2\lambda, \quad K_0 = mc\hbar^{-1}\,. \tag{3}$$

From the form of the potentials (1) and (3) we have

$$2c\hbar\lambda = \mathcal{E} - cp_z$$

where $\mathcal{E}$ is the total energy, p_z is the z component of the generalized momentum. From Eq. (2) for $i=1$ one can find another first integral k,

$$\begin{aligned} &K_0 \dot{x} - \epsilon\gamma y - f_1 = k, \\ &\epsilon = \operatorname{sgn} eH, \quad \gamma = |eH|(c\hbar)^{-1}, \end{aligned} \tag{4}$$

and taking into account Eq. (1), $\hbar k = p_x$ can be found from (4). In order to find the suitable third integral of motion, let the real functions $\chi_1(\xi)$ and $\chi_2(\xi)$ which are defined from the set of equations be introduced as

$$\begin{aligned} &2R\chi_1{}' + 2\gamma\chi_2 - \sqrt{2\gamma}\, f_2 = 0, \\ &2R\chi_2{}' - 2\gamma\chi_1 + \epsilon\sqrt{2\gamma}\,(k + f_1) = 0\,, \; R = 2(\lambda - f_0). \end{aligned} \tag{5}$$

The set of Eqs. (5) may be solved conveniently by using the complex function $\chi(\xi) = \chi_1 + i\chi_2$ (naturally $2\chi_1 = \chi + \chi^*$, $2i\chi_2 = \chi - \chi^*$), which satisfies

$$2iR\chi' + 2\gamma\chi - \sqrt{2\gamma}\,[\epsilon(k + f_1) + i f_2] = 0\,.$$

The solutions of this equation may be easily found to be

$$\begin{aligned} \sqrt{2}\,\chi(\xi) &= \sqrt{\gamma}\, \exp[i\phi(\xi)] \\ &\quad \times \int [R(\xi')]^{-1} \{ f_2(\xi') - i\epsilon[k + f_1(\xi')] \} \\ &\quad \times \exp[-i\phi(\xi')]\, d\xi'\,, \\ \phi(\xi) &= \gamma \int [R(\xi')]^{-1} d\xi'\,. \end{aligned} \tag{6}$$

The variable q can be determined by using the relation

$$\sqrt{2}\,q=\sqrt{2\gamma}\,y+\chi+\chi^* =\sqrt{2\gamma}\,y+2\chi_1 . \quad (7)$$

Then the third integral of motion Q can be found from Eq. (2) for $i=2$ and from Eqs. (3)–(5):

$$K_0{}^2\dot{q}^2+\gamma^2q^2=\gamma^2Q^2 . \quad (8)$$

It should be noted that the interval $s=c\tau$ (τ is the proper time) is determined from (8) as a function of ξ:

$$s=K_0\gamma^{-1}\phi(\xi) .$$

Using (5) and (7) one will get from (4)

$$\epsilon K_0\sqrt{\gamma}\,\dot{x}=\gamma q-\sqrt{2}\,K_0\dot{\chi}_2 . \quad (9)$$

Then from (7)–(9) x and y as functions of ξ can be found

$$\begin{aligned}\epsilon\sqrt{2\gamma}x&=i(\chi-\chi^*)-\sqrt{2}\,Q\cos\phi\\&=-2\chi_2-\sqrt{2}\,Q\cos\phi ,\\ \sqrt{2\gamma}\,y&=\sqrt{2}\,Q\sin\phi-\chi-\chi^*\\&=\sqrt{2}\,Q\sin\phi-2\chi_1 .\end{aligned} \quad (10)$$

Taking into account the identity $g_{ij}\dot{x}^i\dot{x}^j=1$ we obtain x^0 and z as functions of ξ:

$$2x^0-\xi=2z+\xi =\int\{K_0{}^2+|\sqrt{\gamma}\,Q\exp[i\phi(\xi')]-i\sqrt{2\gamma}\,\chi(\xi')-f_2(\xi')+i\epsilon[k+f_1(\xi')]|^2\}\times[R(\xi')]^{-2}d\xi' . \quad (11)$$

Thus, the formulas (10) and (11) determine the classical motion in a parametric form. If $f_0=0$ (so $R=2\lambda$), then from (10) and (11) one has the classical parametric equations of motion for an electron in a constant magnetic field and a plane wave which were discovered first in Ref. 3.

If the solution of the classical Hamilton-Jacobi equation

$$g^{ij}(c\partial_iS+eA_i)(c\partial_jS+eA_j)=m^2c^4,\quad \partial_i=\partial/\partial x^i$$

is defined in the form $S=-\hbar[S_0+F(q)]$, where the function S_0 is determined by the expression

$$4S_0=4\lambda(x^0+z)-4kx+2i\sqrt{2}\,q(\chi-\chi^*)+i(\chi^{*2}-\chi^2)+\int\{2K_0{}^2+2\gamma Q^2-\gamma|\chi(\xi')|^2+|\sqrt{2}\,\epsilon[k+f_1(\xi')]+i\sqrt{2}f_2(\xi')-\sqrt{\gamma}\chi(\xi')|^2\}[R(\xi')]^{-1}d\xi' , \quad (12)$$

then $F(q)$ is the classical action function for a harmonic oscillator $(F')^2=Q^2-q^2$. The equations of motion derived from the action function naturally coincide with (10) and (11).

II. RELATIVISTIC WAVE EQUATIONS

The solution of the Klein-Gordon equation

$$(\mathcal{P}^2-m^2c^2)\Psi_B=0,\quad \mathcal{P}_\mu=i\hbar\partial_\mu-ec^{-1}A_\mu$$

is found in the form (N being the normalization factor)

$$\Psi_B=NR^{-1/2}\exp(-iS_0)\Phi(q,\xi) . \quad (13)$$

For $\Phi(q,\xi)$ one gets the equation

$$(Q^2+\partial_{qq}{}^2-q^2+2iR\gamma^{-1}\partial_\xi)\Phi(q,\xi)=0 . \quad (14)$$

As a solution of this equation one can take the Φ function independent of ξ, that is $\Phi=U(q)$. Then from (14) one has $U=U_n(q)$ ($n=0,1,2,\ldots$). $U_n(q)$ are the Hermite functions which are connected with the Hermite polynomials by the relation

$$U_n(q)=(2^nn!\sqrt{\pi})^{-1/2}\exp(-\tfrac{1}{2}q^2)H_n(q).$$

In this case the integral of motion Q is quantized, $Q^2=2n+1$.

The solution of the Dirac equation

$$(\gamma^\mu\mathcal{P}_\mu-mc)\Psi_D=0$$

can be defined in the "block" form

$$\begin{aligned}\Psi_D&=NR^{-1}\exp(-iS_0)\begin{pmatrix}K_0+R+\sigma_3(\vec{\sigma}\vec{\mathfrak{F}})\\(K_0-R)\sigma_3+(\vec{\sigma}\vec{\mathfrak{F}})\end{pmatrix}\psi(q,\xi) ,\\ \vec{\mathfrak{F}}&=\vec{e}_1(k+\epsilon\gamma y+f_1)-\vec{e}_2(f_2-\sqrt{2\gamma}\,\chi_2+i\partial_y)\\&=\vec{e}_1(k+f_1-\epsilon\sqrt{2\gamma}\,\chi_1+\epsilon q\sqrt{\gamma})-\vec{e}_2(f_2-\sqrt{2\gamma}\,\chi_2+i\sqrt{\gamma}\,\partial_q) ,\end{aligned} \quad (15)$$

where $\vec{e}_1$ and $\vec{e}_2$ are the unit vectors along the x and y axes, $\vec{\sigma}$ are the Pauli matrices, and ψ is a two-component spinor satisfying the equation

$$(2iR\gamma^{-1}\partial_\xi+Q^2+\partial_{qq}{}^2-q^2+\epsilon\sigma_3)\psi=0 .$$

The solution of this equation may be found by two methods. ψ can be considered as the function of q only, and then two independent equations are obtained for the components of ψ. The solutions of these equations may be easily found. However, we shall use another method. If one assumes

$$\psi=\tfrac{1}{2}\{(1+\sigma_3)\exp[\tfrac{1}{2}i\epsilon\phi(\xi)]+(1-\sigma_3)\exp[-\tfrac{1}{2}i\epsilon\phi(\xi)]\}\Phi ,$$

then the equation (14) for Φ will be obtained. The solution of this equation can be written in the form

$$\Phi = U_n(q)\vartheta, \quad Q^2 = 2n+1$$

where ϑ is an arbitrary constant two-component spinor. The existence of this spinor shows that for the Dirac equation the wave function is not defined uniquely by the three integrals of motion λ, k, Q. For its definition a fourth spin integral of motion is necessary. If $H = f_0 = 0$, then such an integral is known from Ref. 4. For $f_i = 0$ $(i = 0, 1, 2)$, $H \neq 0$, such integrals are also known.[5] However, in a general case of nonzero f_i and H we failed to find the spin integral.

By normalization of the functions (13) by the charge density one gets

$$2K_0 = |N|^2 \int R^{-2}\rho_0 U_n^2(q)\, d^3x,$$

$$\rho_0 = K_0^2 + \gamma Q^2 + R^2 + |\epsilon(k+f_1) + if_2 - \sqrt{2\gamma}\,\chi|^2 .$$

It is seen from the above that, if for some $\xi = \xi_0$ it is possible that $R(\xi_0) = 0$, the integrand may have a nonintegrable singularity and the functions (13) are non-normalizable in the general sense. In the same way, the normalization condition for the functions (15) can also be impossible in a general sense,

$$1 = 2|N|^2(\vartheta^\dagger\vartheta) \int R^{-2}(\rho_0 - \epsilon\gamma\bar{\sigma}_3) U_n^2(q)\, d^3x ,$$

$$\bar{\sigma}_3 = (\vartheta^\dagger \sigma_3 \vartheta)(\vartheta^\dagger\vartheta)^{-1} .$$

However, if one takes a new definition of the scalar product following from the formulation of quantum field theory on the null plane, then it is possible to have the normalized solutions (13) and (15) and to prove the orthogonality and completeness relations for them. The main results and bibliography are given in Refs. 6–11. The idea is that the quantum field theory is formulated with the use of a curvilinear coordinate system (ξ, x, y, η), where $\xi = x^0 - x^3$, $x = x^1$, $y = x^2$, $\eta = x^0 + x^3$, and the coordinate ξ is taken to be time so that the scalar product of the fields is defined on the null plane $\xi = $const. A new scalar product can be obtained, if one proceeds from the well-known invariant expressions for scalar products of the Klein-Gordon and Dirac fields on an arbitrary spacelike surface.[12] The surface $\xi = $const is not spacelike but may be considered as a limit of a spacelike surface $\alpha x^0 - x^3 = $const $(\alpha > 1)$ as $\alpha \to 1$. Making use of these considerations one gets

$$(\psi, \phi)_\xi = \int dx\, dy\, d\eta [\psi^*(i\hbar\partial_\eta - ec^{-1}A_\eta)\phi - \phi(i\hbar\partial_\eta + ec^{-1}A_\eta)\psi^*] , \quad (16)$$

$$2A_\eta = A_0 + A_3$$

for the scalar product of Klein-Gordon fields on the null plane, and for the Dirac fields on the null plane one obtains

$$(\psi, \phi)_\xi = \int dx\, dy\, d\eta\, \psi^\dagger P_{(-)}\phi, \quad 2P_{(-)} = 1 - \gamma^0\gamma^3 . \quad (17)$$

It is not difficult to prove that if ψ and ϕ satisfy the corresponding wave equations and behave correctly on the boundaries of a three-dimensional region x, y, η, then (16) and (17) are independent of ξ.

Choosing $2N = \pi^{-3/2}$ in the function (13), from (16) one finds $(m = \{\lambda, k, n\})$

$$(\Psi_B^{m'}, \Psi_B^m)_\xi = \delta(\lambda - \lambda')\delta(k - k')\delta_{n,n'} ,$$

while the completeness condition has the form

$$(ic\hbar\partial_\eta - eA_\eta) \sum_m \Psi_B^m(\xi, x, y, \eta)\Psi_B^{m*}(\xi, x', y', \eta') = c\delta(x - x')\delta(y - y')\delta(\eta - \eta') .$$

Choosing $4N = \pi^{-3/2}$ and $\vartheta = \vartheta_\zeta$ $(\zeta = \pm 1)$ for the Dirac function (15) one can get in the same way

$$(\Psi_D^{m'}, \Psi_D^m)_\xi = \delta(\lambda - \lambda')\delta(k - k')\delta_{n,n'}\delta_{\zeta,\zeta'} \quad (m = \{\lambda, k, n, \zeta\}),$$

$$P_{(-)} \sum_m \Psi_D^m(\xi, x, y, \eta)\Psi_D^{m\dagger}(\xi, x', y', \eta') = P_{(-)}\delta(\eta - \eta')\delta(x - x')\delta(y - y') .$$

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