Tentative relativistic model for neutral Yukawa systems

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Some structurally important results in small as well as in large dimensions are presented concerning a spherical distribution of incoherent dust, the constituents of which are simultaneously sources of gravitational as well as repulsive short-range scalar fields.

I. INTRODUCTION

In investigating the structure of "elementary" particles, it is believed that the effects of gravitational interaction cannot be neglected.[1] In the 1950's, considerable interest was focused on a set of equations in which a real scalar field of long range is coupled with the gravitational field.[2-5] Since the scalar fields which find applications in physics are usually of short range some other physical interpretation is intended for the scalar potential. In the case of a real scalar field with short range, exact solutions are quite difficult to obtain. Duan'-I-Shi[6] first obtained a class of solutions to the equations corresponding to the fields of a point nuclear charge; however, his solutions contain several functions whose forms are not explicitly known. Stephenson[7] presented an approximate static spherically symmetric solution of the Einstein-Maxwell-Yukawa field equations which, he claimed, represents the classical fields of a proton.

It seems worthwhile to investigate the effect of a real scalar field coupled with the gravitational field in the formation of elementary structures. In a recent work the present authors[8] have shown that if one considers the distribution of incoherent dust charged in the scalar sense, in equilibrium under the influence of its own gravitational and long-range repulsive scalar fields, then the only possible solutions are those given by Das[9]; however, for such a distribution one has the matter density equal to the scalar charge density, which cannot be attributed to any known elementary particle.

In this paper we propose to consider a spherical distribution of incoherent dust in static equilibrium. The constituents of the dust are supposed to be the sources of gravitational as well as short-range repulsive Yukawa-type fields. In view of the difficulties in obtaining an exact solution, we have studied the approximate solutions and obtained some interesting results which are structurally important in small as well as in large dimensions.

II. GENERAL EQUATIONS

The Einstein scalar field equations are

$$R^\mu_\nu = -2\epsilon(T^\mu_\nu - \tfrac{1}{2}\delta^\mu_\nu T)/c^2\,, \tag{1}$$

$$S^{;\mu}_{;\mu} + S/l^2 = \epsilon\sigma\,, \tag{2}$$

$$T^\mu_\nu = \rho c^2 u^\mu u_\nu - c^2(2S^{,\mu}S_{,\nu} - \delta^\mu_\nu S^{,\alpha}S_{,\alpha} + \delta^\mu_\nu S^2/l^2)/2\epsilon\,, \tag{3}$$

where

$$\epsilon = 4\pi G/c^2\,, \tag{4}$$

ρ is a matter density, S is a short-range (l) repulsive scalar field, σ is the density of the source of S, and u^μ is the velocity field of ρ and σ. Superscripted and subscripted comma and semicolon mean ordinary and covariant derivatives, respectively.

For static spherically symmetric systems we consider the line element

$$ds^2 = e^{2\eta}dx^{02} - e^{2\alpha}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2\,, \tag{5}$$

with all functions $\rho, \sigma, S, \eta, \alpha$ depending only on r. Since the static condition implies $u^\mu = \delta^\mu_0 e^{-\eta}$, Eqs. (1) and (2) become

$$[\eta_{11} + \eta_1(\eta_1 - \alpha_1) + 2\eta_1/r]\,e^{-2\alpha} = \epsilon\rho + S^2/l^2\,, \tag{6}$$

$$[\eta_{11} + \eta_1(\eta_1 - \alpha_1) - 2\alpha_1/r]\,e^{-2\alpha} = -\epsilon\rho + S^2/l^2 + 2e^{-2\alpha}S_1{}^2\,, \tag{7}$$

$$r^{-2} - [r^{-2} + (\eta_1 - \alpha_1)/r]\,e^{-2\alpha} = \epsilon\rho - S^2/l^2\,, \tag{8}$$

$$r^{-2}e^{-\eta-\alpha}(r^2e^{\eta-\alpha}S_1)_1 - S/l^2 = -\epsilon\sigma\,, \tag{9}$$

where a subscript 1 means d/dr. One obtains from the Bianchi identity $T^\mu_{\nu;\mu} = 0$ the relation

$$\rho\eta_1 + \sigma S_1 = 0\,. \tag{10}$$

We are considering a static spherical distribution of incoherent dust of matter density $\rho(r)$. The gravitational collapse of this dust is supposed to be prevented by a stronger short-range repulsion of a Yukawa-type source density $\sigma(r)$ associated with the dust. Since the four equations (6)–(9) contain five unknowns $(\eta, \alpha, \rho, \sigma, S)$, one additional equation such as

$$\sigma = f\rho, \quad f = \text{const} \tag{11}$$

may be added.

In view of the difficulty in obtaining exact solutions we make use of an approximate method; we expand our three potentials η, α, S and the two densities ρ, σ in integral powers of some small dimensionless constant κ, to be identified later. In the lowest approximation we have taken the densities independent of κ, and differently from Stephenson[7] we take the three potentials proportional to κ. Then (6)–(10) simplify to

$$\eta_{11} + 2\eta_1/r = \epsilon\rho, \tag{12}$$

$$\eta_{11} - 2\alpha_1/r = -\epsilon\rho, \tag{13}$$

$$2\alpha/r - \eta_1 + \alpha_1 = \epsilon\rho r, \tag{14}$$

$$r^{-2}(r^2 S_1)_1 - S/l^2 = -f\epsilon\rho, \tag{15}$$

$$\rho(\eta_1 + fS_1) = 0. \tag{16}$$

III. INTERNAL SOLUTIONS

With $\rho \neq 0$ we get from (12), (15), and (16)

$$S_i = \epsilon(f^2 - 1)f^{-1}l^2\rho, \tag{17}$$

where the subscript i means internal. Substituting (17) into (15) we obtain

$$r^{-2}(r^2\rho_1)_1 + l^{-2}(f^2 - 1)^{-1}\rho = 0. \tag{18}$$

A short reflection shows that we must have $|\sigma| > \rho$, since if this were not the case, the sphere would collapse. So with $f^2 > 1$ in (18) we choose the regular solution

$$\rho = A\nu^2 r^{-1}\sin\nu r, \quad A = \text{const}, \tag{19}$$

$$\nu = l^{-1}(f^2 - 1)^{-1/2}. \tag{20}$$

Then from (17)

$$S_i = \epsilon'(R/fr)\sin\nu r, \tag{21}$$

where R is the radius of the distribution and

$$\epsilon' = \epsilon A/R \tag{22}$$

is a dimensionless constant; and from (16) and (14), respectively,

$$\eta_i = -\epsilon'(B + Rr^{-1}\sin\nu r), \quad B = \text{const}, \tag{23}$$

$$\alpha_i = \epsilon'(Rr^{-1}\sin\nu r - \nu R\cos\nu r). \tag{24}$$

Equation (13) is identically satisfied; the constant B will be fixed by boundary conditions.

IV. EXTERNAL SOLUTIONS; BOUNDARY CONDITIONS

For $\rho = 0$ we easily obtain integrating (12) to (15)

$$\eta_e = -\alpha_e = -Cr^{-1}, \quad S_e = Dr^{-1}e^{-r/l}, \tag{25}$$

where the subscript e means external, and C and D are constants of integration. These potentials have the usual Schwarzschild and Yukawa behavior at infinity.

We now impose that at the boundary of the sphere ($r = R$) the three potentials η, α, S be continuous as well as the radial first derivatives of η and S.

The continuity of S and S_1 not only fixes the constant D, but also prescribes to the radius R the discrete set of values given by

$$\cot\nu R = -(f^2 - 1)^{1/2}. \tag{26}$$

Since $\rho(r)$ in (19) should always be positive, we get from (26) that

$$\tfrac{1}{2}\pi < \nu R < \pi. \tag{27}$$

Then with (26) and (27) we have

$$\sin\nu R = |f|^{-1}, \tag{28}$$

so from (21) and (25)

$$S_i = \epsilon' R f^{-1} r^{-1}\sin\nu r, \qquad S_e = \epsilon'(f|f|r)^{-1}e^{-(r-R)/l} \tag{29}$$

The continuity of η and η_1 at $r = R$ specifies the contants B and C, and the continuity of α is identically satisfied, giving

$$\eta_i = -\epsilon'(Rr^{-1}\sin\nu r - \nu R\cos\nu R), \tag{30}$$

$$\alpha_i = \epsilon'(Rr^{-1}\sin\nu r - \nu R\cos\nu r), \tag{31}$$

$$\eta_e = -\alpha_e = -\epsilon'(1 + R/l)R|f|^{-1}r^{-1}. \tag{32}$$

From equations (29) to (32) it is natural to identify the constant ϵ' with the necessarily small constant κ in terms of which we made our series expansion. This identification implies a restriction on the parameters of the system. Remembering that in Schwarzschild-type systems the mass m is defined by

$$\eta_e = -Gm/c^2 r, \tag{33}$$

we get from (32) that

$$\epsilon' = |f|(1 + R/l)^{-1}(Gm/c^2R); \tag{34}$$

one can verify from (20) and (28) that when $1 < f^2 < \infty$ we have that $\pi^{-1} < |f|(1 + R/l)^{-1} < 1$, so from (34) our approximate solution is valid when

$$Gm/c^2R \ll 1. \tag{35}$$

For practical purposes we define the "effective Yukawa charge" Y of our sphere by

$$c^2 S_e = GYr^{-1}e^{-r/l}; \tag{36}$$

then from (29) we obtain

$$Y = mf^{-1}(1 + R/l)^{-1}e^{R/l}, \tag{37}$$

and we can verify from (20), (27), (28), and (37) that $m< Y/f$ for all values of $f^2 > 1$.

A case which deserves a special consideration is that when $f^2 \gg 1$; then from (27) and (28) one has $\nu R \simeq \pi - |f|^{-1}$, so from (20) $R/l \simeq \pi|f| - 1$; that is, $R \gg l$. The internal quantities become

$$\rho \simeq (m\pi/4R^3)(\gamma^{-1}\sin\gamma - lR^{-1}\cos\gamma), \tag{38}$$

$$\eta_i \simeq -(Gm/c^2R)(1+\gamma^{-1}\sin\gamma), \tag{39}$$

$$\alpha_i \simeq (Gm/c^2R)(\gamma^{-1}\sin\gamma - \cos\gamma), \tag{40}$$

$$S_i \simeq \pm\pi(Gm/c^2R)(l/R)(\gamma^{-1}\sin\gamma + l/R), \tag{41}$$

$$\sigma \simeq \pm\rho R/\pi l, \tag{42}$$

where

$$\gamma = \pi r/R \tag{43}$$

is a radial variable; the external quantities become

$$\eta_e = -\alpha_e = -Gm/c^2r, \tag{44}$$

$$S_e = \pm\pi(Gm/c^2r)(l/R)^2 e^{-(r-R)/l}. \tag{45}$$

V. DISCUSSIONS

Our approximate solution represents a static spherically symmetric system whose only far-reaching interaction is a gravitational one; we did not use any macroscopic concept such as pressure to describe such structure. Since our solution does not show any singularity either in the source densities or in the potentials, it can be accepted as a naive classical model of an uncharged spinless elementary particle.

In our model the mass density $\rho(r)$ has a maximum finite value at the origin, and decreases monotonically to a finite value at the boundary. The condition (35) for the validity of our approximation is usually met both in the very small as well as in the very large physical systems.

A novel feature of our model is the prescription (26) for the radius of the system; all previous solutions[9,10] based on electromagnetic and long-range scalar fields allowed arbitrary values for the radius.

In our approximate solution we have made an expansion in integral powers of the small constant ϵ', and we have taken only the lowest-order term of the fields and sources; as a consequence our model presented all functions $\rho, \sigma, \eta, \alpha, S$ linearly proportional to a same constant A, or equivalently m. A nonlinearity would appear only if higher-order terms in ϵ' were considered.

An interesting feature is the role of the conservation laws in this order of approximation. In our approximation the field equations themselves decouple. One would expect, then, that there should be an independent scalar field existing along with an independent gravitational field in this order of approximation. That this does not occur is a result of the conservation law. Equation (10), which comes from the conservation law, then imposes a condition on the first-order solution because it is static and spherically symmetric.

Our final expressions for the source densities (11) and (19) and for the fields (29) to (32) are not suitable for obtaining the limit of long-range scalar field $l \to \infty$; indeed, it is known[9] that for these fields one must have $f^2 = 1$, and that the internal structure $\rho(r)$ remains undefined in the system (5) to (8).

It would be illustrative to evaluate the constants appearing in our model when applied to an elementary particle. Let us choose the neutral pion, with mass $m = 2.4\times10^{-25}$ g and nuclear range $l = 1.5\times10^{-13}$ cm. Assuming the radius $R = l$, we obtain from (26) $|f| = 1.12$, and we verify from (35) that $\epsilon' \simeq 10^{-42}$. The mass density $\rho(r)$ in (19) decreases from 2.8×10^{13} g/cm^3 at the origin to about half of this value on the boundary. For the "effective Yukawa charge" (37) we get $|Y| = 2.9 \times 10^{-25}$ g.

The case where the radius of the spherical system is much larger than the range of the Yukawa field ($R \gg l$) is particularly interesting: The exterior scalar field S_e [Eq. (45)] is very small (l^2/R^2) relative to the exterior gravitational field η_e [Eq. (44)] on the boundary R, and even smaller at larger distances. If we neglect this S_e, we can replace the two concepts of internal scalar field S_i and its source σ by the single concept of a pressure $p(r)$; since in our lowest order of approximation in ϵ'

$$p_1 + c^2\rho\eta_1 = 0, \tag{46}$$

we get from (38) and (39) on integration

$$p = 2\epsilon' c^2R^3\rho^2. \tag{47}$$

Similarly to the density $\rho(r)$ [Eq. (38)] this pressure has a finite value at the origin and decreases monotonically to zero (for $l/R \to 0$) on the boundary.

[1]A. Einstein and N. Rosen, Phys. Rev. 48, 73 (1935).
[2]G. Szekeres, Phys. Rev. 97, 212 (1955).
[3]O. Bergmann and R. Leipnik, Phys. Rev. 107, 1157 (1957).
[4]H. Yilmaz, Phys. Rev. 111, 1417 (1958).
[5]H. A. Buchdahl, Phys. Rev. 115, 1325 (1959).
[6]Duan'-I-Shi, Zh. Eksp. Teor. Fiz. 31, 1098 (1956) [Sov. Phys. JETP 4, 935 (1957)].
[7]G. Stephenson, Proc. Camb. Philos. Soc. 58, 521 (1962).
[8]I. Wolk, A. F. F. Teixeira, and M. M. Som, Nuovo Cimento Lett. 12, 319 (1975).
[9]A. Das, Proc. R. Soc. London A267, 1 (1962).
[10]U. K. De and A. K. Raychaudhuri, Proc. R. Soc. London A303, 47 (1968).