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Quantum mechanics of extended objects in relativistic field theory*

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We study canonical quantization of certain solutions of nonlinear classical field theories known as extended objects. These solutions are characterized by an energy density confined in a finite region of space for all time. We formulate a quantum theory in terms of the normal modes of oscillation about a static solution and the couplings among these modes. A Feynman diagram prescription is given for calculating Green's functions of the fields in the presence of an extended object. We discuss the extension of the formalism to solutions exhibiting a steady rotation in some internal-symmetry space. Problems that may arise in applying the formalism to bag models for quark confinement are mentioned.

I. INTRODUCTION

The interactions of hadrons have long resisted quantitative theoretical attack by the methods of conventional relativistic perturbation theory. This is primarily due to the large size of the couplings; indeed, the dimensionless pion-nucleon coupling constant satisfies $g^2/4\pi \approx 15$, in sharp contrast to the fine-structure constant of electrodynamics, $e^2/4\pi \simeq 1/137$. While a few terms of an expansion in powers of e accurately describe electrodynamic processes, similar terms of an expansion in g bear little relevance to observed hadronic behavior.

Despite the failure of perturbative techniques in strong-interaction physics, compelling reasons remain for retaining local field theory. First, the astounding success of renormalized quantum electrodynamics indicates that nature has chosen a local theory at least once. Second, local theories require such apparently valid consequences as analyticity properties of scattering amplitudes, the connection between spin and statistics, the existence of antiparticles, and the symmetry of *TCP*. Finally, local field theory is the only presently known way to formulate a quantum-mechanical theory of interacting particles in a relativistic manner.

Faith in local fields has spurred many investigations of alternatives to conventional perturbation theory. In particular, several researchers have speculated that a hadron may be simply described in terms of an appropriately quantized state of a type of classical solution to a field theory known as an "extended object." Solutions of this type are well known in nonlinear classical field theory.[1] For the purposes of this paper, an extended object is defined to be a classical solution to the equations of motion of a local field theory with the property that the energy density is, at all times, localized within a given region of space.[2] A wave packet in a free field theory is not of this type; in general such a wave packet will spread as time evolves.

The appealing feature of these objects arises because on a classical level they are already particlelike, and yet they possess an extended structure. The speculation is that hadrons correspond to such objects and that a perturbative treatment of quantum-mechanical corrections to the classical solution may converge faster than in the conventional approach. These objects also present the exciting possibility of confining quarks. Even classically some of these extended objects trap quarks to such an extent that an isolated quark cannot be produced.[3,4] The successes of the quark model may finally be reconciled with the nonobservation of free quarks.

There is a close relationship between these extended objects and the bag model of Chodos *et al.*[5] for quark confinement. In earlier work we discussed on a classical level the possibility of obtaining this model as a limit of a conventional local theory in which the bags are extended objects in the sense discussed above.[4] Because of difficulties with non-Abelian gauge fields, we were unable to obtain exactly the same model, but we presented a viable alternative based on a single triplet of quarks and an additional scalar constituent.

The purpose of this paper is to develop a formalism for studying the quantum mechanics of extended objects. The canonical approach to this problem has also been discussed by Christ and Lee,[6] Tomboulis,[7] and Faddeev.[8] The procedure consists of first finding the normal modes of oscillation about the lowest state of the classical object, then exactly quantizing these normal modes, and finally doing perturbation theory in the nonlinear couplings among the normal modes. Refinements of the procedure treat conserved quantum numbers such as charge and momentum that may be carried by the object.

As the method is ultimately perturbative about the classical solution it is, in principle, useful only when the classical solution resembles the quantized object. We have not determined whether this is the case for the bag limit discussed above.

Section II of this paper presents some examples of classical extended objects. Section III reviews other approaches to their quantization. Section IV treats the quantization of static classical solutions. Section V formulates a Feynman-diagram expansion for evaluating the expressions of Sec. IV. Section VI generalizes the procedure to include classical solutions with a trivial time dependence related to some conserved quantity such as momentum. Sec VII comments on the bag limit mentioned above. Section VIII summarizes some unanswered questions about these objects.

II. CLASSICAL EXAMPLES

Extended but nonspreading solutions to nonlinear field equations are well known in applied mathematics.[1] In this section we review a few such solutions occurring in Lorentz-invariant theories.

The simplest type of extended, particlelike object occurs in one-space-dimensional scalar field theories with a spontaneously broken discrete symmetry.[1,6-11] In particular, consider the Lagrangian density for a single scalar field

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi(x)\partial_\mu\phi(x)-V(\phi(x)), \tag{2.1}$$

where

$$V(\phi)=\tfrac{1}{2}\lambda\left(\phi^2-\frac{\mu^2}{2\lambda}\right)^2. \tag{2.2}$$

The potential $V(\phi)$ is sketched in Fig. 1. The equations of motion are

$$\Box\phi(x)=-V'(\phi(x))=-2\lambda\phi\left(\phi^2-\frac{\mu^2}{2\lambda}\right). \tag{2.3}$$

The energy density is

$$\mathcal{H}(x)=\tfrac{1}{2}[\pi(x)]^2+\tfrac{1}{2}[\nabla\phi(x)]^2+V(\phi(x)), \tag{2.4}$$

where $\pi(x)=\partial_0\phi(x)$ is the conjugate momentum to $\phi(x)$. Both $\mathcal{H}(x)$ and $\mathcal{L}(x)$ are invariant under the discrete symmetry operation of taking $\phi(x)$ to $-\phi(x)$, however, the lowest-energy configuration does not possess this symmetry and is doubly degenerate with

$$\begin{aligned}\phi(x)&=\pm\left(\frac{\mu^2}{2\lambda}\right)^{1/2},\\ \pi(x)&=0,\\ \mathcal{H}(x)&=0.\end{aligned} \tag{2.5}$$

The static solution for the extended object of interest is

$$\phi(x)=\left(\frac{\mu^2}{2\lambda}\right)^{1/2}\tanh\left[\left(\frac{\mu^2}{2}\right)^{1/2}x_1\right], \tag{2.6}$$

where x_1 is the space component of x_μ. This solution is sketched in Fig. 2. As x goes from $-x$ to $+x$, $\phi(x)$ goes from one minimum of $V(\phi)$ to the other. The energy density becomes

$$\mathcal{H}(x)=\frac{\mu^4}{4\lambda}\cosh^{-4}\left[\left(\frac{\mu^2}{2}\right)^{1/2}x\right] \tag{2.7}$$

and is plotted in Fig. 3. The energy is localized in a region characterized by the dimension μ^{-1}. Integrating (2.7) gives the total classical energy of the object

$$H=\int_{-\infty}^{\infty}dx\,\mathcal{H}(x)=\frac{\sqrt{2}\,\mu^3}{3\lambda}. \tag{2.8}$$

The stability of this solution is guaranteed by the topology of its behavior at infinity.[12]

This example is readily generalized to any $V(\phi)$ with several totally degenerate minima. The field can start in one mimimum at $x=-\infty$ and shift into another as x goes to $+\infty$. One extensively studied case has[1,13]

$$V(\phi)=\frac{\mu^4}{12\lambda}\left[1+\cos\left(\frac{2(3\lambda)^{1/2}}{\mu}\phi\right)\right]. \tag{2.9}$$

This potential yields the sine-Gordon equation

$$\Box\phi(x)=\frac{\mu^3}{2(3\lambda)^{1/2}}\sin\left(\frac{2(3\lambda)^{1/2}}{\mu}\phi(x)\right). \tag{2.10}$$

The solution for the extended object is

$$\phi(x)=\frac{\mu}{(3\lambda)^{1/2}}\arctan(\sinh\mu x_1). \tag{2.11}$$

The interest in the sine-Gordon equation arises because many solutions are analytically known. Coleman[14] has discussed the quantized version of this theory, obtaining the remarkable result that the theory is equivalent to the massive Thirring[15] model. Apparently upon quantization the above extended object becomes a fermion. Mandelstam[16]

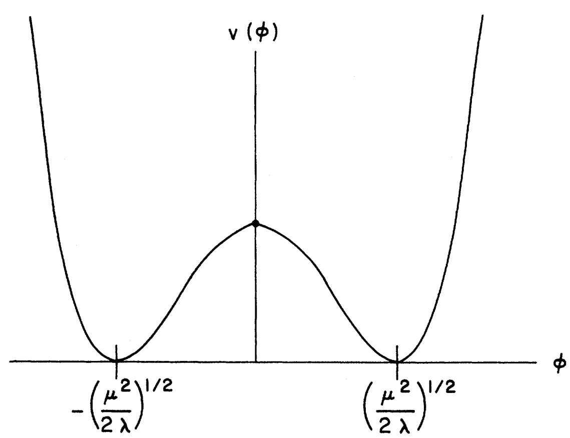

FIG. 1. The potential $V(\phi)$ in Eq. (2.2).

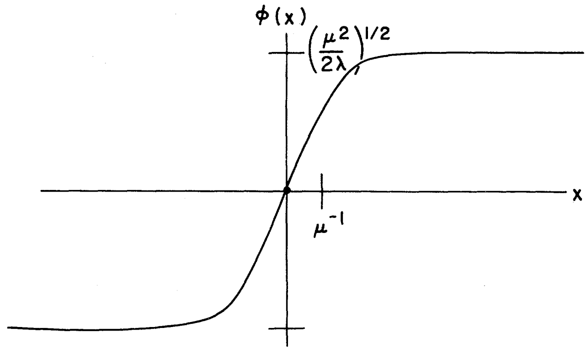

FIG. 2. The static solution in Eq. (2.6).

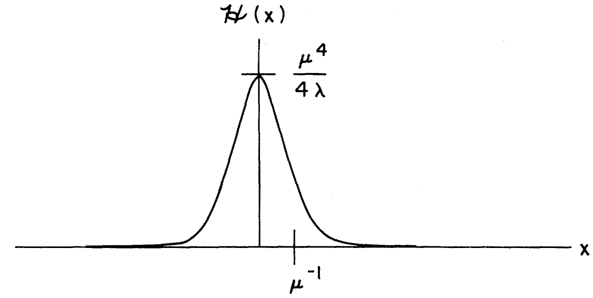

FIG. 3. The energy density of the extended object in Eq. (2.6).

has given an expression for the anticommuting Fermi field in terms of the scalar field $\phi(x)$; this Fermi field satisfies the Thirring-model equations of motion.

These one-dimensional examples owe their stability to the topology of the field configuration. A natural attempt to generalize this to higher spatial dimensions involves adding more fields. Consider, for example, in three space dimensions a three-component field in a theory with spontaneously broken O(3) symmetry. We might look for an extended-object type of solution with the asymptotic behavior

$$\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\\ \phi_3(x)\end{pmatrix} \underset{|\vec{x}|\to\infty}{\sim} \frac{c}{|\vec{x}|}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}, \tag{2.12}$$

where c^2 is the value of $\phi_1{}^2+\phi_2{}^2+\phi_3{}^2$ in the vacuum state of constant fields. However, this fails because the term in the energy $\int d^3x(\vec{\nabla}\phi_i)^2$ diverges at large $|\vec{x}|$. Consequently, the energy of such a solution must be infinite. This divergence at large distances is associated with the massless Goldstone bosons[17] of the theory. By introducing vector mesons and eliminating the Goldstone bosons via the Higgs[18] mechanism, 't Hooft[19] has obtained a solution of finite energy with the topology of Eq. (2.12). In this construction a massless vector field (a photon) remains, and the extended object is a magnetic source with respect to this field.

Topological properties of the field configuration at large distances from the extended object stabilize the above examples.[12] An alternative type of stable extended object occurs in hadronic "bag" models.[3,4,20,21] Here a scalar field is distorted from its vacuum value and held in this distorted configuration by other fields, i.e., the quarks, carrying conserved quantum numbers. These conserved quantum numbers insure the stability of the object. In general, these solutions will not be truly static, for associated with the conserved charges will be a continuing rotation in the internal-symmetry space generated by the charges. This rotation is trivial in the sense that observables invariant under the internal symmetry are time independent. In particular, this includes the Hamiltonian density. These solutions we call "quasistatic."

There are several reasons to study quasistatic extended objects. First, these objects occur in three dimensions in theories involving only scalar fields. The trivial time dependence evades a result of Derrick,[22] namely, that stable static solutions of finite energy above the vacuum do not exist in more than one space dimension, for theories involving only scalar fields. Second, realistic hadronic models have hadronic constituents that do carry conserved charges. Finally, momentum is a conserved quantity which can be treated in the same manner as other charges; this allows us to study moving extended objects, even with relativistic momenta.

Our definition of extended object includes much more than the above static and quasistatic solutions. Indeed, the field configuration can have any complicated time dependence as long as the energy does not spread with time. Such solutions are known for the sine-Gordon equation; they correspond roughly to two of the above static solutions near to each other but lacking sufficient energy to separate. These solutions are periodic in time and are well suited to the quantization technique of Dashen, Hasslacher, and Neveu.[9,13] Our discussion is not sufficiently sophisticated to treat these solutions, although Christ and Lee[6] have shown how canonical procedures may also be applied here.

We must remark on the spontaneous symmetry breaking which is essential to some of the above examples. In quantum mechanics, a symmetry cannot be broken by the ground state unless there is a "sufficiently infinite" number of degrees of freedom.[23] A discrete symmetry, such as possessed by the Lagrangian of Eq. (2.1), can be broken only if there is at least one infinite spatial dimension. A continuous symmetry requires at least two spatial dimensions. The examples discussed have sufficient spatial dimension for the requisite symmetry breaking to still occur quantum mechanically.

III. OTHER APPROACHES TO QUANTIZATION

Presumably, unique answers exist to such questions as "do these extended classical objects correspond to particles in the quantum theory, and, if so, what are their masses?" Insofar as they are applied correctly, all methods should yield the same answers and the preferred technique might be regarded as a matter of taste. However, the complexity of the respective procedures may vary with the problem at hand. In particular, for periodic solutions which are neither static nor quasistatic in the sense of the previous section, it appears that only the method of Dashen, Hasslacher, and Neveu[9,13] is simple. We feel that the procedures presented in the later sections of this paper are conceptually the simplest treatments of the static and quasistatic objects. This section briefly comments on some of the alternative approaches. We have already mentioned the works of Coleman[14] and Mandelstam[16] on the sine-Gordon theory.

A. WKB methods

In an extensive series of papers, Dashen, Hasslacher, and Neveu[9,13] have made semiclassical approximations on a Feynman path integral representation for the trace of the resolvent operator

$$G(E)=\operatorname{tr}\left(\frac{1}{H-E}\right). \qquad (3.1)$$

Poles in this function occur at the energies of the physical states of the theory. By considering paths near a periodic classical solution Dashen *et al.* are able to isolate such a pole when the classical orbit satisfies a "quantization condition." The procedure is essentially a generalization of the WKB[24] method to an infinite-degree-of-freedom problem. This method was applied successfully to the theory of Eqs. (2.1) and (2.2) as well as the sine-Gordon theory. In the latter problem the mathematics simplifies enough that Dashen *et al.* can exhibit an extremely rich spectrum of states. Recently Rajaraman and Weinberg[25] have extended the discussion to problems with internal symmetry. This extension covers the quasistatic problem, and bears some resemblance to our Sec. VI.

B. Coherent-state variational technique

Coherent states provide a useful technique in the study of the classical limit of a quantum field theory. Several authors have attempted to utilize such states to mimic the classical extended objects under consideration.[21,26] The parameters of these states are varied to minimize the expectation value of the Hamiltonian. In this way a classical variational problem is obtained. The hope is that this will provide a state which is a good approximation to the properly quantized state of the extended object.

In our opinion this approach has some inherent problems. First, it is difficult to quantify the validity of the approximations; indeed, there does not seem to be a systematic procedure for improving the calculation. Second, it is unclear how translation invariance enters the scheme; the Heisenberg uncertainty principle implies that the position of the extended object at rest cannot be specified in the quantum theory. Third, renormalization effects are difficult to treat; the bare coupling constants appearing in the Lagrangian are usually divergent quantities, and these divergences should not appear in physical quantities.

C. Generating function variational techniques

Goldstone and Jackiw[10] have pursued a more promising variational procedure. In a theory with a single field $\phi(x)$, an external source term $J(x)\phi(x)$ is added to the full interacting Lagrangian.[27] The probability amplitude for the ground state $|G>$ to remain the ground state in the presence of such a source is

$$W(J)=\langle G|T(\exp[i\textstyle\int d^4x\,J(x)\phi(x)])|G\rangle. \qquad (3.2)$$

The state $|G\rangle$ is in turn thought of as the vacuum or as the ground state of the extended object at rest. In the standard manner the generating function for connected Green's functions is defined

$$Z(J)=\ln W(J). \qquad (3.3)$$

This is Legendre transformed[28]

$$\phi_c(x)=\frac{\delta Z(J)}{\delta J(x)},$$
$$\Gamma(\phi_c)=Z(J)-\int d^4x\,\phi_c(x)J(x). \qquad (3.4)$$

The functional $\Gamma(\phi_c)$ satisfies

$$\frac{\delta\Gamma(\phi_c)}{\delta\phi_c(x)}=-J(x). \qquad (3.5)$$

If we now turn off the external source, we have

$$\left.\frac{\delta\Gamma(\phi_c)}{\delta\phi_c(x)}\right|_{J=0}=0. \qquad (3.6)$$

Given $\Gamma(\phi_c)$, it becomes a variational problem to find a $\phi_c(x)$ corresponding to the theory without an external source. From $\Gamma(\phi_c)$, one can calculate the Green's functions of the theory. By considering the appropriate $\phi_c(x)$ we can find the Green's functions for the field in the presence of the extended object. This includes the energy and form factor of the state. The calculation of Γ proceeds as an expansion in the number of loops in a Feynman-diagram expression. Symmetries of the Lagrangian essential to some of the extended solutions are preserved in such an expansion.

Translation invariance complicates the procedure by introducing infrared divergences. As mentioned before, the position of the extended object cannot be specified when it is at rest. Reference 10 shows how to treat this problem, and the method should give identical results to the techniques of this paper.

D. Path integrals

Gervais and Sakita and Callan and Gross[11] have considered the Green's functions of quantum fields in the presence of an extended object. They proceed using path integrals and expanding the action about the action of the classical path. The equivalence of canonical quantization with path integrals has been formally shown by Feynman.[29] Indeed the Feynman rules of Sec. V should be alternately obtainable through the procedures of Ref. 8. We feel that the path-integral method will show its greatest strength in the treatment of fluctuations about classical solutions with nontrivial time dependence, where the canonical discussion becomes rather complicated.

IV. QUANTIZATION OF STATIC EXTENDED OBJECTS

In this section we study the quantum fluctuations about a static classical solution to a field theory. We expand the Hamiltonian in the difference between the quantum field and classical solution. The quadratic terms in this expansion provide an exactly soluble quantum theory. Higher-order terms are treated using standard perturbation theory. Before beginning, we remark that in this discussion physical quantities such as energies are to be measured relative to the vacuum; thus, at the end of the calculation one must make vacuum subtractions. Also, the parameters appearing in the Lagrangian are bare couplings and masses; consequently, one must re-express these in terms of physical parameters calculated in the conventional manner. Technically the theory should be cut off at high frequencies until these renormalizations are made. With a cutoff we do not need to define a normal ordering, and thus all products in this section are *not* normal ordered.

For simplicity we consider a single Hermitian scalar field with dynamics controlled by the Lagrangian density

$$\mathcal{L}(x)=\tfrac{1}{2}\partial_\mu\phi(x)\partial_\mu\phi(x)-V(\phi(x)). \qquad (4.1)$$

We will write expressions in four-dimensional space-time although, as already mentioned, static solutions of finite energy require nore than just scalar fields. The essence of the procedure should apply in more realistic situations.

The equations of motion implied by Eq. (4.1) are

$$\Box\phi(x)=-V'(\phi(x)), \qquad (4.2)$$

where

$$V'(\phi)=\frac{\partial V(\phi)}{\partial\phi}. \qquad (4.3)$$

The Hamiltonian operator is

$$H(x)=\int d^3x\{\tfrac{1}{2}[\partial_0\phi(x)]^2+\tfrac{1}{2}[\vec{\nabla}\phi(x)]^2+V(\phi(x))\}. \qquad (4.4)$$

The canonical equal-time commutation relations are

$$[\partial_0\phi(\vec{x},t),\phi(\vec{x}\,',t)]=-i\delta^3(\vec{x}\,'-\vec{x}), \qquad (4.5)$$

$$[\phi(\vec{x},t),\phi(\vec{x}\,',t)]=[\partial_0\phi(\vec{x},t),\partial_0\phi(\vec{x}\,',t)]=0. \qquad (4.6)$$

We now assume that we are given a classical function $\phi_c(x)$ which is a static solution to Eq. (4.2)

$$-\vec{\nabla}^2\phi_c(x)+V'(\phi_c(x))=0, \qquad \partial_0\phi_c(x)=0. \qquad (4.7)$$

We wish to study the quantum field when it is in some sense near to $\phi_c(x)$. This suggests we write

$$\phi(x)=\phi_c(x)+\epsilon(x), \qquad (4.8)$$

where $\epsilon(x)$ is a quantum-mechanical operator. We expand the potential $V(\phi)$

$$V(\phi)=V(\phi_c+\epsilon)=V(\phi_c)+\epsilon V'(\phi_c)+\tfrac{1}{2}\epsilon^2V''(\phi_c)+R(\epsilon,\phi_c). \qquad (4.9)$$

This equation defines $R(\epsilon,\phi_c)$, which is of order ϵ^3 for small ϵ. Using Eq. (4.7), we can eliminate terms linear in ϵ from the Hamiltonian to obtain

$$H=H_c+\int d^3x\{\tfrac{1}{2}[\partial_0\epsilon(x)]^2+\tfrac{1}{2}[\nabla\epsilon(x)]^2+\tfrac{1}{2}[\epsilon(x)]^2V''(\phi_c(x))+R(\epsilon(x),\phi_c(x))\}, \qquad (4.10)$$

where H_c is the energy of the classical solution

$\phi_c(x)$. We split this Hamiltonian into two parts

$$H = H_0 + H_1, \tag{4.11}$$

where

$$H_0 = H_c + \int d^3x \left\{\tfrac{1}{2}[\partial_0 \epsilon(x)]^2 + \tfrac{1}{2}[\vec{\nabla}\epsilon(x)]^2 + \tfrac{1}{2}\epsilon^2 V''(\phi_c(x))\right\} \tag{4.12}$$

and

$$H_1 = \int d^3x \, R(\epsilon(x), \phi_c(x)). \tag{4.13}$$

We will solve the quantum theory defined by H_0 exactly, and then treat H_1 as a perturbation using standard perturbation theory. Note that in the one space dimensional theory of Eqs. (2.1) and (2.2), H_1 is of order $\lambda^{1/2}$. Thus the perturbation expansion requires the weak-coupling limit of small λ.

The operators and states of the solution to H_0 are denoted by the superscript "in." To find this solution, expand $\epsilon^{\text{in}}(x,t)$

$$\epsilon^{\text{in}}(x,t) = \sum_n q_n^{\text{in}}(t)\psi_n(\vec{x}), \tag{4.14}$$

where the ψ_n are a real, orthonormal, complete set of functions

$$\psi_n(\vec{x}) = \psi_n^*(\vec{x}),$$
$$\int d^3x\, \psi_n(\vec{x})\psi_{n'}(\vec{x}) = \delta_{nn'}, \tag{4.15}$$

$$\sum_n \psi_n(\vec{x})\psi_n(\vec{x}\,') = \delta^3(\vec{x}\,' - \vec{x}). \tag{4.16}$$

and the $q_n(t)$ are Hermitian quantum operators. Because of the structure of H_0, a particularly convenient complete set of functions is the set of solutions to the Schrödinger-type equation

$$[-\tfrac{1}{2}\vec{\nabla}^2 + \tfrac{1}{2}V''(\phi_c(x))]\,\psi_n(x) = \tfrac{1}{2}\omega_n{}^2\psi_n(x). \tag{4.17}$$

Stability of the classical solution requires the $\omega_n{}^2$ to be non-negative; we assume this is the case. With this choice for the ψ_n, $H_0(\epsilon^{\text{in}})$ assumes a particularly simple form

$$H_0(\epsilon^{\text{in}}) = H_c + \sum_n [\tfrac{1}{2}(\dot{q}_n^{\text{in}})^2 + \tfrac{1}{2}\omega_n{}^2(q_n^{\text{in}})^2], \tag{4.18}$$

where $\dot{q}_n^{\text{in}}$ means $\partial_0 q_n^{\text{in}}$. The equal-time commutation relations among the q_n^{in} and $\dot{q}_n^{\text{in}}$ are

$$[\dot{q}_n^{\text{in}}(t), q_m^{\text{in}}(t)] = -i\delta_{nm},$$
$$[q_n^{\text{in}}(t), q_m^{\text{in}}(t)] = [\dot{q}_n^{\text{in}}(t), \dot{q}_m^{\text{in}}(t)] = 0. \tag{4.19}$$

The equations of motion are

$$\ddot{q}_n^{\text{in}}(t) = -\omega_n{}^2 q_n^{\text{in}}(t). \tag{4.20}$$

The problem has now been reduced to a set of independent harmonic oscillators; indeed, we have rewritten the dynamics in terms of normal modes of oscillation.

In general, Eq. (4.17) will have both discrete and continuum eigenvalues. The discrete solutions correspond to vibrations of the extended object, i.e., excited states. The continuum solutions correspond to moving waves of the ϕ field in the presence of the extended object. These moving waves represent the mesons of the conventionally treated ϕ field. This formalism allows one to discuss meson scattering on the various states of the extended object. Any sum over the modes n is actually a discrete sum over the discrete eigenvalues plus an integral over the continuum eigenvalues.

Translation invariance of the original Lagrangian implies that if the classical object is translated in any spatial direction, there is no restoring force. This means that there is a normal mode with $\omega_n = 0$ for each spatial dimension.[30] These modes can be found by taking the gradient of Eq. (4.7)

$$-\vec{\nabla}^2\vec{\nabla}\phi_c + V''(\phi_c)\vec{\nabla}\phi_c = 0. \tag{4.21}$$

Thus $\vec{\nabla}\phi_c(x)$ is a solution to the Schrödinger equation (4.16) with $\omega_n = 0$. We write this triply degenerate basis function as a vector

$$\vec{\psi}_0 = N\vec{\nabla}\phi_c, \tag{4.22}$$

where N is a normalization given by

$$N^2 \int d^3x (\vec{\nabla}\phi_c)^2 = 3. \tag{4.23}$$

This integral can be evaluated in terms of H_c via a "virial" relation between the gradient and potential terms in the Hamiltonian

$$\int d^3x [\vec{\nabla}\phi_c(x)]^2 = -6\int d^3x\, V(\phi_c(x)) = 3H_c. \tag{4.24}$$

This relation is proved in Appendix A. The minus sign in the potential term of Eq. (4.24) is a manifestation of the remark made earlier that scalar fields alone cannot produce static extended objects in more than one spatial dimension.[31] We ignore this problem because it can be evaded in the ways discussed in Sec. II. Using Eq. (4.24), we find

$$\vec{\psi}_0 = \frac{1}{(H_c)^{1/2}}\vec{\nabla}\phi_c. \tag{4.25}$$

The association of this mode with translation is evident from

$$\phi_c(\vec{x} + \vec{\delta}) = \phi_c(\vec{x}) + (H_c)^{1/2}\vec{\delta}\cdot\vec{\psi}_0 + O(\delta^2). \tag{4.26}$$

The theory defined by H_0 can now be solved by writing

$$q_n^{\text{in}}(t) = \frac{1}{(2\omega_n)^{1/2}}(a_n^{\text{in}} e^{-i\omega_n t} + a_n^{\text{in}\dagger} e^{+i\omega_n t}), \quad n > 0$$
$$\vec{q}_0^{\,\text{in}}(t) = \vec{q}_0^{\,\text{in}}(0) + t\dot{\vec{q}}_0^{\,\text{in}}, \tag{4.27}$$

where $\dot{\vec{q}}_0$ is time independent. The commutation relations become

$$[a_n^{\text{in}}, a_m^{\text{in}\dagger}] = \delta_{nm}^{\text{in}},$$
$$[a_n^{\text{in}}, a_m^{\text{in}}] = [a_n^{\text{in}\dagger}, a_m^{\text{in}\dagger}] = [q_0^{\text{in}}(t), a_n^{\text{in}}] = 0. \tag{4.28}$$

The states of the theory are defined as eigenstates of $\dot{q}_0^{\text{in}}$ and by the occupation numbers of the various modes with $n > 0$

$$\dot{\vec{q}}_0^{\text{in}} |\vec{s}; n_1, n_2, \ldots, n_i, \ldots\rangle = \vec{s}|\vec{s}; n_1, \ldots\rangle, \tag{4.29}$$

$$a_i^{\text{in}\dagger} |\vec{s}; n_1, \ldots, n_i, \ldots\rangle = (n_i + 1)^{1/2} |\vec{s}; n_1, \ldots, n_i + 1, \ldots\rangle, \tag{4.30}$$

$$a_i^{\text{in}} |\vec{s}; n_1, \ldots, n_i, \ldots\rangle = (n_i)^{1/2} |\vec{s}; n_1, \ldots, n_i - 1, \ldots\rangle. \tag{4.31}$$

We impose the normalization

$$\langle \vec{s}'; n_1', \ldots, n_i', \ldots |\vec{s}; n_1, \ldots, n_i, \ldots\rangle = (2\pi)^3 \delta^3(\vec{s}' - \vec{s}) \prod_{i=1}^{\infty} \delta_{n_i n_i'}. \tag{4.32}$$

These states diagonalize $H_0(\epsilon^{\text{in}})$

$$H_0(\epsilon^{\text{in}})|\vec{s}; n_1, \ldots\rangle = \left(H_c + \sum_{i=1}^{\infty} (n_i + \tfrac{1}{2})\omega_i + \tfrac{1}{2}\vec{s}^{\,2}\right)|\vec{s}; n, \ldots\rangle. \tag{4.33}$$

To understand the physical significance of $\vec{s}$, consider the momentum operator

$$\begin{aligned}\vec{P} &= -\int d^3x [\partial_0 \phi(x)][\vec{\nabla}\phi(x)] \\ &= -\int d^3x [\partial_0 \epsilon(x)] \vec{\nabla}\phi_c(x) + O(\epsilon^2) \\ &= -(H_c)^{1/2} \dot{\vec{q}}_0 + O(\epsilon^2).\end{aligned} \tag{4.34}$$

This gives

$$\vec{P}|\vec{s}; n_1, \ldots\rangle = [-(H_c)^{1/2}\vec{s} + O(\epsilon^2)]|\vec{s}; n_1, \ldots\rangle \tag{4.35}$$

Remembering that $H_1 = O(\epsilon^3)$ and $\vec{P} = O(\epsilon)$, we have

$$H|\vec{s}; n_1, \ldots\rangle = \left(H_c + \frac{\vec{P}^2}{2H_c} + \sum_{i=1}^{\infty} (n_i + \tfrac{1}{2})\omega_i + O(\epsilon^3)\right)|\vec{s}; n_1, \ldots\rangle. \tag{4.36}$$

This represents the beginning of an expansion of the energy in the momentum and quantum fluctuations. Note that the underlying relativistic nature of the problem has required the rest mass to equal the kinematic mass of the $\vec{P}^2$ term. The expansion is nonrelativistic because we have chosen to quantize in a particular frame, the rest frame of the classical extended object. In Sec. VI we discuss quantization about a moving extended solution; this maintains the relativistic relation between energy and momentum.

As mentioned earlier there are both discrete and continuum normal modes. To restore a continuum notation we would write the states

$$|\vec{s}; n_1, \ldots, n_m; k_1, \ldots, k_l\rangle_{\text{in}}, \tag{4.37}$$

where there are l conventional mesons of momenta k_1 to k_l in the presence of an excited extended object characterized by the occupation numbers n_1 to n_m of the m discrete normal modes. Note that even a single discrete eigenmode gives rise to an infinity of excited states corresponding to the infinity of allowed occupation numbers for that level. Of course, when the interactions between modes are introduced, high excitations become unstable to meson emission.

The perturbation theory in $R(\epsilon, \phi_c)$ is formulated in a standard manner. Using the adiabatic theorem,[32] eigenstates of the full Hamiltonian can be found by gradually turning on the interaction from $t = -\infty$

$$R(\epsilon(x), \phi_c(x)) \to R(\epsilon(x), \phi_c(x))e^{-\delta|t|}, \tag{4.38}$$

where δ is infinitesimal. The interacting field is given by

$$\epsilon(x) = T_A\left(\exp\left(i\int_{-\infty}^{t} dt' H_1(\epsilon^{\text{in}}, t')\right)\right)\epsilon^{\text{in}}(x) T\left(\exp\left(i\int_{-\infty}^{t} dt' H_1(\epsilon^{\text{in}}, t')\right)\right), \tag{4.39}$$

where T (T_A) is the time-ordering (anti-time-ordering) operation, and $H_1(\epsilon^{\text{in}}, t')$ represents H_1 evaluated for ϵ^{in} at time t'. Inserting this field into Eq. (4.8) defining ϵ, we have a formal solution to the equations of motion (4.2) satisfying the commutation relations (4.5) and (4.6). The states defined in Eqs. (4.29) through (4.32) are, by the adiabatic theorem, eigenstates of the full Hamiltonian $H(\epsilon)$. The eigenvalues are given by

$$\begin{aligned}E(\vec{s}; n_1, \ldots)(2\pi)^3\delta^3(\vec{s}' - \vec{s}) &= {}_{\text{in}}\langle \vec{s}'; n_1, \ldots |H(\epsilon)|\vec{s}; n_1, \ldots\rangle_{\text{in}} \\ &= {}_{\text{in}}\left\langle \vec{s}'; n_1, \ldots \left| T_A\left(\exp\left(i\int_{-\infty}^{\infty} dt' H_1(\epsilon^{\text{in}}, t')\right)\right)\right.\right. \\ &\quad \times T\left(H(\epsilon^{\text{in}}, t=0)\exp\left(-i\int_{-\infty}^{\infty} dt' H_1(\epsilon^{\text{in}}, t')\right)\right)\left|\vec{s}; n_1, \ldots\right\rangle_{\text{in}}.\end{aligned} \tag{4.40}$$

Perturbation theory consists of an expansion of these equations in powers of H_1. To understand the physical significance of the "in" states, restore the continuum notation as in Eq. (4.37). The "in" state represents the configuration which at $t=-\infty$ has the extended object characterized by $\vec{s}$ and the excitation numbers n_1 to n_m, while widely separated from this object are mesons of momenta k_1 to k_l. One can also define "out" states representing the same configuration at $t=+\infty$.

$$|\vec{s};n_1,\ldots\rangle_{\text{out}}=T_A\left(\exp\left(+i\int_{-\infty}^{\infty}dt'H_1(\epsilon^{\text{in}},t')\right)\right)|\vec{s};n_1,\ldots\rangle_{\text{in}}\,. \tag{4.41}$$

When the energy is below the meson continuum, the discrete "in" and "out" states of the extended object can differ only by a phase. In this case Eq. (4.40) becomes

$$E(s;n_1,\ldots,n_m)=\int\frac{d^3s'}{(2\pi)^3}\,{}_{\text{in}}\Big\langle\vec{s}';n_1,\ldots\Big|T\Big[H(\epsilon^{\text{in}},t=0)\exp\Big(-i\int_{-\infty}^{\infty}dt'H_1(\epsilon^{\text{in}},t')\Big)\Big]\Big|\vec{s};n_1,\ldots\Big\rangle_{\text{in}}$$
$$\times\left(\int\frac{d^3s'}{(2\pi)^3}\,{}_{\text{out}}\langle\vec{s}';n_1,\ldots|\vec{s}';n_1,\ldots\rangle_{\text{in}}\right)^{-1}. \tag{4.42}$$

The last factor removes disconnected Feynman diagrams in the expansion for the ground-state energy. In the next section we develop the Feynman rules, taking careful account of the $\omega=0$ translation mode.

V. FEYNMAN RULES

Equation (4.42) provides a perturbative scheme for calculating the energy levels of the extended object. Equation (4.39) allows calculation of matrix elements of other operators between these states. In practice, however, such calculations are rather tedious due to the complicated nature of the time-ordered products that must be evaluated. A degree of simplification is possible through the use of Wick's theorem[33] to reduce these time-ordered products to a diagrammatic expansion in Feynman graphs.[34] Were it not for the $\omega=0$ transition mode, such a procedure would be standard and no elaboration would be necessary. The purpose of this section is to include this mode in the usual expansion.

The Feynman expansion is most compactly formulated using functional techniques.[35] In this way the ground-state matrix elements of a time-ordered product in the quantum-mechanical Hilbert space is expressed as a matrix element of an ordinary product of commuting operators in a new space. These commuting operators are creation operators for the ends of lines in a Feynman graph, while the matrix element connects these ends with propagators in all possible ways. We call this new space the "end-of-line space." States in this space are denoted by curving bras and kets $|\psi)$ as opposed to states in the quantum-mechanical space denoted $|\psi\rangle$.

We start with the set of "in" states and operators solving H_0 as in the last section. For each mode $q_n^{\text{in}}(t)$ we define an operator $c_n(t)$ in the new end-of-line space. These $c_n(t)$ satisfy the commutation relations[36]

$$[c_n(t),c_{n'}^\dagger(t')]=\delta_{nn'}\delta(t'-t),\qquad [c_n(t),c_{n'}(t')]=0. \tag{5.1}$$

The states of this space are generated by applying the operators $c_n^\dagger(t)$ to a state $|0)$ satisfying

$$c_n(t)|0)=0 \tag{5.2}$$

and normalized

$$(0|0)=1. \tag{5.3}$$

We label these states

$$|n_1,t_1;n_2,t_2;\cdots n_i,t_i)\equiv c_{n_1}^\dagger(t_1)\cdots c_{n_i}^\dagger(t_i)|0). \tag{5.4}$$

In a Feynman graph, the operator $c_n^\dagger(t)$ creates the end of a line corresponding to the mode n at time t. The state $|0)$ represents a configuration with no lines.

We now define the generating state for the "in" fields by

$$(W_0(\vec{s}',\vec{s})|=(0|\left[{}_{\text{in}}\Big\langle\vec{s}';n_i=0\Big|T\exp\Big(\int_{-\infty}^{\infty}dt\sum_n q_n^{\text{in}}(t)c_n(t)\Big)\Big|\vec{s};n_i=0\Big\rangle_{\text{in}}\right]. \tag{5.5}$$

The dependence on s' and s must be kept in order to handle the zero-frequency mode. In conventional discussions of functional techniques, the state $|\vec{s};n_i=0\rangle$ is replaced with the ground state of the system. The utility of the generating state becomes apparent with the relation

$$(W_0(\vec{s};\vec{s})|c_{n_1}^\dagger(t_1)\cdots c_{n_i}^\dagger(t_i)|0)={}_{\text{in}}\langle\vec{s}';0,\ldots|T(q_{n_1}^{\text{in}}(t_1)\cdots q_{n_1}^{\text{in}}(t_i))|\vec{s};0,\ldots\rangle_{\text{in}}\,. \tag{5.6}$$

Thus a time-ordered product of operators in the physical Hilbert space is related to an ordinary product of commuting operators in the "end-of-line" space. The state $(W_0(\vec{s}', \vec{s})|$ serves to connect the lines created by the $c_n^\dagger(t)$. The contribution of the nonzero-frequency modes to $(W_0(s', s)|$ is well known[35] to be

$$(W_0(\vec{s}', \vec{s})| = (0|\exp\left(\frac{i}{2}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} dt\,dt' c_n(t)\Delta_n(t,t')c_n(t')\right)\,{}_{\text{in}}\langle\vec{s}';0,\ldots|T\exp[\textstyle\int dt\,\vec{q}_0^{\,\text{in}}(t)\cdot\vec{c}_0(t)]|\vec{s};0,\ldots\rangle_{\text{in}}\,, \tag{5.7}$$

where $\Delta_n(t, t')$ is the Feynman "propagator"

$$\Delta_n(t,t') = \frac{-i}{2\omega_n}e^{-i\omega_n|t'-t|}, \quad n>0. \tag{5.8}$$

In Appendix B we evualuate the zero-frequency contribution to $(W_0(\vec{s};\vec{s})|$ with the result[37]

$$(W_0(\vec{s}', \vec{s})| = (0|(2\pi)^3\delta^3\left(\vec{s}' - \vec{s} + i\int dt\,\vec{c}_0(t)\right)$$
$$\times\exp\left(\frac{i}{2}\sum_{n=1}^{\infty}\int dt\,dt' c_n(t)\Delta_n(t,t')c_n(t') - \frac{i}{4}\int dt\,dt'|t-t'|\vec{c}_0(t)\cdot\vec{c}_0(t') + \frac{(\vec{s}+\vec{s}')}{2}\cdot\int dt\,t\vec{c}_0(t)\right). \tag{5.9}$$

Associated with c_0 is a "propagator"

$$\Delta_0(t,t') = -\tfrac{1}{2}|t-t'| \tag{5.10}$$

but we also have additional dependence of $(W_0(\vec{s}', \vec{s})|$ on $c_0(t)$. These additional factors allow for zero-frequency lines in a Feynman diagram to disappear. In some sense they are being absorbed by the classical solution. Equation (5.9) tells us precisely how to calculate this effect.

The Feynman expansion can be formulated in coordinate space by defining

$$c(x_\mu) = \sum_n \psi_n(x)c_n(t). \tag{5.11}$$

This gives

$${}_{\text{in}}\langle\vec{s}';0,\ldots|T(\epsilon^{\text{in}}(x_1)\cdots\epsilon^{\text{in}}(x_n))|\vec{s};0,\ldots\rangle_{\text{in}} = (W_0(\vec{s}',\vec{s})|c^\dagger(x_1)\cdots c^\dagger(x_n)|0),$$

where $(W_0(\vec{s}', \vec{s})|$ can be written

$$(W_0(\vec{s}', \vec{s})| = (0|\exp\left(\frac{i}{2}\int d^4x\,d^4x' c(x)\Delta(x,x')c(x')\right)$$
$$\times\exp\left[\left(\frac{\vec{s}+\vec{s}'}{2}\right)\cdot\int d^4x\,t\vec{\psi}_0(x)c(x)\right](2\pi)^3\delta\left(\vec{s}'-\vec{s}+i\int d^4x\,\vec{\psi}_0(x)c(x)\right). \tag{5.12}$$

Here we have defined the propagator $\Delta(x, x')$ for the ϵ^{in} field to have contributions from all modes

$$\Delta(x,x') = \sum_{n=0}^{\infty}\psi_n(\vec{x})\Delta_n(t,t')\psi_n(\vec{x}'), \tag{5.13}$$

with Δ_n given in Eqs. (5.8) and (5.10). We can now define a generating state for the interacting fields

$$(W(\vec{s}', \vec{s})| = (W_0(\vec{s}', \vec{s})|\exp\left[i\int d^4x\,R(c^\dagger(x), \phi_c(x))\right]. \tag{5.14}$$

This gives the main result of this section:

$${}_{\text{in}}\langle\vec{s}';0,\ldots|T\{\epsilon^{\text{in}}(x_1)\cdots\epsilon^{\text{in}}(x_n)\exp[i\textstyle\int d^4x\,R(\epsilon^{\text{in}}(x),\phi_c(x))]\}|\vec{s};0,\ldots\rangle_{\text{in}}$$
$$= {}_{\text{in}}\langle\vec{s}';0,\ldots|T(\epsilon(x_1)\cdots\epsilon(x_n))|\vec{s}';0,\ldots\rangle_{\text{in}}\int\frac{d^3s''}{(2\pi)^3}\,{}_{\text{out}}\langle\vec{s}'';0|\vec{s}';0\rangle_{\text{in}} = (W(s',s)|c^\dagger(x_1)\cdots c^\dagger(x_n)|0). \tag{5.15}$$

Feynman perturbation theory is an expansion of the exponential in Eq. (5.14) in powers of R. This expansion generates the vertices which become connected with propagators by $(W_0(\vec{s}', \vec{s})|$. The main change from conventional Feynman rules comes from zero-frequency lines being absorbed into the nonpropagator terms of Eq. (5.12).

The expression of Eqs. (5.12) to (5.15) suffice to calculate perturbatively a large class of Green's functions for the ϵ field in the presence of the extended object. A slight complication arises for some operators of physical interest, including the Hamiltonian, which involve time derivatives of $\epsilon(x)$. As time derivatives do not commute with time orderings, some care is necessary. In Appendix C we extend the end-of-line space to treat time derivatives of fields. Here we show that in the case of the energy density the complication only enters the lowest-order loop diagram. Such a modification is formally necessary to obtain the correct zero-point energy for the normal modes. This extension of the end-of-line space also allows one to discuss Feynman rules with derivative couplings.

If all we wish to calculate is the ground-state energy of the extended object with $\vec{s}=0$, then the formalism simplifies considerably. First, we can eliminate the δ-function term in Eq. (5.12) by integrating over $\vec{s}\,'-\vec{s}$ at fixed $\vec{s}+\vec{s}\,'$. Then we consider $\vec{s}+\vec{s}\,'=\vec{0}$ so the term in Eq. (5.12) with an exponential linear in $c(x)$ will drop out. Thus all new terms in the Feynman rules involving absorption of zero-frequency lines by the extended object do not contribute. We only need consider a conventional diagrammatic expansion using the propagator (5.13) and vertices read from $R(\epsilon, \phi_c)$. This yields the ground-state rest energy

$$E(\vec{0};0,\dots)=H_c+\left(0\left|\exp\left(\frac{i}{2}\int d^4x\,d^4x'\,c(x)\Delta(x,x')c(x')\right)\exp\left(i\int d^4x\,R(c^\dagger(x),\phi_c(x))\right)\right.\right.$$
$$\left.\left.\times\int d^3x\left\{\tfrac{1}{2}[\partial_0c^\dagger(x)\partial_0c^\dagger(y)-\delta^4(x-y)]\big|_{y=x}+\tfrac{1}{2}[\vec{\nabla}c^\dagger(x)]^2+\tfrac{1}{2}[c^\dagger(x)]^2V''(\phi_c(x))+R(c^\dagger(x),\phi_c(x))\right\}\right|0\right)_{\text{connected}}. \tag{5.16}$$

The $\delta^4(x-y)$ term comes from the time-derivative terms in the Hamiltonian as discussed in Appendix C. The subscript "connected" indicates that we only include connected Feynman diagrams. Of course, the conventional vacuum energy still needs to be subtracted and the coupling constants need renormalization.

VI. SYMMETRIES AND CONSERVED QUANTUM NUMBERS

In this section we generalize our discussion to treat the quasistatic objects discussed in Sec. II. We wish to quantize a solution to a classical field theory where this solution minimizes the energy with specified values for certain conserved quantities such as momentum and electric charge. These conserved quantities are assumed to be commuting; i.e., they can be simultaneously specified. Our goal is to define coordinates which are conjugate to each of these commuting charges. By restricting the states to particular values for the charges, these coordinates will be eliminated, leaving a quantum-mechanical system with the number of degrees of freedom reduced by the number of conserved charges. This section is essentially an elaboration of ideas discussed by Rajaraman and Weinberg.[25] The procedure is completely analogous to the usual treatment of angular momentum in the two-degrees-of-freedom problem defined by the Lagrangian

$$L=\tfrac{1}{2}\dot{x}^2+\tfrac{1}{2}\dot{y}^2-V((x^2+y^2)^{1/2}). \tag{6.1}$$

Here one changes variables

$$x=r\cos\theta\,,\qquad y=r\sin\theta\,. \tag{6.2}$$

and eliminates the variable θ from the problem by specifying the angular momentum

$$l=x\dot{y}-y\dot{x}=r^2\dot{\theta}\,. \tag{6.3}$$

This gives a one-degree-of-freedom problem with r as the dynamical variable. The energy in terms of r and l is

$$H=\tfrac{1}{2}\dot{r}^2+\tfrac{1}{2}\frac{(l^2-\frac{1}{4})}{r^2}+V(r)\,. \tag{6.4}$$

The term proportional to r^{-2} is well-known "centrifugal barrier." The commutation relations satisfied by $r(t)$ are

$$[\dot{r}(t),r(t)]=-i\,. \tag{6.5}$$

The equation of motion is

$$\ddot{r}=\frac{l^2-\frac{1}{4}}{r^3}-V'(r)\,. \tag{6.6}$$

In the classical problem the centrifugal-barrier term would be $\frac{1}{2}l^2r^{-2}$ rather than $\frac{1}{2}(l^2-\frac{1}{4})r^{-2}$; the difference is introduced by the quantum-mechanical change of variables because θ and $\dot{\theta}$ do not commute. The remainder of this section generalizes Eqs. (6.1)–(6.6) to the field theory problem with an infinite number of degrees of freedom and a finite number of commuting symmetries.

For simplicity we restrict ourselves to scalar fields and begin with the Lagrangian density

$$\mathcal{L}(x)=\tfrac{1}{2}\sum_a\partial_\mu\phi_a(x)\partial_\mu\phi_a(x)-V(\phi)\,, \tag{6.7}$$

where the index a labels the respective fields. Without loss of generality we take the fields $\phi_a(x)$ to be Hermitian. To keep the algebra under control, we introduce a notation where the space variable $\vec{x}$ and the field index a are combined into a single discrete index i. To do this explicitly, expand the fields $\phi_a(x)$ in a complete orthonormal set

$$\phi_a(x) = \sum_i q_i(t)\psi_{ia}(\vec{x}) . \tag{6.8}$$

The functions $\psi_{ia}(x)$ are chosen to be real functions so that the $q_i(t)$ are Hermitian operators. Orthonormality implies

$$\sum_a \int d^3x\, \psi_{ia}(\vec{x})\psi_{ja}(\vec{x}) = \delta_{ij} \tag{6.9}$$

and completeness requires

$$\sum_i \psi_{ia}(\vec{x})\psi_{ib}(\vec{y}) = \delta_{ab}\delta^3(\vec{x}-\vec{y}) . \tag{6.10}$$

The $q_i(t)$ are given by

$$q_i(t) = \sum_a \int d^3x\, \psi_{ia}(\vec{x})\phi_a(x) . \tag{6.11}$$

In this notation the Lagrangian becomes

$$L = \int d^3x\, \mathcal{L}(x) = \sum_i \tfrac{1}{2}[\dot{q}_i(t)]^2 - V(q) , \tag{6.12}$$

where we have included the spatial gradient terms in the definition

$$V(q) \equiv \int d^3x \left(V(\phi(x)) + \tfrac{1}{2}\sum_a [\vec{\nabla}\phi_a(x)]^2\right) \tag{6.13}$$

Finally we define

$$V_{i_1\cdots i_n}(q) = \frac{\partial^n}{\partial q_{i_1}\cdots \partial q_{i_n}} V(q) . \tag{6.14}$$

In this notation the equations of motion are

$$\ddot{q}_i(t) = -V_i(q(t)) . \tag{6.15}$$

The Hamiltonian is

$$H = \tfrac{1}{2}\sum_i [p_i(t)]^2 + V(q(t)), \tag{6.16}$$

where

$$p_i(t) = \dot{q}_i(t) . \tag{6.17}$$

The commutation relations of the theory are

$$[p_i(t), q_j(t)] = -i\delta_{ij} . \tag{6.18}$$

We now assume that the theory possesses k symmetries generated by matrices M^α_{ij} ($\alpha = 1, \ldots, k$) such that

$$V\left(\left[\exp\left(i\sum_\alpha M^\alpha \gamma^\alpha\right)\right]_{ij} q_j\right) = V(q_i) \tag{6.19}$$

for every set of k numbers γ_α. A matrix sum over j is understood. The generators are taken to be Hermitian, imaginary, and antisymmetric

$$M^\alpha_{ij} = M^{\alpha\dagger}_{ij} = -M^{*}_{ij} = -M^\alpha_{ji} , \tag{6.20}$$

where † denotes Hermitian conjugation and * denotes complex conjugation. Associated with each symmetry is a conserved charge

$$\begin{aligned} l^\alpha &= ip_i M^\alpha_{ij} q_j , \\ \partial_0 l^\alpha &= 0 . \end{aligned} \tag{6.21}$$

As we desire to simultaneously diagonalize these charges, we assume the generators commute

$$M^\alpha_{ij} M^\beta_{jk} = M^\beta_{ij} M^\alpha_{jk} , \tag{6.22}$$

implying that the operators l^α commute

$$[l^\alpha, l^\beta] = 0 . \tag{6.23}$$

A relation that will prove useful follows from first differentiating Eq. (6.19) by γ_α at $\gamma_\alpha = 0$

$$V_i(q) i M^\alpha_{ij} q_j = 0 \tag{6.24}$$

and then differentiating with respect to q_i

$$V_{ij}(q) i M^\alpha_{jk} q_k - i M^\alpha_{ij} V_j(q) = 0 . \tag{6.25}$$

At this point we digress and give some physical examples of possible M^α. In particular, consider the momentum operator

$$\vec{P} = -\int d^3x\, \partial_0\phi_a \vec{\nabla}\phi_a . \tag{6.26}$$

Comparison with Eq. (6.21) shows that M^α represents the gradient operator. In terms of the complete set introduced in Eq. (6.8)

$$\vec{M}_{ij} = i\sum_a \int d^3x\, \psi_{ia}(x)\vec{\nabla}\psi_{ja}(x) . \tag{6.27}$$

In the two-degrees-of-freedom problem discussed at the beginning of this section

$$l = (p_x,\ p_y)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and thus for this problem

$$M = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} . \tag{6.28}$$

Returning to the general case, we now study the classical solution about which we wish to quantize. Considering the p's and q's as classical functions, we adjust them at a particular time to minimize H given l^α. As H and l^α are all conserved, this minimization will be maintained as time evolves. Introducing Lagrange multipliers λ^α, we demand that the variation of $H - \Sigma_\alpha \lambda^\alpha l^\alpha$ vanish:

$$\delta\left(\tfrac{1}{2}\tilde{p}_i\tilde{p}_i + V(\tilde{q}) - \sum_a \lambda^\alpha i\tilde{p}_i M^\alpha_{ij}\tilde{q}_j\right) = 0 . \tag{6.29}$$

Henceforth, quantities with tildes refer to the classical, quasistatic solution. Varying $\tilde{p}_i$ gives

$$\tilde{p}_i - i\lambda \cdot M_{ij}\tilde{q}_j = 0\,, \tag{6.30}$$

where $\lambda \cdot M$ denotes $\Sigma_\alpha \lambda^\alpha M^\alpha$. Varying $\tilde{q}_i$ gives (M^α_{ij} is antisymmetric)

$$V_i(\tilde{q}) + i\lambda \cdot M_{ij}\tilde{p}_j = 0\,. \tag{6.31}$$

Using the equations of motion to determine the time evolution of the solution, we obtain

$$\tilde{q}_i(t) = (e^{i\lambda \cdot Mt})_{ij}\tilde{q}_j(0)\,, \tag{6.32}$$

where $\tilde{q}_j(0)$ is a solution to the equation

$$V_i(\tilde{q}(0)) + (i\lambda \cdot M)_{ij}{}^2\tilde{q}_j(0) = 0 \tag{6.33}$$

and the λ^α are determined implicitly by

$$l^\alpha = -\tilde{q}_i(0) iM^\alpha_{ij}(i\lambda \cdot M)_{jk}\tilde{q}_k(0)\,. \tag{6.34}$$

The system of equations (6.33) and (6.34) is the generalization of Eq. (4.7) to the quasistatic problem. Note that all time dependence of the solution has been isolated in the exponential factor of Eq. (6.32). If we multiply Eq. (6.33) by M^α and use Eq. (6.25), we obtain

$$[V_{ij}(\tilde{q}) + (i\lambda \cdot M_{ij}{}^2)][iM^\alpha\tilde{q}(0)]_j = 0\,. \tag{6.35}$$

For each of the charges l^α we have an eigenvector $iM^\alpha\tilde{q}(0)$ of zero eigenvalue for the matrix $V_{ij}(\tilde{q}) + i(\lambda \cdot M)_{ij}{}^2$. These are the generalizations of the zero-frequency translation mode found in Sec. III.

Equations (6.32) and (6.33) have a simple physical interpretation when l is the momentum and M the gradient operator. In this case Eq. (6.32) becomes

$$\tilde{\phi}_a(\vec{x}, t) = e^{-t(\vec{\lambda}\cdot\vec{\nabla})}\tilde{\phi}_a(\vec{x}, 0) = \tilde{\phi}_a(\vec{x} - \vec{\lambda}t, 0)\,. \tag{6.36}$$

We see that the Lagrange multiplier $\vec{\lambda}$ is just the velocity of the extended object. Using Eq. (6.13), we can write Eq. (6.33) as

$$-[\vec{\nabla}^2 - (\vec{\lambda}\cdot\vec{\nabla})^2]\tilde{\phi}_a(\vec{x}, 0) + \frac{\partial}{\partial\tilde{\phi}_a(\vec{x}, 0)}V(\tilde{\phi}(\vec{x}, 0)) = 0\,. \tag{6.37}$$

The form of the differential operator explicitly exhibits the Lorentz contraction of the extended object along the direction of the velocity $\vec{\lambda}$.

We will find it useful to introduce a $k \times k$ symmetric matrix

$$\tilde{A}^{\alpha\beta} = -\tilde{q}_i iM^\alpha_{ij} iM^\beta_{jk}\tilde{q}_k = \tilde{A}^{\beta\alpha}\,. \tag{6.38}$$

Equation (6.34) then becomes

$$l^\alpha = \tilde{A}^{\alpha\beta}\lambda^\beta\,,$$

where a sum over β is understood. We also define the infinite-dimensional matrix

$$\tilde{S}_{ij} = -iM^\alpha_{ik}\tilde{q}_k(0)(\tilde{A}^{-1})^{\alpha\beta}\tilde{q}_l(0) iM^\beta_{lj}\,. \tag{6.39}$$

If the $M^\alpha_{ij}q_j(0)$ are linearly dependent, $\tilde{A}^{-1}$ will not exist; we assume the charges are chosen to avoid such degeneracies. One can readily check that $\tilde{S}$ is a projection

$$\tilde{S}_{ik}\tilde{S}_{kl} = \tilde{S}_{il}\,. \tag{6.40}$$

Indeed, $\tilde{S}$ projects onto the space spanned by the $M^\alpha_{ij}\tilde{q}_j$; this is the k-dimensional space of zero eigenvalue to $V_{ij}(\tilde{q}) + (i\lambda \cdot M)_{ij}{}^2$.

We now turn to the quantum-mechanical problem and consider $q_i(t)$ which are in some sense near to the solution $\tilde{q}_i(t)$. By analogy with the two-degrees-of-freedom problem mentioned at the beginning of this section, we change variables to a generalized cylindrical coordinate system by writing

$$q_i(t) = \{\exp[i\Sigma_\alpha\theta^\alpha(t)M^\alpha]\}_{ij}r_j(t)\,. \tag{6.41}$$

In introducing k "angular" coordinates θ^α we must place k constraints on r_j to maintain the original number of degrees of freedom. We choose to require $r_i(t)$ to be outside the k-dimensional subspace spanned by the vectors $M^\alpha_{ij}\tilde{q}_j(0)$; so, our k constraints are

$$\tilde{q}_i(0)M^\alpha_{ij}r_j(t) = 0, \quad \alpha = 1, \ldots, k\,. \tag{6.42}$$

or, in terms of the projector $\tilde{S}$,

$$\tilde{S}_{ij}r_j(t) = 0\,. \tag{6.43}$$

The quasistatic solution $\tilde{q}(t)$ has $\tilde{r}(t) = \tilde{q}(0)$ and $\tilde{\theta}^\alpha(t) = \lambda^\alpha t$.

For $q_i(t)$ near to $\tilde{q}_i(t)$, these variables are unique. For large fluctuations the coordinates $\theta^\alpha(t)$ and $r_i(t)$ should be sufficient to describe the configuration, but their values may be ambiguous. In particular, with fluctuations so large that the system is better described in terms of several extended objects, there will be an ambiguity in the definition of the coordinate corresponding to the object's position. As our calculations are ultimately perturbative, this non-uniqueness plays no role in the following discussion. Note that no similar ambiguity occurred in Sec. III; there the coordinates $q_n(t)$ including $q_0(t)$ were unique for any field configuration.

In analogy to $\tilde{A}^{\alpha\beta}$ and $\tilde{S}_{ij}$ we define the matrices

$$A^{\alpha\beta} = -r_i iM^\alpha_{ij} iM^\beta_{jk}r_k \tag{6.44}$$

and

$$S_{ij} = -iM^\alpha_{ik}r_k(A^{-1})^{\alpha\beta}r_l iM^\beta_{lj}\,. \tag{6.45}$$

These objects are quantum operators. In a perturbative treatment $(A^{-1})^{\alpha\beta}$ appearing in Eq. (6.45) is to be expanded as a power series about $(\tilde{A}^{-1})^{\alpha\beta}$. The matrix S_{ij} is a projector onto the space spanned by $iM^\alpha_{ij}r_j$.

We need the conjugate momenta to the variables r_i and θ^α. For θ^α we find

$$l^\alpha = \frac{\delta L}{\delta \dot\theta^\alpha} = p_i iM^\alpha_{ij} q_j = -q_i iM^\alpha_{ij} p_j . \quad (6.46)$$

Thus the k conserved charges l^α are conjugate variables to the θ^α. To determine the conjugate momenta for r_i we must maintain the constraint (6.43). This gives

$$\pi_i = (1-\tilde S)_{ij} \frac{\delta L}{\delta \dot r_j}$$
$$= \tfrac{1}{2}(1-\tilde S)_{ij}[(e^{-iM\cdot\theta})_{ik} p_k + p_k (e^{+iM\cdot\theta})_{kj}] , \quad (6.47)$$

where $M\cdot\theta \equiv \Sigma_\alpha M^\alpha \theta^\alpha$. In terms of these variables the equal-time commutation relations of the theory are

$$[l^\alpha, \theta^\beta] = -i\delta^{\alpha\beta} , \quad (6.48)$$

$$[\pi_i, r_j] = -i(1-\tilde S)_{ij} , \quad (6.49)$$

$$[r_i, r_j] = [\pi_i, \pi_j] = 0 , \quad (6.50)$$

$$[\theta^\alpha, \theta^\beta] = [l^\alpha, l^\beta] = 0 , \quad (6.51)$$

$$[r_i, \theta^\alpha] = [r_i, l^\alpha] = [\pi_i, \theta^\alpha] = [\pi_i, l^\alpha] = 0 . \quad (6.52)$$

Note that $(1-\tilde S)_{ij}$ is the unit matrix on the subspace to which r_i and π_i are restricted. To write the Hamiltonian in terms of the new variables requires that we know p_i in terms of r_i, θ^α, π_i, and l^α. In Appendix D we work out this relation taking proper account of the noncommuting nature of the quantum operators. The result is

$$p_i = (e^{iM\cdot\theta})_{ij}[B_{jk}(\pi_k + iC_k) + (1+B\tilde S)_{jk} l^\alpha (A^{-1})^{\alpha\beta} iM^\beta_{kl} r_l]$$
$$= [(\pi_k - iC_k)B_{kj} - r_l iM^\alpha_{lk}(A^{-1})^{\alpha\beta} l^\beta (1+\tilde S B)_{kj}](e^{-iM\cdot\theta})_{ji} , \quad (6.53)$$

where

$$B_{ij} = \{[1-(1-S)\tilde S]^{-1}(1-S)\}_{ij} = B_{ji} \quad (6.54)$$

and

$$C_i = -\tfrac{1}{2}[(1-\tilde S) iM^\alpha (1+B\tilde S)(A^{-1})^{\alpha\beta} iM^\beta]_{ij} r_j . \quad (6.55)$$

The object C_i comes from the noncommuting nature of the quantum operators and does not arise in the classical problem. Because our change of variables does not involve any explicit time dependence, the Hamiltonian for the new variables is unchanged. Thus we can insert Eqs. (6.41) and (6.53) into (6.16) to obtain

$$H = \tfrac{1}{2}[\pi_i - l^\alpha (A^{-1})^{\alpha\beta} r_l iM^\beta_{lk} \tilde S_{ki} - iC_i]$$
$$\times B_{ij}{}^2 [\pi_j + \tilde S_{jm} iM^\alpha_{mn} r_n (A^{-1})^{\alpha\beta} l^\beta + iC_j]$$
$$+ \tfrac{1}{2} l^\alpha (A^{-1})^{\alpha\beta} l^\beta + V(r) . \quad (6.56)$$

The terms involving l^α are the generalization of the centrifugal barrier in Eq. (6.4). The terms linear in C and the conjugate momenta can be combined using the commutation relations into terms independent of π_i and l^α. All terms involving C represent the quantum correction to the centrifugal barrier mentioned after Eq. (6.6). The equations of motion of the theory are easily found by forming commutators with the Hamiltonian.

As before, perturbation theory provides a formal solution to this problem. This begins with an expansion of such quantities as A^{-1}, S, and B about their classical values. Because of the nonclassical terms involving C in Eq. (6.56), an expansion of H about $\tilde q$ will have terms linear in $r - \tilde r$. Such terms effectively shift the position of the minimum of the quantum-mechanical Hamiltonian in Eq. (6.56) with respect to the minimum of the classical Hamiltonian. An alternative perturbative procedure is to expand about a modified classical Hamiltonian containing the terms involving C in Eq. (6.56). The two procedures should be equivalent, although low-order calculations can differ. The difference can be important with compact symmetries where the charges are quantized and near their minimum quantized value. The classical limit is approached only when the charges are large and their discrete nature unimportant. For example, in the two-degrees-of-freedom problem of Eq. (6.1), the difference between the classical centrifugal barrier $\frac{1}{2}l^2/r^2$ and the quantum-mechanical $\frac{1}{2}(l^2 - \frac{1}{4})/r^2$ is only important when l^2 is of order unity. [On restoration of $\hbar$ to the theory, $(l^2 - \frac{1}{4}) \to (l^2 - \hbar^2/4)$.] In bag models of hadrons with only a small number of quarks in a hadron, these corrections may be nontrivial.

We will not actually carry out the perturbative expansion here, but merely note some unusual features. Suppose we expand r about the classical $\tilde r_i = \tilde q_i(0)$

$$r_i = \tilde r_i + \rho_i . \quad (6.57)$$

The matrix $A^{\alpha\beta}$ of Eq. (6.44) becomes

$$A^{\alpha\beta} = \tilde A^{\alpha\beta} - 2\rho_i iM^\alpha_{ij} iM^\beta_{jk} \tilde r_k - \rho_i iM^\alpha_{ij} iM^\beta_{jk} \rho_k . \quad (6.58)$$

Thus $(A^{-1})^{\alpha\beta}$, which appears in H, is

$$(A^{-1})^{\alpha\beta} = (\tilde A^{-1})^{\alpha\beta} + 2(\tilde A^{-1})^{\alpha\gamma} \rho_i iM^\gamma_{ij} iM^\delta_{jk} \tilde r_k (\tilde A^{-1})^{\delta\beta}$$
$$+ (\tilde A^{-1})^{\alpha\gamma} \rho_i iM^\gamma_{ij} (1-4\tilde S)_{jk} iM^\delta_{kl} \rho_l (\tilde A^{-1})^{\delta\beta}$$
$$+ O(\rho^3) . \quad (6.59)$$

If we restore continuum notation, ${A^{-1}}_{\alpha\beta}$ is not simply related to a local operator. Rather, it is a power series in integrals of local operators. The reason we have been forced to consider such nonlocal objects is rooted in our constraining the charges. In a study of fluctuations of the fields at some space point, a change in the charge density there must be compensated by changes at other points in space. Our constraints are nonlocal in that they only fix the global charge. Thus our reduced Hamiltonian is not an integral of a local density.

A second peculiar feature of the perturbation theory for this problem is that the couplings will involve time derivatives of the coordinates. This comes about because B, occurring in H via the term $\frac{1}{2}\pi_i B^2{}_{ij}\pi_j$, is itself a power series in r. Thus there will be interaction terms involving factors of π_i. In Appendix C we indicated how to handle such couplings in the end-of-line formalism.

We have formally defined the quantum-mechanical problem with the number of degrees of freedom reduced by the number of conserved charges. Unfortunately the procedure has become rather complicated. Although the relativistic connection between energy and momentum should follow, in practice it is simpler to work at zero momentum and treat translations as in Sec. III. The formalism of this section appears unavoidable if we wish to treat other conserved charges such as are carried by quarks in bag models.

VII. THE BAG MODEL

In Ref. 4 we showed on a classical level how the bag model for quark confinement of Chodos *et al.*[5] is related to a limit of an extended object in a local field theory. Except for some difficulties with non-Abelian gauge fields, the classical model of Ref. 5 could be obtained exactly in this limit. To evade the problems with the gauge fields, we presented a viable model with one Abelian gauge field confined in the bag. The hadron constituents in this model are a single triplet of quarks and one scalar parton. The mesons are quark-antiquark bound states while the baryons are made of three quarks plus the scalar parton. In this model, quarks are permanently bound even before taking the limit giving the model of Ref. 5.

In Ref. 4 we took certain coupling constants in our local theory to infinity. The methods of this paper have involved a perturbation series in a coupling; suggesting that the perturbative treatment of quantum fluctuations could fail for this system. We conjecture, however, that the perturbative treatment might be valid for the low excited states of the bag. Indeed, when the number of quarks in a bag is large and the bag itself is large, we intuitively expect a classical behavior of the system.

The reason that perturbation theory may work in our limit is that the modes which possess strong coupling are also taken to infinite frequency. In particular, the scalar mesons responsible for the phase transition producing the bag both acquire a a strong coupling among themselves and a large mass. On a classical level these strongly coupled modes will not be seen if we only study the bag at finite energies; there only the low-frequency modes are relevant, and these modes can be weakly coupled. This is why the classical model is so closely related to the Chodos *et al.*[5] bag.

The difficulty in the quantum theory is that high-frequency modes can enter as virtual intermediate states in a study of low-energy properties. Thus, we must study loop diagrams. If there were only a finite number of high-energy modes, there would still not be a problem because a simple counting of masses in propagator denominators and coupling constants from vertices indicates that in the limit of Ref. 4 the high-frequency modes do not contribute to low-energy properties. However, because of possible divergences in sums over the infinite number of high-frequency modes, in general one connot factor masses in high-frequency propagators out of loop diagrams. Because of these questions, we have been unable to determine if the perturbative techniques of this paper apply to the limiting theory of Ref. 4.

We remark that for studies of the bag at low excitations one should define the renormalization constants of the theory not in terms of the two-, three-, and four-point functions of the fields acquiring large masses, but rather in terms of physical bag parameters. These include the bag skin thickness, skin energy density, and volume energy density. Indeed, in the Lagrangian we have a large number of counterterms at our disposal which can absorb many, if not all, effects of the high-frequency modes. Our conjecture is that the renormalized theory can be defined in terms of finite physical bag parameters. If so, the bag models of Ref. 4 and Ref. 5 should still be closely related on the quantum level.

VIII. REMAINING QUESTIONS

We have studied the problem of quantization of extended solutions of classical field theories. Using canonical methods, we exhibited a perturbation solution for the properties of these objects, the perturbation being essentially in fluctuations of the quantum solution about the classical one. Translation invariance required a careful separation of this degree of freedom. Several interesting questions remain.

How should Fermi fields be treated? Unlike the Bose case, anticommutation relations are altered by adding to a Fermi field a classical solution to the Dirac equation. A possible approach might be to look for "classical" solutions in some anticommuting algebra. Another interpretation has been proposed by Chang, Ellis, and Lee.[38]

How can one study states with several extended objects? If we have two extended objects there will be forces between them and the solution is necessarily time dependent. Furthermore, when they are close together the forces will be strong and the validity of a perturbative approach is unclear. Inevitably this leads us to the problem of creation of extended objects and pairs of extended objects. For the bag models this is central to understanding $e^+e^- \to$ hadrons.

Can one find a local field for an extended object? As mentioned earlier Coleman[14] and Mandelstam[16] have discussed a connection between the extended solutions of the sine-Gordon equation and the Fermions of the massive Thirring model. It may be that hadrons can equivalently be described as extended objects or in terms of local fields.

What are the statistics of the quantized extended particles? The results for the sine-Gordon equation emphasize that we should expect the unexpected. The bag solutions are unlikely to acquire peculiar statistics, but the 't Hooft monopole[19] has unusual properties under rotations and something strange may occur on quantization.

Indeed, the list of questions seems endless. How should gauge fields be handled? Does the bag limit of Ref. 4 make sense? How should one treat theories with both magnetic and electric couplings to gauge fields?[3,4] Can one obtain a spontaneously broken symmetry by making the symmetric vaccuum unstable to extended object formation? Can quarks themselves be extended objects?

APPENDIX A: THE VIRIAL THEOREM

The proof of the virial theorem (4.24) is a sequence of partial integrations. Here we prove it in d spatial dimensions

$$\int d^d x\, V(\phi_c(x)) = \frac{1}{d}\int d^d x\, V(\phi_c(x))\vec{\nabla}\cdot\vec{x} = -\frac{1}{d}\int d^d x\, V'(\phi_c(x))\vec{x}\cdot\vec{\nabla}\phi_c(x). \tag{A1}$$

Using the equation of motion (4.7) gives

$$\begin{aligned}\int d^d x\, V(\phi_c(x)) &= -\frac{1}{d}\int d^d x[\vec{\nabla}^2\phi_c(x)]\vec{x}\cdot\vec{\nabla}\phi_c(x)\\ &= +\frac{1}{d}\int d^d x\{[\vec{\nabla}\phi_c(x)]^2 + [\vec{\nabla}\phi_c(x)]\cdot(\vec{x}\cdot\vec{\nabla})\vec{\nabla}\phi_c(x)\}\\ &= \frac{1}{d}\int d^d x\{[\vec{\nabla}\phi_c(x)]^2 + \tfrac{1}{2}(\vec{x}\cdot\vec{\nabla})(\vec{\nabla}\phi_c)^2\}\\ &= \frac{1}{d}\left(1-\frac{d}{2}\right)\int d^d x(\vec{\nabla}\phi_c)^2.\end{aligned} \tag{A2}$$

Then the general theorem reads

$$\int d^d x(\nabla\phi_c)^2 = \frac{2d}{2-d}\int d^d x\, V(\phi_c(x)) = H_c d. \tag{A3}$$

Note that only for $d<2$ do the gradient and potential terms in H_c have the sam sign. This is an alternative proof to the statement that stable static extended solutions do not exist in scalar field theories in two or more space dimensions.

APPENDIX B: THE ZERO-FREQUENCY GENERATING STATE

Here we work out the zero-frequency contribution to $(W_0(\vec{s}',\vec{s})|$. Since all higher modes separate, we consider the Hamiltonian

$$H = \tfrac{1}{2}\dot{\vec{q}}^{\,2}. \tag{B1}$$

The equation of motion

$$\ddot{\vec{q}}(t) = 0 \tag{B2}$$

implies

$$\vec{q}(t) = \vec{q}(0) + \dot{\vec{q}}t. \tag{B3}$$

The commutation relations are

$$[\dot{\vec{q}}_i(t), \vec{q}_j(t)] = -i\delta_{ij}. \tag{B4}$$

The states $|\vec{s}\rangle$ satisfy

$$\langle\vec{s}'|\vec{s}\rangle = (2\pi)^3\delta^3(\vec{s}'-\vec{s}), \tag{B5}$$

$$\dot{\vec{q}}|\vec{s}\rangle = \vec{s}|\vec{s}\rangle. \tag{B6}$$

The relative phases of the states $|\vec{s}\rangle$ are fixed by

$$|\vec{s}\rangle = e^{i\vec{s}\cdot\vec{q}(0)}|\vec{0}\rangle. \tag{B7}$$

We wish to evaluate

$$(W_0(\vec{s}',\vec{s})| = (0|\langle\vec{s}'|T\{\exp[\textstyle\int dt\,\vec{q}(t)\cdot\vec{c}(t)]\}|\vec{s}\rangle. \tag{B8}$$

Define the quantity

$$|\psi(\vec{s},\vec{c})\rangle = T\left\{\exp\left(\int dt\,\vec{q}(t)\cdot\vec{c}(t)\right)\right\}|\vec{s}\rangle. \tag{B9}$$

This object is a state in Hilbert space and an operator in end-of-line space. Using Eqs. (B3), (B4), and (B6), we have

$$\dot{\vec{q}}\,|\psi(\vec{s},\vec{c})\rangle = \left(\vec{s} - i\int dt\,\vec{c}(t)\right)|\psi(\vec{s},\vec{c})\rangle. \tag{B10}$$

This implies

$$|\psi(\vec{s},\vec{c})\rangle = F(\vec{s},\vec{c})\left|\vec{s} - i\int dt\,\vec{c}(t)\right\rangle, \tag{B11}$$

where $F(\vec{s},\vec{c})$ is to be determined. Taking $\vec{q}(0)$ on the state gives

$$\vec{q}(0)|\psi(\vec{s},\vec{c})\rangle = \left(+i\int dt\,t\vec{c}(t) - i\vec{\nabla}_s\right)|\psi(\vec{s},\vec{c}(t))\rangle. \tag{B12}$$

With Eq. (B11) we see

$$-i\vec{\nabla}_s F(\vec{s},\vec{c}) = -i\int dt\, t\vec{c}(t) F(\vec{s},\vec{c}) \qquad \text{(B13)}$$

or

$$F(\vec{s},\vec{c}) = A(\vec{c})\exp\left(\vec{s}\cdot\int dt\, t\vec{c}(t)\right), \qquad \text{(B14)}$$

where $A(\vec{c})$ is to be determined. Now consider the commutator

$$\begin{aligned}[|\psi(\vec{s},\vec{c})\rangle, \vec{c}^{\dagger}(t)] &= T\left\{\exp\left(\int dt'\vec{q}(t')\cdot\vec{c}(t')\right)q(t)\right\}|\vec{s}\rangle \\ &= \left(+i\int_{-\infty}^{t} dt' t'\vec{c}(t') - it\int_{-\infty}^{t} dt'\vec{c}(t') - i\vec{\nabla}_s + t\vec{s}\right)|\psi(\vec{s},\vec{c})\rangle \\ &= [A(\vec{c}), \vec{c}^{\dagger}(t)]\exp\left(\vec{s}\int dt' t'\cdot\vec{c}(t')\right)|\vec{s} - i\int dt\,\vec{c}(t)\rangle + \left(t\vec{s} - i\vec{\nabla}_s + i\int_{-\infty}^{\infty} dt' t'\vec{c}(t')\right)|\psi(\vec{s},\vec{c})\rangle. \qquad \text{(B15)}\end{aligned}$$

This implies

$$\begin{aligned}[A(\vec{c}), \vec{c}^{\dagger}(t)] &= \left(-i\int_{t}^{\infty} dt' t'\vec{c}(t') - i\int_{-\infty}^{\infty} dt' t\vec{c}(t')\right)A(\vec{c}) \\ &= -i\int_{-\infty}^{\infty} dt'\tfrac{1}{2}(t + t' + |t - t'|)\vec{c}(t')A(\vec{c}) \qquad \text{(B16)}\end{aligned}$$

or

$$A(\vec{c}) = B\exp\left(-\frac{i}{4}\int dt\,dt'\vec{c}(t)\cdot\vec{c}(t')(t + t' + |t - t'|)\right). \qquad \text{(B17)}$$

The constant B is easily shown to be unity. Thus we have

$$|\psi(\vec{s},\vec{c})\rangle = \exp\left(-\frac{i}{4}\int dt\,dt'\vec{c}(t)\cdot\vec{c}(t')(t + t' + |t - t'|)\right)\exp\left(\vec{s}\cdot\int dt\, t\vec{c}(t)\right)|\vec{s} - i\int dt\,\vec{c}(t)\rangle \qquad \text{(B18)}$$

and

$$(W(\vec{s}',\vec{s})| = (0|(2\pi)^3\delta^3\left(\vec{s}' - \vec{s} + i\int dt\vec{c}(t)\right)\exp\left(-\frac{i}{4}\int dt\,dt'\vec{c}(t)\cdot\vec{c}(t')(t + t' + |t - t'|) + \vec{s}\cdot\int dt\, t\,\vec{c}(t)\right). \qquad \text{(B19)}$$

This can be written in the more symmetric form

$$(W(\vec{s}',\vec{s})| = (0|(2\pi)^3\delta^3\left(\vec{s}' - \vec{s} + i\int dt\,\vec{c}(t)\right)\exp\left(-\frac{i}{4}\int dt\,dt'\vec{c}(t)\cdot\vec{c}(t')|t - t'|\right)\exp\left(\frac{\vec{s} + \vec{s}'}{2}\cdot\int dt\, t\,\vec{c}(t)\right). \qquad \text{(B20)}$$

Restoring the non-zero-frequency modes gives Eq. (5.9).

APPENDIX C: TIME DERIVATIVES AND END-OF-LINE SPACE

To treat time-ordered products of time derivatives of field requires care in the end-of-line formation because time derivatives do not commute with time orderings. Since the full Hamiltonian involves time derivatives we must study this problem to calculate the energy of an extended object. This treatment becomes all the more essential in theories with derivative couplings.

To illustrate the procedure, consider a single mode with Hamiltonian

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2, \qquad \text{(C1)}$$

where $p = \dot{q}$. As in Eq. (4.27), we write

$$q(t) = \frac{1}{(2\omega)^{1/2}}(ae^{-i\omega t} + a^{\dagger}e^{+i\omega t}) \qquad \text{(C2)}$$

and consider states $|n\rangle$ such that

$$\begin{aligned}|n\rangle &= \left(\frac{1}{n!}\right)^{1/2}(a^{\dagger})^n|0\rangle, \\ a|0\rangle &= 0, \qquad \text{(C3)} \\ \langle 0|0\rangle &= 1.\end{aligned}$$

We desire to find an end-of-line space expression for

$$\langle 0|T\{q(t_1)\cdots q(t_n)p(t_{n+1})\cdots p(t_{n+m})\}|0\rangle, \qquad \text{(C4)}$$

where for simplicity we consider only first time derivatives. We extend the end-of-line space by introducing a creation operator for end of lines corresponding to time derivatives of the coordinate. Thus consider a new operator $d(t)$ such that

$$\begin{aligned}&[d(t'), d^{\dagger}(t)] = \delta(t' - t), \\ &d(t)|0) = 0, \\ &[d(t), d(t')] = [d(t), c(t')] = [d(t), c^{\dagger}(t')] = 0,\end{aligned}$$

$$|t_1, \ldots, t_n; t_{n+1}, \ldots, t_{n+m}) = c^\dagger(t_1)\cdots c^\dagger(t_n)d^\dagger(t_{n+1})\cdots d^\dagger(t_{n+m})|0). \tag{C5}$$

The generating state is now defined

$$(W_0| = (0\left|\left\langle 0|T \exp\left(\int dt\, c(t)q(t) + \int dt\, d(t)p(t)\right)\right|0\right\rangle. \tag{C6}$$

With this definition

$$\langle 0|T\{q(t_1)\cdots q(t_n)p(t_{n+1})\cdots p(t_{n+m})\}|0\rangle = (W_0|c^\dagger(t_1)\cdots c^\dagger(t_n)d^\dagger(t_{n+1})\cdots d^\dagger(t_{n+m})|0). \tag{C7}$$

The expression for $(W_0|$ can be evaluated using Wick's theorem[33] to give

$$(W_0| = (0|\exp\left\{\frac{i}{2}\int_{\infty}^{\infty} dt\,dt'\left[c(t)c(t')\Delta(t,t') + 2d(t)c(t')\frac{\partial}{\partial t}\Delta(t,t') + d(t)d(t')\left(-\frac{\partial^2}{\partial t^2}\Delta(t,t') - \delta(t'-t)\right)\right]\right\}, \tag{C8}$$

where

$$\Delta(t,t') = -\frac{i}{2\omega}e^{-i\omega|t'-t|}. \tag{C9}$$

Equation (C8) gives the correct two-point functions

$$\langle 0|T\{q(t)q(t')\}|0\rangle = i\Delta(t,t'), \tag{C10}$$

$$\langle 0|T\{p(t)q(t')\}|0\rangle = i\partial_0\Delta(t,t'), \tag{C11}$$

$$\langle 0|T\{p(t)p(t')\}|0\rangle = i\partial_0{}^2\Delta(t,t') - i\delta(t-t'). \tag{C12}$$

The term involving $\delta(t-t')$ in Eq. (C12) comes from the equal-time commutator of p with q.

To see the importance of keeping the term involving $\delta(t'-t)$ in Eq. (C8), consider the ground-state energy

$$\begin{aligned} E_0 = \langle 0|H|0\rangle = \langle 0|T\{H\}|0\rangle &= (W|\tfrac{1}{2}d^{\dagger 2} + \tfrac{1}{2}\omega^2 c^{\dagger 2}|0) \\ &= \frac{i}{2}[-\partial_0{}^2\Delta(t,t') - \delta(t-t') + \omega^2\Delta(t,t')]\big|_{t=t'} \\ &= \frac{1}{4\omega}\{[+2\omega^2 + i\delta(t'-t)]e^{-i\omega|t'-t|} - \delta(t-t')\}\big|_{t=t'} = \tfrac{1}{2}\omega. \end{aligned} \tag{C13}$$

The purpose of the $\delta(t-t')$ is to subtract a divergent piece of $\partial_0{}^2\Delta(t,t')$ thus giving the correct zero-point energy. In momentum space the calculation proceeds,

$$\Delta(q) = \int d(t-t')e^{iq(t-t')}\Delta(t,t') = \frac{1}{q^2-\omega^2+i\epsilon}, \tag{C14}$$

$$\begin{aligned} E_0 &= \frac{i}{2}\int_{-\infty}^{\infty}\frac{dq}{2\pi}[q^2\Delta(q) - 1 + \omega^2\Delta(q)] \\ &= i\int_{-\infty}^{\infty}\frac{dq}{2\pi}\left(\frac{\omega^2}{q^2-\omega^2+i\epsilon}\right) = \tfrac{1}{2}\omega. \end{aligned} \tag{C15}$$

In interacting theories without derivative couplings, the equal-time term $\delta(t-t')$ in Eq. (C8) only need be considered for this lowest-order zero-point energy. All higher-order corrections will not involve a contraction of two $d^\dagger$ operators; consequently, the Feynman rules are the naive ones. This entire discussion is similar for the zero-frequency mode, where Δ becomes the propagator in Eq. (5.10).

APPENDIX D: DERIVATION OF EQ. (6.53)

Equation (6.47) can be rewritten

$$\pi_i + iC_i = (1-\tilde{S})_{ij}(e^{-iM\cdot\theta})_{jk}p_k, \tag{D1}$$

where

$$-iC_i = \tfrac{1}{2}(1-\tilde{S})_{ij}[p_k, (e^{iM\cdot\theta})_{kj}]. \tag{D2}$$

We will work out this commutator later. From Eq. (6.46) using Eq. (6.45) we obtain

$$l^\alpha(A^{-1})^{\alpha\beta}iM^\beta_{ij}r_j = S_{ij}(e^{-iM\cdot\theta})_{jk}p_k. \tag{D3}$$

Between Eqs. (D1) and (D3) we know what S and $1-\tilde{S}$ do to $(e^{-iM\cdot\theta})_{jk}p_k$. To invert this we first prove the following identity

$$1 = S - B(1-\tilde{S})S + B(1-\tilde{S}), \tag{D4}$$

where B is a matrix representing the inverse of $1-\tilde{S}$ restricted to the space defined by $1-S$

$$B = [1-(1-S)\tilde{S}]^{-1}(1-S). \tag{D5}$$

Matrix indices and sums are understood in these equations. To prove (D4) let X be an arbitrary vector and write

$$(1-S)(1-\tilde{S})X = (1-S)(1-\tilde{S})SX+(1-S)(1-\tilde{S})(1-S)X = (1-S)(1-\tilde{S})SX+[1-(1-S)\tilde{S}](1-S)X. \quad \text{(D6)}$$

This implies

$$(1-S)X=[1-(1-S)\tilde{S}]^{-1}(1-S)[(1-\tilde{S})X-(1-\tilde{S})SX] = B(1-\tilde{S})X-B(1-\tilde{S})SX \quad \text{(D7)}$$

or

$$X=[S-B(1-\tilde{S})S+B(1-\tilde{S})]X. \quad \text{(D8)}$$

Since X is arbitrary, this is equivalent to Eq. (D4).

Combining Eqs. (D1), (D3), and (D4) we obtain

$$p_i=(e^{iM\cdot\theta})_{ij}[(1-B(1-\tilde{S}))_{jk}l^\alpha(A^{-1})^{\alpha\beta}iM^\beta_{kl}r_l + B_{jk}(\pi_k+iC_k)]. \quad \text{(D9)}$$

This simplifies slightly since $(1-S)_{ij}iM^\alpha_{jk}r_k=0$. Thus we have

$$p_i=(e^{iM\cdot\theta})_{ij}[(1+B\tilde{S})_{jk}l^\alpha(A^{-1})^{\alpha\beta}iM^\beta_{kl}r_l + B_{jk}(\pi_k+iC_k)], \quad \text{(D10)}$$

which is Eq. (6.53). Using (D10) and (6.48)–(6.52) we can determine C_i:

$$C_i=-\tfrac{1}{2}[(1-\tilde{S})iM^\alpha(1+B\tilde{S})(A^{-1})^{\alpha\beta}iM^\beta]_{ij}r_j, \quad \text{(D11)}$$

which is Eq. (6.55).

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[1]A. Scott, F. Chu, and D. McLaughlin, Proc IEEE 61, 1443 (1973).

[2]To be precise, a solution of a field theory is an extended object if for every $\epsilon>0$ there exists an $x(t)$ and an $r(\epsilon)$ such that for all times t

$$\frac{\int_S d^3x\,\mathcal{H}(\vec{x},t)}{\int d^3x\,\mathcal{H}(\vec{x},t)}>1-\epsilon,$$

where S is a sphere of radius $r(\epsilon)$ centered on $\vec{x}(t)$, and $\mathcal{H}(\vec{x},t)$ is the energy density. We consider the energy relative to the vacuum so $\mathcal{H}(x)$ is a non-negative quantity.

[3]Y. Nambu, in *Proceedings of the Johns Hopkins Workshop on Current Problems in High Energy Particle Theory, 1974*, edited by G. Domokos *et al.* (Johns Hopkins Univ., Baltimore, 1974); G. Parisi, Phys. Rev. D 11, 970 (1975); M. Kobayashi, Prog. Theor. Phys. 51, 1636 (1974); P. Olesen, invited lecture at IX Balaton Symposium on Particle Physics, Hungary, 1974 (unpublished); M. Creutz, Phys. Rev. D 10, 2696 (1974); S. Mandelstam, Phys. Lett. 53B, 476 (1975); A. Jevicki and P. Senjanovic, CCNY report, 1975 (unpublished).

[4]M. Creutz, Phys. Rev. D 10, 1749 (1974); M. Creutz and K. S. Soh, *ibid.* 12, 443 (1975).

[5]A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. W. Weisskopf, Phys. Rev. D 9, 3471 (1974).

[6]N. H. Christ and T. D. Lee, Phys. Rev. D 12, 1606 (1975).

[7]E. Tomboulis, Phys. Rev. D 12, 1678 (1975).

[8]L. D. Faddeev, Zh. Eksp. Teor. Fiz. Pis'ma Red. 21, 141 (1975) [JETP Lett. 21, 64 (1975)].

[9]R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4114 (1974); 10, 4139 (1974); 10, 4128 (1974); for an excellent review see R. Rajaraman, Inst. for Advanced Study Report No. AEC-2220-47 (unpublished).

[10]J. Goldstone and R. Jackiw, Phys. Rev. D 11, 1486 (1975); see also G. Rosen, Phys. Rev. 160, 1278 (1967); S. J. Chang, Phys. Rev. D 12, 1071 (1975); S. J. Chang and J. A. Wright, *ibid.* 12, 1595 (1975); R. J. Rivers, Imperial College Report No. ICTP/74/5, 1974 (unpublished); A. Klein and F. Krejs, Phys. Rev. D 12, 3112 (1975).

[11]J.-L Gervais and B. Sakita, Phys. Rev. D 11, 2943 (1975); C. G. Callan and D. J. Gross, Princeton Univ. report, 1975 (unpublished).

[12]J. Arafune, P. G. O. Freund, and C. J. Goebel, J. Math. Phys. 16, 433 (1975).

[13]R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 11, 3424 (1975), and references therein.

[14]S. Coleman, Phys. Rev. D 11, 2088 (1975).

[15]W. Thirring, Ann. Phys. (N. Y.) 3, 91 (1958).

[16]S. Mandelstam, University of California at Berkeley report, 1975 (unpublished).

[17]J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[18]P. W. Higgs, Phys. Rev. Lett. 13, 508 (1964).

[19]G. 't Hooft, Nucl. Phys. B79, 276 (1974).

[20]P. Vinciarelli, Lett. Nuovo Cimento 4, 905 (1972); Nucl. Phys. B89, 463 (1975); T. D. Lee and G. C. Wick, Phys. Rev. D 9, 3471 (1974).

[21]W. A. Bardeen, M. S. Chanowitz, S. D. Drell, M. Weinstein, and T.-M. Yan, Phys. Rev. D 11, 1094 (1975).

[22]G. H. Derrick, J. Math. Phys. 5, 1252 (1964).

[23]S. Coleman, Commun. Math. Phys. 31, 259 (1973).

[24]G. Wentzel, Z. Phys. 38, 518 (1926); H. A. Kramers, *ibid.* 39, 828 (1926); L. Brillouin, C. R. Acad. Sci. 183, 24 (1926); more recent references are quoted in Refs. 9 and 13.

[25]R. Rajaraman and Erik J. Weinberg, Phys. Rev. D 11, 2950 (1975).

[26]P. Vinciarelli, CERN Report No. TH1993, 1975 (unpublished); K. Cahill, Phys. Lett. 53B, 174 (1974).

[27]Goldstone and Jackiw actually use a slightly more sophisticated form of Γ, including not only a source for $\phi(x)$ but in addition a bilinear source of form $\int dx\,dy\,\phi(x)f(x,y)\phi(y)$.

[28]A good review of these variational techniques is con-

tained in E. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).

[29]R. P. Feynman, Rev. Mod. Phys. 20, 367 (1948).

[30]This mode does not appear when discussing the vacuum with $\phi_i(x)$ constant.

[31]For a finite-energy stable solution $V(\phi(x))$ must always exceed $V(\phi_c(\infty))=0$. In Appendix A the virial theorem (4.24) is given in arbitrary spatial dimension; the gradient and potential terms are both positive in one space dimension.

[32]M. Born and V. Fock, Z. Phys. 51, 165 (1928).

[33]G. C. Wick, Phys. Rev. 80, 268 (1950).

[34]R. P. Feynman, Phys. Rev. 76, 749 (1949).

[35]H. M. Fried, *Functional Methods and Models in Quantum Field Theory* (MIT Press, Cambridge, Mass., 1972). Our notation differs somewhat.

[36]For a theory with Fermi fields these would become anticommutation relations.

[37]The δ function containing $\vec{c}_0(t)$ in Eq. (5.9) is defined as an operator in end-of-line space with the properties

$$\delta^3(\vec{s}\,' - \vec{s} + i\int dt\, \vec{c}_0(t))|\,0\rangle = \delta^3(\vec{s}\,' - \vec{s})|\,0\rangle,$$

$$[\delta^3(\vec{s}\,' - \vec{s} + i\int dt\, \vec{c}_0(t)),\ \vec{c}_0^{\,\dagger}(t)] = -i\vec{\nabla}_s\,\delta^3(\vec{s}\,' - \vec{s} + i\int dt\, \vec{c}_0(t)),$$

$$[\delta^3(\vec{s}\,' - s + i\int dt\, \vec{c}_0(t)),\ \vec{c}_n^{\,\dagger}(t)] = 0,\quad n > 0$$

where $\vec{\nabla}_s$ is the gradient with respect to $\vec{s}$. Repeated application of these relations gives the effect of this δ function on the states of end-of-line space. For more rigor, one can smear the states with a test function.

[38]S. J. Chang, S. D. Ellis, and B. W. Lee, Phys. Rev. D 11, 3572 (1975).