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Lorentz-covariant quantization of nonlinear waves*

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The quantum theory of a real scalar field in one space dimension with quartic coupling is studied as an example of a field theory possessing exact classical solutions representing extended distributions of energy and momentum. It is shown how formal canonical quantization can be implemented in practice to obtain self-consistent, Lorentz-covariant descriptions of restricted portions of Hilbert space. This is done by deriving dynamical and kinematical sum rules from the equations of motion and commutation relations, respectively. These sum rules are shown to have a variational basis, which guarantees the consistency of the approach. Overall, we derive Lorentz-covariant generalizations of results found previously by Goldstone and Jackiw using the same approach, as well as some new results.

I. INTRODUCTION

This paper is concerned with the quantum theory of nonlinear field equations known to possess classical solutions variously characterized as kinks, solitons, or simply nonlinear waves.[1-3] Since detailed introductory remarks can be found in essentially all the references cited in the next paragraph, we shall assume the reader to be conversant with this material.

It is indicative of the interest it has generated that the study of this problem has already elicited from the theoretical community a truly impressive (though to the budding author a correspondingly discouraging) versatility in technique. These include WKB methods,[4,5] coherent-state variational principles,[5a,6-8] generating-function variational methods,[9,10] path-integral quantization,[11] canonical quantization,[12-15] and generalized self-consistent field methods, based directly on the field equations.[9,16-18]

It is the purpose of this paper, as well as of those with which we intend to follow it, to develop more fully the last of these methods, particularly as described by Goldstone and Jackiw.[9] In this early phase, it would appear to be useful to develop any method which is, in its *formulation,* theoretically complete and capable of systematic exploitation and improvement. We aim to show how this is true for the method under study, which was developed originally to provide a theory of collective motion for nonrelativistic many-body systems.[19]

Let us understand from the beginning that in the new development one gets nowhere unless one has some *a priori* notions of the physics one is seeking. Conventional, i.e., previous, field theory may *almost* be defined by the statement that the fundamental particles are the fundamental fields in the Lagrangian density. The kinks or solitons, by contrast, are coherent, nonlinear superpositions of the fundamental fields. The existence of the new objects as *quantum objects* is signaled by the discovery of exact solutions to the classical field equations. It is in how this signal is utilized that the various methods differ. In most of the quantization methods only the difference between the original field operator and the classical solution is quantized. In our method, no such *a priori* separation is made. Formal quantization is carried out quite conventionally, and novelty enters only when the structure of the *assumed* Hilbert space is studied from matrix elements of the field equations. Assumptions concerning the structure of this space are verified self-consistently.

In the example studied in Ref. 9 and also in this paper, where the classical solution is time independent, the latter is proved, from the quantum theory, to be the Fourier transform of the form factor of a new particle. (In the next paper of this series, we shall show that certain time-dependent solutions of the classical field equations can be interpreted as *generating functions* of form factors of a one-dimensional array of states.)

Since, in fact, the structure of our approach has been elegantly exposed in Ref. 9, we must become more specific in order to demonstrate how the practice can be carried further than it was in that work. We deal therefore, as an example, with the Lagrangian density describing a ϕ^4 model in one-plus-one dimension,

$$\mathcal{L}(x,t)=\tfrac{1}{2}[\partial_t\phi(x,t)]^2-\tfrac{1}{2}[\partial_x\phi(x,t)]^2-U(\phi), \quad (1.1)$$

where

$$U(\phi)=\frac{1}{2\lambda}(m^2-\lambda\phi^2)^2. \quad (1.2)$$

The Hamiltonian is

$$H=\int \mathcal{H}(x,t)dx, \quad (1.3)$$

$$\mathcal{H}(x,t)=\tfrac{1}{2}\pi^2(x,t)+\tfrac{1}{2}[\partial_x\phi(x,t)]^2+U(\phi). \qquad (1.4)$$

The theory is taken, as conventionally, to be defined by (1.3), (1.4), and the canonical commutation relations

$$[\phi(x,t),\pi(y,t)]=i\delta(x-y), \qquad (1.5)$$

which together yield the field equations

$$\partial_t{}^2\phi(x,t)-\partial_x{}^2\phi(x,t)+U'(\phi(x,t))=0. \qquad (1.6)$$

Goldstone and Jackiw[9] have indicated how to obtain consistent c-number equations from (1.5) and (1.6) which characterize the soliton and certain excited states in the same sector of Hilbert space. We refer the reader to their paper for the general derivation of the equations which we shall exploit in the sequel. (We shall, however, give a summary of ideas in Secs. II, III.) They have shown that the static classical solution

$$\phi_c(x-x_0)=\frac{m}{\sqrt{\lambda}}\tanh m(x-x_0) \qquad (1.7)$$

describes the Fourier transform of the form factor of a particle at rest, whose mass is in lowest approximation

$$M=\frac{4}{3}\frac{m^3}{\lambda}, \qquad (1.8)$$

and that this solution is valid provided $M\gg m$, i.e., $m^2\gg\lambda$. The derivation of this result requires, as we analyze in Sec. II, two assumptions, namely, $(m/M)\ll 1$ and $(p/M)\ll 1$, where p is the momentum of the particle. Goldstone and Jackiw have established, however, that the corrections of order $(p/M)^2$ are as expected, and thus they have provided the approximate relativistic invariance of the theory. For the soliton, the double expansion in (m/M) and (p/M) is studied to lowest order in Sec. II, the purpose of this study being to set the stage for the core of the work.

In Sec. III we develop the first new result of our work. We consider the exact c-number equations derivable from (1.5) and (1.6) by assuming all sums over intermediate states are exhausted by the soliton, an excited bound state of the same intrinsic structure, and the one-soliton–one-meson continuum states. As established,[9] the resulting equations contain correctly all terms of order λ^{-1} and λ^0. We prove that these equations can be solved to all orders in (p/M) provided we always neglect (m/M) compared to (p/M). We thereby establish the relativistic covariance of the c-number equations and of their solutions. The consistency of the commutation relations (1.5) is also established in the same approximation.

In Sec. IV we show how the c-number equations of motion can be derived from a novel variational principle, involving the trace of the Hamiltonian, H. This novel variational principle has been known within other contexts for some time,[20] and a concise account of it has been given quite recently within the context of particle quantum mechanics.[21] The general principle is described and the functional which yields the c-number equations of Sec. II is derived and discussed. An important consequence of the variational principle is that the Hamiltonian is diagonal within the space of states utilized in the trace. It is seen that this yields a set of consistency conditions which must be the general consistency conditions yearned for in Ref. 9.

These last are illustrated in Sec. V, where we also establish that energy and momentum of the soliton transform like the components of a vector. This is shown by use of a generalized virial theorem.

We conclude this introduction by emphasizing that the method under study here is both theoretically complete and systematically exploitable. The latter property has been questioned,[11] and we therefore address ourselves to this point in particular: (i) Every observable can be calculated from the c-number equations derivable from (1.5) and (1.6). (ii) The basic elements of these equations are the matrix elements of single field operators $\phi(x,t)$ between various states. Each of these can be represented by a Laurent series in (λ^2/m). The dominant dependence for small λ is determined by self-consistency requirements. (iii) Assuming the leading order terms in a given equation can be obtained exactly, higher-order corrections can be calculated by straightforward perturbation methods.

The one really new element compared to previous applications to the nonrelativistic many-body problem is the problem of renormalization, but the work already done indicates[4] that this can be handled without excessive difficulty.

II. ONE-SOLITON SECTOR: LEADING CORRECTIONS TO THE STATIC LIMIT

The most rudimentary properties of the soliton are determined from the equation

$$\{-(q-p)^2+[E(q)-E(p)]^2+2m^2\}\langle p|\phi|q\rangle =2\lambda\int\frac{dr}{2\pi}\frac{dr'}{2\pi}\langle p|\phi|r\rangle\langle r|\phi|r'\rangle\langle r'|\phi|q\rangle, \qquad (2.1)$$

where $E(q)=(q^2+M^2)^{1/2}$, $M=(4m^3/3\lambda)$ is the soliton mass in lowest approximation, $\phi=\phi(x=0)$, and $|q\rangle$ is the soliton state vector with momentum q.

The approximation in (2.1) arises from evaluation of the matrix element $\langle p \mid \phi^3 \mid q\rangle$ from a sum rule which includes only soliton intermediate states.

We form the Fourier transform

$$\Phi_p(x) = \frac{1}{2\pi}\int dq e^{i(q-p)x}\langle p \mid \phi \mid q\rangle\,, \tag{2.2}$$

and thereby transform (2.1) into the equation ($\hat{p} = -i\partial_x$)

$$\{-\hat{p}^2\Phi_p(x) + [E(\hat{p}+p) - E(p)]^2 + 2m^2\}\Phi_p(x)$$
$$= 2\lambda(2\pi)^{-1}\int dr dy e^{i(r-p)(x-y)}\Phi_r(x)\Phi_r(y)\Phi_p(y)\,, \tag{2.3}$$

where we have assumed that $\Phi_p(x)$ is real. [Incidentally, if $|X\rangle$ is the localized state

$$|X\rangle = (2\pi)^{-1/2}\int dp \mid p\rangle \; e^{-ipX}, \tag{2.4}$$

then

$$\Phi_p(x) = \langle p \mid \phi(x) | X=0\rangle \;(2\pi)^{-1/2}.\,] \tag{2.5}$$

The emergence of a static solution, $\phi_c(x)$, to (2.3) requires the two approximations

$$\langle [E(\hat{p}+p) - E(p)]^2\rangle \ll \langle -\hat{p}^2\rangle\,, \tag{2.6}$$

$$\Phi_p(x) \cong \Phi_0(x) \equiv \phi_c(x), \tag{2.7}$$

the former pertaining to an average with respect to the wave function $\phi_c(x)$, the latter condition becoming nontrivial when p refers to one of the variables of integration. The application of (2.6) and (2.7) to (2.3) yields the now well-known result

$$[-\hat{p}^2 + 2m^2 - 2\lambda\phi_c^2(x)]\,\phi_c(x) = 0, \tag{2.8}$$

$$\phi_c(x) = (m/\lambda^{1/2})\tanh m(x-x_0), \tag{2.9}$$

as well as the interpretation of ϕ_c as Fourier transform of a form factor.

In seeking to improve the solution (2.8), (2.9), the essential observation is that there are two expansion parameters. This is immediately evident if we consider the neglected terms,

$$[E(\hat{p}+p) - E(p)]^2 = M^{-2}(p^2\hat{p}^2 + p\hat{p}^3 + \tfrac{1}{4}\hat{p}^4) + O(\text{`}p\text{'}^6), \tag{2.10}$$

where 'p' $= p$ or $\hat{p}$. Since $\langle \hat{p}\rangle \sim m$, the contributions neglected so far will generate a double series, in the velocity $v\sim(p/M)$ and in the mass ratio (m/M).

We explore this double series by expanding the difference between $\Phi_p(x)$ and $\phi_c(x)$ in a complete set,

$$\Phi_p(x) = \phi_c(x) + \sum_n [(p/M)^2 c_n + (pm/M^2)d_n + (m/M)^2 e_n]\psi_n(x) + \cdots. \tag{2.11}$$

The orthonormal set $\psi_n(x)$ is that determined by the process of linearization of the field equations about the soliton solution, interpreted quantum mechanically as excited states in the soliton sector. The $\psi_n(x)$ thus satisfy the equation,[9] rederived in Sec. III B below,

$$[-\hat{p}^2 + 2m^2 - 6\lambda\phi_c^2(x)]\psi_n(x) = \omega_n^2\psi_n(x), \tag{2.12}$$

and include, with proper normalization, the function $i\hat{p}\phi_c(x) \equiv \phi_c'(x)$ for which $\omega = 0$. (We designate this function ψ_0 below.) Without loss of generality we can and do choose functions ψ_n which are real.

Substituting the expansion (2.11) into (2.3), equations can be derived for the coefficients c_n, d_n, e_n by standard perturbative procedure. We find (choosing the ψ_n real)

$$\omega_n^2 c_n = -\int \psi_n(x)\hat{p}^2\phi_c(x), \tag{2.13}$$

$$\omega_n^2 d_n = -m^{-1}\int \psi_n(x)\hat{p}^3\phi_c(x) + (2\lambda/m)\int \{\psi_n(x)\hat{p}\phi_c^2(x) - 2[\hat{p}\psi_n(x)]\;\phi_c^2(x)\}\sum_{n'} c_{n'}\psi_{n'}(x), \tag{2.14}$$

$$e_n\omega_n^2 = -(2m)^{-2}\int \psi_n(x)\hat{p}^4\phi_c(x) + (2\lambda/m^2)\int \{\psi_n(x)[\hat{p}^2\phi_c^2(x)] + \phi_c(x)[\hat{p}^2\phi_c(x)\psi_n(x)]\}\sum_{n'} c_{n'}\psi_{n'}(x)$$
$$+ (2\lambda/m)\int \{\psi_n(x)[\hat{p}\phi_c(x)]\,\phi_c(x) - [\hat{p}\psi_n(x)]\;\phi_c^2(x)\}\sum_{n'} d_{n'}\psi_{n'}(x). \tag{2.15}$$

The increasing complexity of these equations results from integrations by parts, which demonstrate that the momenta r and r' which occur as variables of integration in (2.3) yield contributions both to the expansion in powers of v and in powers of (m/M). Our equations furthermore do not determine the coefficients c_0, d_0, e_0. These may be taken zero, either for reasons of symmetry

(d_0) or because nonvanishing values of c_0 and e_0 simply determine a shift in the position of the center of mass of the soliton away from the origin.

Further discussion of (2.13)–(2.15) is, for our purposes, unnecessary as we see by consideration of the energy for which we have the tentative expression (also discussed more thoroughly in Secs. III and V)

$$\langle p|\mathcal{H}(x)|p\rangle = E(p) \cong M + \frac{p^2}{2M} = \left\langle p\left|\int dx\left[\tfrac{1}{2}\pi^2(x) + \tfrac{1}{2}\left(\frac{\partial\phi(x)}{\partial x}\right)^2 + \frac{1}{2\lambda}[m^2 - \lambda\phi(x)]^2\right]\right|p\right\rangle. \tag{2.16}$$

This is evaluated in the same approximation (sum over intermediate soliton states only) as used to derive (2.3) (see Sec. IV for discussion of this connection), using unit "volume" for the system, and substituting $\langle p|\pi(0)|q\rangle = i[E(p)-E(q)]\langle p|\phi|q\rangle$. We thus obtain

$$E(p) = \tfrac{1}{2}\int dx\Phi_p(x)\{[E(\hat{p}+p) - E(p)]^2 + \hat{p}^2 - 2m^2\}\Phi_p(x)$$
$$+\tfrac{1}{2}\lambda\int \frac{dr}{d\pi} dx dy e^{i(p-r)(x-y)}\Phi_p(x)\Phi_r(x)\Phi_r(y)\Phi_p(y) + (m^4/2\lambda). \tag{2.17}$$

When we insert (2.11) into (2.17), keeping only first-order corrections to the previous static limit, we characterize the result according to powers of p. For the p-independent term we find

$$M = (4m^3/3\lambda)[1 + O((m/M)^2)]. \tag{2.18}$$

The designated order of the corrections can be seen from a typical contribution which arises from the first term of (2.17), namely,

$$\frac{1}{8M^2}\int \phi_c(x)\hat{p}^4\phi_c(x) \sim \left(\frac{m}{M}\right)^2\int \phi_c(x)\hat{p}^2\phi_c(x)$$
$$= (m/M)^2 \times M. \tag{2.19}$$

A detailed evaluation of these terms will not be presented. The terms linear in p, typically

$$\frac{1}{2M^2}\int \phi_c(x)\hat{p}^3\phi_c(x) = 0,$$

vanish by inversion symmetry, as they should.

Finally for the terms in p^2, we find with the help of (2.8) that only the first term of (2.17) contributes,

$$\frac{p^2}{2M} = \tfrac{1}{2}\left(\frac{p}{M}\right)^2\int \phi_c(x)\hat{p}^2\phi_c(x), \tag{2.20}$$

which is self-consistent to this order provided

$$M = \int \phi_c(x)\hat{p}^2\phi_c(x), \tag{2.21}$$

which is known to be so from a virial theorem,[9] generalized in Sec. V.

We have thus verified the approximate Lorentz invariance of the theory. The method used in this section is cumbersome, however, because we are trying to do two jobs at once. In the next section we demonstrate how these tasks may be separated.

III. LORENTZ-INVARIANT FORMULATION

A. One-soliton sector

The essential observation which underlies this section is that Eq. (2.3) can be solved exactly in the limit $(m/M) \to 0$, $v = [p/E(p)]$ finite. This can be seen by considering more closely the perturbation due to the originally neglected difference

$$E(\hat{p}+p) - E(p) = \frac{p\hat{p} + \frac{1}{2}\hat{p}^2}{E(p)} - \frac{1}{8}\frac{(2\hat{p}p + \hat{p}^2)^2}{E(p)^3} + \cdots. \tag{3.1}$$

The first term on the right-hand side is $v\hat{p}$. *Every other term of (3.1) contains at least one additional factor of order (m/M) when acting on* $\phi_c(x)$. With the notation

$$\Phi_p(x) \equiv \Phi(x, v), \tag{3.2}$$

the left-hand side of (2.3) becomes, in the limit under consideration,

$$\partial_x^2(1 - v^2)\Phi(x,v) + 2m^2\Phi(x,v). \tag{3.3}$$

What of the right-hand side? We show that the right-hand side becomes $2\lambda\Phi^3(x,v)$. Toward this end we write

$$\Phi_r(x) = \sum_{\nu=0}^{\infty} (r/M)^\nu \chi^{(\nu)}(x) \tag{3.4}$$

It is understood that

$$\partial_x \chi^{(\nu)}(x) \sim m\chi^{(\nu)}(x). \tag{3.5}$$

Substituting (3.4) into (2.3), we see that integration by parts effectively replaces each factor of r by $(p + i\partial_y)$, where ∂_y has the effect indicated in (3.5). Ignoring the latter we indeed reach the desired equation

$$\partial_x{}^2(1-v^2)\Phi(x,v)+2m^2\Phi(x,v)=2\lambda\Phi^3(x,v). \quad (3.6)$$

This has the solution

$$\Phi(x,v)=\phi_c(x'),\quad x'=x/(1-v^2)^{1/2}. \quad (3.7)$$

Corrections in powers of (m/M) could also be developed systematically with the help of (3.4), but are of no interest here.

It is straightforward now to demonstrate that for the limit considered, Eq. (2.17) yields the correct relativistic energy. We have, after taking the limit $(m/M)\to 0$ and making a trivial rearrangement,

$$\begin{aligned}E(p)&=\langle p\,|\mathcal{H}(x)\,|p\rangle\\&=\tfrac{1}{2}\int\Phi(x,v)(1-v^2)(-\partial_x{}^2)\Phi(x,v)dx\\&\quad+\int U(\Phi(x,v))\,dx+v^2\int\Phi(x,v)(-\partial_x{}^2)\Phi(x,v)\\&=M(1-v^2)^{1/2}+\frac{Mv^2}{(1-v^2)^{1/2}}=M/(1-v^2)^{1/2}. \quad (3.8)\end{aligned}$$

In this evaluation we make use of (3.7) and of the virial theorem[9]

$$\int U(\phi_c(x))\,dx=\tfrac{1}{2}\int\phi_c(x)(-\partial_x{}^2)\phi_c(x)=\tfrac{1}{2}M. \quad (3.9)$$

The basis for calculating corrections to (3.8), i.e., corrections to the value of M will be provided in the next subsection, but the actual calculation is postponed to Sec. V.

B. Excited states

We next consider states which are continuum one-soliton-one-meson states. As we know from previous work,[4] there is a bound excited state of the soliton, which must be considered at the same level of approximation. We first study the continuum states $|\,q,k\rangle$, which satisfy the approximate equation (k,ω referring to the meson),

$$\begin{aligned}\{-(q+k-p)^2+[E(q)+\omega(k)-E(p)]^2+2m^2\}\langle p|\phi\,|\,q;k\rangle=2\lambda\int\frac{dr}{2\pi}\frac{dr'}{2\pi}(&\langle p|\phi|r\rangle\langle r|\phi|r'\rangle\langle r'|\phi|\,q;k\rangle\\&+\langle p|\phi|r\rangle\langle r\;|\phi\,|r';k\rangle\langle r'|\phi\,|\,q\rangle\\&+\langle p|\phi|r;k\rangle\langle r|\phi|r'\rangle\langle r'|\phi|\,q\rangle).\end{aligned} \quad (3.10)$$

[We describe briefly the derivation of the right-hand side of this equation. First we write

$$\begin{aligned}\langle p|\phi^3|\,q;k\rangle\cong\sum(&\langle p|\phi|r\rangle\langle r|\phi|r'\rangle\langle r'|\phi|\,q;k\rangle+\langle p|\phi|r\rangle\langle r|\phi|r';k'\rangle\langle r';k'|\phi|q;k\rangle\\&+\langle p|\phi|r;k'\rangle\langle r;k'|\phi\,|r';k''\rangle\langle r';k''|\phi|\,q;k\rangle).\end{aligned} \quad (3.11)$$

We then assume, for example,

$$\langle r;k|\phi|r';k'\rangle=2\pi\delta(k-k')\langle r|\phi\,|\,r'\rangle+\text{smaller scattering terms.}] \quad (3.12)$$

We shall take the Fourier transform of (3.10) and in so doing find it convenient to consider two different transforms, namely,

$$\Phi(x;p,k)\equiv X(x,v_p,\omega)=\int\frac{dq}{2\pi}e^{i(q+k-p)x}\langle p|\phi|\,q;k\rangle, \quad (3.13)$$

$$\overline{\Phi}(x;q,k)\equiv\overline{X}(x;v_q,\omega)=\int\frac{dp}{2\pi}e^{-i(q+k-p)x}\langle p|\phi|\,q;k\rangle. \quad (3.14)$$

With the aid of these definitions, Eq. (3.10) can be transformed to

$$\left\{\frac{d^2}{dx^2}+[E(\hat{p}+p-k)+\omega(k)-E(p)]^2+2m^2\right\}\Phi(x;p,k)$$

$$=2\lambda\int\frac{dr}{2\pi}dy\,\{e^{i(r-p)(x-y)}\Phi_p(y)\Phi_r^*(y)\Phi(x;r,k)+e^{i(r+k-p)(x-y)}[\Phi_p(y)\overline{\Phi}(y;r,k)\Phi_r(x)+\Phi(y;p,k)\Phi_r^*(y)\Phi_r(x)]\}. \quad (3.15)$$

This equation simplifies in the limit

$$v=[p/E(p)]=O(1),\quad u=[k/\omega(k)]=O(1),\quad (m/M)\text{ and }(\omega/E)\to 0. \tag{3.16}$$

We then have

$$E(\hat{p}+p-k)+\omega-E(p)\cong \omega+[p/E(p)](\hat{p}-k) =\omega(1-uv)+\hat{p}v, \tag{3.17}$$

which will simplify the left-hand side of (3.15). To simplify the right-hand side, we introduce the expansions

$$\Phi_r(y)=\sum (r/M)^\nu \chi^{(\nu)}(y),\qquad \Phi(x;r,k)=\sum (r/M)^\nu \eta^{(\nu)}(x;k), \tag{3.18}$$

integrate by parts, and take the limit $(m/M)\to 0$. The three terms on the right-hand side of (3.15) become

$$2\lambda[\Phi_p{}^2(x)X(x;v,\omega)+\Phi_p(x)\Phi_{p-k}(x)\overline{\Phi}(x;p-k,k) +X(x;v,\omega)\Phi_{p-k}{}^2(x)]. \tag{3.19}$$

A further simplification can be achieved, however. First, we notice that

$$p-k=E(p)[v-u(\omega/E)]\cong E(p)v, \tag{3.20}$$

so that we can neglect k compared to p. Less trivially, we find from a study of the equation for $\overline{\Phi}$, which we choose not to reproduce, that we can set [*in our limit* (3.16)]

$$\overline{\Phi}(x;p,k)=\Phi(x;p,k). \tag{3.21}$$

The right-hand side (3.19) has now been reduced to

$$6\lambda\Phi^2(x,v)X(x;v,\omega). \tag{3.22}$$

A final simplification is achieved by the substitution

$$X(x;v,\omega)=\exp[i\omega v(1-uv)x/(1-v^2)]\,Y(x;v,\omega), \tag{3.23}$$

which eliminates a term linear in $\hat{p}$ from the differential operator.

With the help then of (3.17), (3.22), and (3.23), we find that (3.15) simplifies to

$$\left[-(1-v^2)\frac{d^2}{dx^2}-2m^2+6\lambda\Phi^2(x,v)\right]Y(x;v,\omega) =(\omega')^2Y(x;v,\omega), \tag{3.24}$$

where, in fact,

$$\omega'=\omega(1-uv)/(1-v^2)^{1/2} \tag{3.25}$$

is the energy of the meson relative to the soliton of momentum p.

Equation (3.24) tells us that if $Y_0(x;\omega)$ is the solution of the same equation for $v=0$, then

$$Y(x;v,\omega)=C(v,\omega)Y_0(x';\omega'), \tag{3.26}$$

where $x'=x/(1-v^2)^{1/2}$. The constant $C(v,\omega)$ will be considered below. By definition the function Y_0 is $(2\omega)^{-1/2}$ times the normalized solution ψ_n considered in Sec. II. This means that the set $|p;k\rangle$ satisfies standing-wave boundary conditions.

We conclude this section with a brief account of the corresponding development for the bound state $|p^*\rangle$. We will make note only of differences compared to the previous considerations. The analog of (3.11) is

$$\langle p|\phi^3|q^*\rangle\cong\sum(\langle p|\phi|r\rangle\langle r|\phi|r'\rangle\langle r'|\phi|q^*\rangle+\langle p|\phi|r\rangle\langle r|\phi|r'^*\rangle\langle r'^*|\phi|q^*\rangle+\langle p|\phi|r^*\rangle\langle r^*|\phi|r'^*\rangle\langle r'^*|\phi|q^*\rangle). \tag{3.27a}$$

Corresponding to (3.12) we have

$$\langle p^*|\phi|q^*\rangle\cong\langle p|\phi|q\rangle. \tag{3.27b}$$

We define

$$E^*(p)=[p^2+(M+\omega_B)^2]^{1/2}. \tag{3.28}$$

Thus

$$E^*(\hat{p}+p)-E(p)\cong\hat{p}v+\omega_B(1-v^2)^{1/2} \tag{3.29}$$

The analogs of (3.13) and (3.14) are called $\Psi_p(x)=X(x;v_p,\omega_B)$ and $\overline{\Psi}_q(x)=\overline{X}(x;v_q,\omega_B)$ and may be taken equal in the limit (3.16) when $v_p=v_q$.

Corresponding to (3.15), we then find

$$\left\{\frac{d^2}{dx^2}+[E^*(\hat{p}+p)-E(p)]^2+2m^2\right\}X(x;v,\omega_B) =2\lambda\int\frac{dr}{2\pi}\,dy\,e^{i(r-p)(x-y)}[\Phi_p(y)\Phi_r^*(y)\Psi_r(x)+\Phi_p(y)\overline{\Psi}_r(y)\Phi_r(x)+\Psi_p(y)\Phi_r^*(y)\Phi_r(x)]. \tag{3.30}$$

Proceeding as before and with the substitution

$$\Psi_p(x) = X(x; v, \omega_B) = \exp[i\omega v x/(1-v^2)^{1/2}]\, Y(x; v, \omega_B), \tag{3.31}$$

We find the equation

$$\left[-(1-v^2)\frac{d^2}{dx^2} - 2m^2 + 6\lambda\Phi^2(x,v) \right] Y(x; v, \omega_B) = \omega_B{}^2 Y(x; v, \omega_B). \tag{3.32}$$

Again if $Y_0(x;\omega_B)$ is the solution of (3.32) for $v=0$, then

$$Y(x; v, \omega_B) = C_B(v, \omega_B) Y_0(x', \omega_B), \tag{3.33}$$

and the constant C_B is to be determined under the condition $(2\omega_B)^{1/2}Y_0$ is normalized.

We shall utilize these solutions in the remaining sections, but we must complete our understanding of them by a consideration of questions of normalization, completeness, and their relation to the commutation relations.

C. Commutation relations

We study the commutation relation

$$\left\langle p \left| \left[\phi(x,t), \frac{\partial\phi(y,t)}{\partial t} \right] \right| q \right\rangle \Bigg|_{t=0} = i\langle p | [\phi(x,0), [H, \phi(y,0)]] | q\rangle = i\langle p|q\rangle \delta(x-y). \tag{3.34}$$

First, consider the contribution from soliton intermediate states, namely,

$$i\int \frac{dr}{2\pi} \{\langle p|\phi(x)|r\rangle\langle r|\phi(y)|q\rangle[E(r)-E(q)] + \langle p|\phi(y)|r\rangle\langle r|\phi(x)|q\rangle[E(r)-E(p)]\}. \tag{3.35}$$

Introducing (2.2) in the form

$$\langle p|\phi(x)|r\rangle = \int dx_0 e^{i(r-p)x_0}\Phi_p(x-x_0), \tag{3.36}$$

(3.35) becomes

$$i\int dx_0\, \{e^{i(q-p)x_0}\Phi_p(x-x_0)[E(q-\hat{p}_y) - E(q)]\Phi_q(y-x_0) + e^{+i(q-p)x_0}\Phi_q(x-x_0)[E(p+\hat{p}_y) - E(p)]\Phi_p(y-x_0)\}. \tag{3.37}$$

The simplest way to extract the information we are after is to take the limit $q\to p$ and to utilize the expansion (3.1) The leading contributions are quadratic in p. We thereby find

$$i\int dx_0 \frac{(1-v^2)^{3/2}}{M}\, \partial_x \Phi_p(x-x_0)\partial_y\Phi_p(y-y_0). \tag{3.38}$$

The contributions from the excited intermediate states studied in the previous subsection can be obtained following the same outline. In terms of the solutions given in (3.26) and (3.33), the contributions analogous to (3.38) are

$$i\int dx_0 \left\{ \int \frac{dk}{2\pi}\frac{2\omega(1-uv)}{1-v^2}\, Y(x-x_0, v, \omega)Y(y-x_0, v, \omega) + \frac{2\omega_B}{(1-v^2)^{1/2}} Y(x-x_0, v, \omega_B)Y(y-y_0, v, \omega_B) \right\}. \tag{3.39}$$

The sum of (3.38) and (3.39) is to be equated to the right-hand side of (3.34) in the limit $q\to p$. Here we must remember that we are not using an invariant normalization. Consequently, we must write

$$\lim_{q\to p}\langle p|q\rangle = \langle p|p\rangle = (1-v^2)^{1/2}\langle 0|0\rangle = (1-v^2)^{1/2}L, \tag{3.40}$$

where L is the "volume" of the system at rest. The factor in (3.40) is understood from the circumstance that $|p\rangle$ transforms as $[E(p)]^{-1/2}$. Let us further note that

$$\langle p|p\rangle = \lim_{q\to p} 2\pi\delta(p-q) = \lim \int dx_0 e^{ix_0(p-q)}. \tag{3.41}$$

Comparison of (3.40) and (3.41) implies that we must interpret $\int dx_0$ as $L(1-v^2)^{1/2}$ when we integrate over a surface at constant time in the frame in which the soliton has momentum p.

The two sides of (3.34) can be consistent only if the integrands of (3.38) and (3.39) sum to a δ function,

$$\delta(x-y)=\frac{(1-v^2)^{3/2}}{M}\partial_x\Phi(x,v)\partial_y\Phi(y,v)+\int\frac{dk}{2\pi}\frac{2\omega(1-uv)}{1-v^2}Y(x;v,\omega)Y(y;v,\omega)+\frac{2\omega_B}{(1-v^2)^{1/2}}Y_B(x;v,\omega_B)Y_B(y;v,\omega_B)\,. \tag{3.42}$$

This is to be compared with the known completeness relation for $v=0$, namely,

$$\delta(x-y)=\frac{1}{M}\partial_x\phi_c(x)\partial_y\phi_c(y)+\int\frac{dk}{2\pi}2\omega Y_0(x,\omega)Y_0(y,\omega)+2\omega_B Y_{B0}(x,\omega_B)Y_{B0}(y,\omega_B)\,. \tag{3.43}$$

We shall show that these are the same relation provided we make suitable choices of the normalization constants in (3.26) and (3.33).

Starting from (3.42), we substitute $x=x'(1-v^2)^{1/2}$ and utilize (3.7), (3.26), and (3.33). We find easily, taking care about what is held constant

$$\delta(x'-y')=\frac{1-v^2}{M}\partial_{x'}|_t\phi_c(x')\partial_{y'}|_t\phi_c(y')+\int\frac{dk}{2\pi}2\omega' Y_0(x',\omega')Y_0(y',\omega')C^2(v,\omega)+2\omega_B Y_{B0}(x',\omega_B)Y_{B0}(y',\omega_B)C_B^2(v,\omega)\,. \tag{3.44}$$

It is an elementary exercise in change of variable to show that

$$\begin{aligned}\partial_{x'}|_{t'}\phi_c(x')&=\partial_x|_t\phi_c(x')\\&=(1-v^2)^{1/2}\partial_{x'}|_t\phi_c(x'),\end{aligned} \tag{3.45}$$

where $(t'=0)$

$$x'=\frac{x-vt}{(1-v^2)^{1/2}},\quad x=\frac{x'}{(1-v^2)^{1/2}},\qquad t=\frac{vx'}{(1-v^2)^{1/2}}\,. \tag{3.46}$$

We also note that since $dk\omega'=dk'\omega$ (dk/ω is invariant) we have proved the equivalence of (3.42) and (3.43) provided

$$C_B^2=1,\qquad C^2(v,\omega)=(\omega'/\omega). \tag{3.47}$$

An alternative derivation of (3.42) or (3.44) making use directly of Lorentz transformation of the field operators is left as an exercise to the reader.

IV. VARIATIONAL PRINCIPLE FOR THE EQUATION OF MOTION

The general equations of motion of our method, of which Eq. (2.3) is a special case are derivable from a variational principle which will prove useful in the sequel. Writing a general state as $|n\rangle$, which includes the momentum p_n, we compute by sum-rule methods the *trace* of H over all included states and obtain up to an additive constant

$$\mathrm{Tr}H=\sum_n\langle n|H|n\rangle=\sum_{nn'}\tfrac{1}{2}\{|\langle n|\pi|n'\rangle|^2+[(p_n-p_{n'})^2-2m^2]|\langle n|\phi|n'\rangle|^2\}+\tfrac{1}{2}\lambda\sum_{n,n',n'',n'''}\langle n|\phi|n'\rangle\langle n'|\phi|n''\rangle\langle n''|\phi|n'''\rangle\langle n'''|\phi|n\rangle\,. \tag{4.1}$$

We require that this expression be stationary with respect to variations of $\langle n|\pi|n'\rangle$ and $\langle n|\phi|n'\rangle$ subject to constraints imposed by the commutation relations

$$\delta\sum_{n''}[\langle n|\phi|n''\rangle\langle n''|\pi|n'\rangle-\langle n|\pi|n''\rangle\langle n''|\phi|n'\rangle]=0. \tag{4.2}$$

To exploit (4.2), we multiply by a Lagrange multiplier $i\langle n'|h|n\rangle$, sum over n and n', and subtract from the trace of (4.1) to form a master variational principle

$$\delta\,\mathrm{Tr}\{H-i\pi[h,\phi]\}=\delta\,\mathrm{Tr}\{H-i\phi[\pi,h]\}=0. \tag{4.3}$$

Here the Hermiticity of H implies that h is also an Hermitian matrix. From (4.1) and (4.3), we derive the "equations of motion"

$$\delta(\mathrm{Tr}H)/\delta(\langle n'|\pi|n\rangle) = i\langle n|[h,\phi]|n'\rangle = \langle n|\pi|n'\rangle, \tag{4.4}$$

$$\delta(\mathrm{Tr}H)/\delta\langle n'|\phi|n\rangle = i\langle n|[\pi,h]|n'\rangle = [(p_n - p_{n'})^2 - 2m^2]\langle n|\phi|n'\rangle + 2\lambda \sum_{n'',n'''} \langle n|\phi|n''\rangle\langle n''|\phi|n'''\rangle\langle n'''|\phi|n'\rangle. \tag{4.5}$$

These are the matrix elements of the Heisenberg equations of motion provided $h = H$. The equation (2.3) within the one-soliton sector is then a special case of (4.4) and (4.5).

It is furthermore amusing to observe that consequent upon the identification $h = H$, (4.3) can be interpreted as a variational principle for the trace of the quantum action in which $\partial_t \phi$ has been replaced by $(-i)[\phi, H]$. This observation renders our result somewhat less surprising.

The variational origin of the equations of motion guarantees that a consistent solution of these equations and of the commutation relations within a given subspace renders the Hamiltonian diagonal within the same subspace. The statements

$$\langle n|H|n'\rangle = 0, \quad n \neq n', \tag{4.6}$$

evaluated by sum-rule methods, should provide the general consistency conditions conjectured in Ref. 9. This point will be explored briefly in Sec. V.

We next apply the variational principle in detail. We first consider the one-soliton subspace in the limit $m/M \to 0$, v finite. In this subspace the constrained variational principle (4.3) can be cast into a form familiar as the "cranking" model in many-body theory.[20] Evaluation of the first term of (4.3), using the methods of the previous section, yields

$$\mathrm{Tr}\mathcal{H} = \sum_v \tfrac{1}{2} \int \Phi(x,v)(1-v^2)(-\partial_x{}^2)\Phi(x,v)dx + \int U(\Phi(x,v))\,dx + v^2 \int \Phi(x,v)(-\partial_x{}^2)\Phi(x,v). \tag{4.7}$$

The second term, similarly evaluated becomes

$$-\sum_v v^2 \int \Phi(x,v)(-\partial_x{}^2)\Phi(x,v) = -\mathrm{Tr}\, v\hat{\mathcal{P}}, \tag{4.8}$$

where $\hat{\mathcal{P}}$ is the momentum density operator

$$\hat{P} = -\tfrac{1}{2}\{\pi(x), \partial_x\phi(x)\}_+. \tag{4.9}$$

The equality (4.8) is again confined to the limit under consideration as the reader will verify.

Adding (4.7) and the term (4.8) arising from the constraints, the variation can be performed and leads to Eq. (3.6).

Several conclusions can be drawn from (4.7)–(4.9):

(i) The summation over v may now be discarded and we have as a result the "cranking" variational principle

$$\delta\left(\langle\hat{\mathcal{H}}\rangle - v^2 \int \Phi(x,v)\hat{p}^2\Phi(x,v)\right) = \delta\langle\hat{\mathcal{H}} - v\hat{\mathcal{P}}\rangle = 0. \tag{4.10}$$

This simple "cranking" form of the variational principle applies only to the translation of the soliton and does not extend to the higher sectors where excited states are included.

(ii) The implication that there is indeed a trial state for which (4.10) is the functional can be read off from the considerations found in Refs. 6–8.

(iii) By scaling arguments a virial theorem can be derived from the variational principle as has been shown by Goldstone and Jackiw for the soliton sector. Beyond the one-soliton sector a more general virial theorem can be obtained by the same method if all matrix elements including those which arise from the bound and continuum states are rescaled and then varied simultaneously. In Sec. V we shall derive this virial theorem in a different way by using the equations of motion.

To carry these considerations further to the sectors of Hilbert space, which include the excited states, we first examine the contribution to the variational principle of the bound excited state. To consider the change in soliton energy due to the inclusion of this state, we in turn calculate the contribution to the various terms in $\mathcal{H}$. Using the reasoning and notation of Sec. III, the term $\frac{1}{2}\pi^2$ changes by

$$\tfrac{1}{2}\langle p|\pi^2|p\rangle_{\text{bound}} = \tfrac{1}{2}\sum_q [E^*(q) - E(p)]^2 |\langle p|\phi|q^*\rangle|^2 = \tfrac{1}{2}\int dx X^*(x,v,\omega_B)[v\hat{p} + \omega(1-v^2)^{1/2}]^2 X(x,v,\omega_B). \tag{4.11}$$

Substituting Eq. (3.31) and remembering that Y is now real and has a definite parity, we obtain

$$\tfrac{1}{2}\langle p|\pi^2|p\rangle_{\text{bound}} = \tfrac{1}{2}\int dx\, Y_B(x;v,\omega_B)\left(v^2\hat{p}^2 + \frac{\omega^2}{1-v^2}\right) Y_B(x;v,\omega_B). \tag{4.12}$$

The remaining terms in $\mathcal{H}$ are similarly evaluated to yield

$$\tfrac{1}{2}\langle p|\,(\partial_x\phi)^2|p\rangle_{\text{bound}}=\tfrac{1}{2}\int dx\,Y_B(x;v,\omega_B)\left(\hat{p}^2+\frac{\omega_B^{\,2}v^2}{1-v^2}\right)Y_B(x;v,\omega_B), \tag{4.13}$$

and

$$\langle p|U(\phi)|p\rangle_{\text{bound}}=-m^2\int dx\,Y_B^2(x;v,\omega_B)+3\lambda\int dx\,Y_B^2(x;v,\omega_B)\Phi^2(x,v). \tag{4.14}$$

From Eqs. (4.12–(4.14) we can obtain the contribution to the soliton energy as well as to Tr$\mathfrak{L}$, i.e., to the variational principle, in the space of the soliton and the bound excited state. The latter is found by first summing each term over p and then subtracting twice the sum of (4.13) and (4.14) from twice (4.12). This gives

$$(\mathrm{Tr}\mathfrak{L})_{\text{bound}}=-\sum_v\int dx\,Y_B(x;v,\omega_B)[(1-v^2)\hat{p}^2-2m^2+6\lambda\Phi^2(x,v)]\,Y_B(x;v,\omega_B)+\int dx\,Y_B(x;v,\omega_B)\omega_B^{\,2}Y_B(x;v,\omega_B). \tag{4.15}$$

(The reason for the factor of 2 is that there is an equal contribution from terms where the trace is over the momenta of the bound excited state and the soliton is the intermediate state.)

As remarked above, the constrained variational principle (4.3) is equivalent to an unconstrained variational principle of the trace of the quantum action. Indeed if (4.15) is varied with respect to Y_B it gives the correct equation of motion for Y_B, Eq. (3.32).

By summing (4.12), (4.13), and (4.14) the bound-state contribution to the soliton energy is found to be

$$\langle p|\mathcal{H}|p\rangle_{\text{bound}}=\tfrac{1}{2}\int dx\,Y_B(x;v,\omega_B)\left((1+v^2)\hat{p}^2-2m^2+6\lambda\Phi^2(x,v)+\omega_B^{\,2}\,\frac{1+v^2}{1-v^2}\right)Y_B(x;v,\omega_B). \tag{4.16}$$

Because Y_B is a solution of Eq. (3.32), the expression (4.15) vanishes for each v separately. This can be used to rewrite the energy contribution in a simpler form

$$\langle p|\mathcal{H}|p\rangle_{\text{bound}}=\langle p|\,(\pi^2-\mathfrak{L})|p\rangle_{\text{bound}}=\langle p|\pi^2|p\rangle_{\text{bound}}=\int dx\,Y_B(x,v,\omega_B)\left(v^2\hat{p}^2+\frac{\omega_B^{\,2}}{1-v^2}\right)Y_B(x;v,\omega_B). \tag{4.17}$$

At the same level of approximation we must also include the contribution from those continuum states which consist of a soliton and a meson. Their contributions can be computed in a manner completely analogous to the bound-state contribution. We shall only quote the results here. With the notation of Sec. III, we find for the individual terms which make up the Hamiltonian density

$$\tfrac{1}{2}\langle p|\pi^2|p\rangle_{\text{cont}}=\tfrac{1}{2}\sum_k\int dx\,Y(x;v,\omega)\left(v^2\hat{p}^2+\frac{\omega^2(1-uv)^2}{(1-v^2)^2}\right)Y(x;v,\omega), \tag{4.18}$$

$$\tfrac{1}{2}\langle p|\,(\partial_x\phi)^2|p\rangle_{\text{cont}}=\tfrac{1}{2}\sum_k\int dx\,Y(x;v,\omega)\left(\hat{p}^2+\frac{v^2\omega^2(1-uv)^2}{(1-v^2)^2}\right)Y(x;v,\omega), \tag{4.19}$$

$$\langle p|U(\phi)|p\rangle_{\text{cont}}=-m^2\sum_k\int dx\,Y^2(x;v,\omega)+3\lambda\sum_k\int dx\,\Phi^2(x,v)Y^2(x;v,\omega). \tag{4.20}$$

The variational expression can be formed as before including a factor of 2 and we obtain

$$\sum_v\sum_k\int dx\,Y(x;v,\omega)\left(-(1-v^2)\hat{p}^2+2m^2-6\lambda\Phi^2(x,v)+\frac{\omega^2(1-uv)^2}{1-v^2}\right)Y(x;v,\omega). \tag{4.21}$$

Variation of (4.21) gives again the correct equation for $Y(x;v,\omega)$, (3.24). The contribution of the soliton-meson states to the soliton energy is then found by adding (4.18)–(4.20)

$$\begin{aligned}\langle p|\mathcal{H}|p\rangle_{\text{cont}}&=\tfrac{1}{2}\sum_k\int dx\,Y(x;v,\omega)\left((1+v^2)\hat{p}^2-2m^2+6\lambda\Phi^2(x,v)+\omega^2\,\frac{(1+v^2)(1-uv)^2}{(1-v^2)^2}\right)Y(x;v,\omega)\\&=\sum_k\int dx\,Y(x;v,\omega)\left(v^2\hat{p}^2+\omega^2\,\frac{(1-uv)^2}{(1-v^2)^2}\right)Y(x;v,\omega),\end{aligned} \tag{4.22}$$

where in the last equality we again used the fact that for a solution of Eq. (3.24), (4.23) vanishes identically and we have

$$\langle p|\mathcal{H}|p\rangle_{\rm cont} = \langle p|\pi^2 - \mathcal{L}|p\rangle_{\rm cont} = \langle p|\pi^2|p\rangle_{\rm cont}\ . \tag{4.23}$$

Let us notice finally that because (4.15) and (4.21) depend on $\Phi(x,v)$ to this order Eq. (3.6) is generalized to

$$\{(1-v^2)\partial_x{}^2 + 2m^2 - 2\lambda\Phi^2(x,v) - 6\lambda[Y_B^2(x;v,\omega_B) + \int (dk/2\pi)Y^2(x;v,\omega)]\}\Phi(x,v) = 0. \tag{4.24}$$

The terms in square brackets in (4.24) are the "one-loop" contributions and must be renormalized.[4]

V. FURTHER COMMENTS

In this section we shall round out the material already presented by some additional remarks. In part A we derive a virial theorem and in part B we establish the proper Lorentz transformation properties of the soliton. Finally in part C we comment on consistency relations guaranteed by the variational principle.

A. Virial theorem

For the soliton sector a virial theorem was derived by scaling arguments in Ref. 9. By starting from the equations of motion we shall now extend this to the space which includes, besides the soliton also the excited bound and one-soliton–one-meson states. Multiplying the equations of motion (4.24), (3.24), and (3.32) by $\partial_x\phi(x,v)$, $\partial_x Y_B(x;v,\omega_B)$, and $\partial_x Y(x;v,\omega)$, respectively, they can be written as

$$(1-v^2)[\partial_x\Phi(x,v)][\partial_x^2\Phi(x,v)] = \frac{\partial U(\phi, Y_B, Y)}{\partial\Phi(x,v)}[\partial_x\Phi(x,v)], \tag{5.1}$$

$$(1-v^2)[\partial_x Y_B(x,v,\omega_B)][(\partial_x{}^2+\omega_B^2)Y_B(x,v,\omega_B)] = \frac{\partial U(\phi,Y_B,Y)}{\partial Y_B(x,v,\omega_B)}[\partial_x Y_B(x,v,\omega_B)], \tag{5.2}$$

$$(1-v^2)[\partial_x Y(x,v,\omega)]\left[\left(\partial_x{}^2 + \frac{\omega^2(1-uv)^2}{1-v^2}\right)Y(x,v,\omega)\right] = \frac{\partial U(\phi,Y_B,Y)}{\partial Y(x,v,\omega)}[\partial_x Y(x,v,\omega)]. \tag{5.3}$$

If (5.3) is integrated over meson momenta and if these equations are then summed, a first integral can be obtained immediately as the right-hand side is just dU/dx. The result is

$$\tfrac{1}{2}\left[(1-v^2)\{[\partial_x\phi(x,v)]^2 + [\partial_x Y_B(x,v,\omega_B)]^2\} + \omega_B^2 Y_B^2(x,v,\omega_B) + \int\frac{dk}{2\pi}\left((1-v^2)[\partial_x Y(x,v,\omega)]^2 + \frac{\omega^2(1-uv)^2}{1-v^2}Y^2(x,v,\omega)\right)\right] = U(\phi,Y_B,Y). \tag{5.4}$$

Taking this equation in the rest frame, integrating over all space and performing an integration by parts in the derivative terms, we arrive at a virial theorem

$$\tfrac{1}{2}\int dx\left[\phi_c(x)(-\partial_x{}^2)\phi_c(x) + Y_{0B}(x,\omega_B)(-\partial_x{}^2+\omega_B^2)Y_{0B}(x,\omega_B) + \int(dk/2\pi)Y_0(x,\omega)(-\partial_x{}^2+\omega^2)Y_0(x,\omega)\right] = \int dx\, U(\phi,Y_B,Y). \tag{5.5}$$

From the above derivation it is clear that this virial theorem can be readily extended if higher excited states are included.

B. Lorentz transformation properties

If the soliton is to describe an extended particle we have to insist that its energy and momentum transform like the components of a vector. That this is indeed the case we shall now show.

The mass of the soliton is defined as the expectation value of the Hamiltonian density in the rest frame. If excited bound as well as one-soliton–one-meson states are included, it is no longer given by (2.21), but rather by

$$M = \langle 0|\mathcal{H}(x,t)|0\rangle$$
$$= \int dx\left\{\phi_c(x)(-\partial_x{}^2)\phi_c(x) + Y_{0B}(x,\omega_B)(-\partial_x{}^2+\omega_B{}^2)Y_{0B}(x,\omega_B) + \int(dk/2\pi)Y_0(x,\omega)[(-\partial_x{}^2)+\omega^2]Y_0(x,\omega)\right\}, \tag{5.6}$$

where again the volume has been taken to be unity and the virial theorem (5.5) has been used to eliminate the integral over the potential.

Let us then compute the energy of a soliton moving with momentum p. Using (3.8), (4.16), and (4.22) and [by means of (3.26), (3.33) and (3.47)] expressing all quantities in the rest frame we find

$$E=\langle p|\mathcal{H}(x,t)|p\rangle$$
$$=\tfrac{1}{2}(1-v^2)^{1/2}\int dx\Big[\phi_c(x)(-\partial_x^2)\phi_c(x)+Y_{0B}(x,\omega_B)(-\partial_x^2+\omega_B^2)Y_{0B}(x,\omega_B)+\int(dk/2\pi)Y_0(x,\omega)(-\partial_x^2+\omega^2)Y_0(x,\omega)\Big]$$
$$+(1-v^2)^{1/2}\int U(\phi_c,Y_{0B},Y_{0C})+\frac{v^2}{(1-v^2)^{1/2}}\int dx\Big[\phi_c(x)(-\partial_x^2)\phi_c(x)+Y_{0B}(x,\omega_B)(-\partial_x^2+\omega_B^2)Y_{0B}(x,\omega_B)$$
$$+\int(dk/2\pi)Y_0(x,\omega)(-\partial_x^2+\omega^2)Y_0(x,\omega)\Big]$$
$$=(1-v^2)^{1/2}M+\frac{v^2}{(1-v^2)^{1/2}}M=\frac{M}{(1-v^2)^{1/2}},\tag{5.7}$$

where the virial theorem (5.5) and (5.6) have been used. The soliton momentum can be evaluated in the same way and using (4.9) we find

$$P=-\tfrac{1}{2}\langle p|\{\pi(x,t),\partial_x\phi(x,t)\}_+|p\rangle$$
$$=\frac{Mv}{(1-v^2)^{1/2}}.\tag{5.8}$$

Energy and momentum of the soliton therefore do transform as appropriate for a particle.

Desirable as this result is, it has to be reconciled with the fact that $\mathcal{H}(x,t)$ and the momentum density $\mathcal{P}$ are elements of the energy-momentum tensor, and thus transform as a tensor of second rank. To understand why there is no inconsistency, let us calculate the energy (5.7) and the momentum (5.8) in a different way. We do this first for the energy density by evaluating it in the rest frame and then boosting to a frame where the soliton moves with momentum p.

For the scalar field the symmetric energy-momentum tensor has the form

$$\mathcal{H}(x,t)=T_{00}(x,t)$$
$$=\tfrac{1}{2}\pi^2(x,t)+\tfrac{1}{2}[\partial_x\phi(x,t)]^2+U(\phi),\tag{5.9}$$
$$T_{01}(x,t)=T_{10}(x,t)$$
$$=\tfrac{1}{2}\{\pi(x,t),\partial_x\phi(x,t)\}_+,\tag{5.10}$$
$$T_{11}(x,t)=\tfrac{1}{2}\pi^2(x,t)+\tfrac{1}{2}[\partial_x\phi(x,t)]^2-U(\phi).\tag{5.11}$$

Let U_p be the unitary operator which mediates the Lorentz transformation from the rest frame to the frame where the soliton has momentum p. With the normalization explained below (3.41), we find

$$\langle p|\mathcal{H}(x,t)|p\rangle=(1-v^2)^{1/2}\langle 0|U_p^\dagger T_{00}(x,t)U_p|0\rangle,\tag{5.12}$$

where

$$U_p^\dagger T_{00}(x,t)U_p=\frac{1}{1-v^2}[T_{00}(x',t')-2vT_{10}(x',t')+v^2T_{11}(x',t')].\tag{5.13}$$

Similarly we find for the momentum density

$$\langle p|\mathcal{P}(x,t)|p\rangle=-(1-v^2)^{1/2}\langle 0|U_p^\dagger T_{01}(x,t)U_p|0\rangle,\tag{5.14}$$

with

$$U_p^\dagger T_{01}(x,t)U_p=\frac{1}{1-v^2}[-vT_{00}(x',t')+(1+v^2)T_{01}(x',t')-vT_{11}(x',t')].\tag{5.15}$$

It can readily be verified that $\langle 0|T_{01}(x,t)|0\rangle$ vanishes and since $\langle 0|T_{00}(x,t)|0\rangle=M$ it is necessary and sufficient for the energy and momentum density (5.12) and (5.14) to transform like a vector [cf. (5.7) and (5.8)] that $\langle 0|T_{11}(x,t)|0\rangle$ vanish. This is indeed the case by virtue of the virial theorem (5.5) as we now show. From (5.11) we find with the help of (2.17), (4.12), (4.13), (4.18), and (4.19) and using the definition (5.6) for M

$$\langle p|T_{11}(y,t)|p\rangle$$
$$=(1-v^2)^{1/2}\Big[\tfrac{1}{2}M-\int dx\,U(\phi_c,Y_{0B},Y_0)\Big]+\frac{v^2M}{(1-v^2)^{1/2}},\tag{5.16}$$

where again all quantities have been expressed in the rest frame. The first term vanishes by virtue of the virial theorem (5.5) and so does the second term in the rest frame where $v=0$. Thus

$$\langle 0|T_{11}(x,t)|0\rangle=0,\tag{5.17}$$

and we recover (5.7) and (5.8) from (5.12) and (5.14). Thus (5.17), which is a consequence of the virial theorem (5.5), guarantees that the expectation value of the energy and momentum density does indeed transform like a vector as is required by the interpretation of the soliton as a particle.

C. Consistency conditions

The equations of motion (2.1), (3.24), and (3.32) for the various matrix elements of the field operator investigated in this work can be derived by taking matrix elements of the Heisenberg equations of motion for these operators as has been shown in Ref. 9. In Sec. IV we pointed out that these same equations can be derived from a trace variational principle, which also guaranteed that the solutions of these equations would render the Hamiltonian diagonal in the space considered. This gives a set of additional conditions satisfied by the solutions of the equations of motion, namely,

$$\langle n|H|n'\rangle = 0, \quad n' \neq n. \tag{5.18}$$

That our solutions do indeed satisfy these consistency conditions we shall now briefly illustrate for $\langle p|\mathcal{H}(x,t)|p^*\rangle$.

To that end we evaluate the various terms of the Hamiltonian and find

$$\tfrac{1}{2}\langle p|\pi^2(x,t)|p\rangle = \tfrac{1}{2}\sum_q [\langle p|\pi(x,t)|q\rangle\langle q|\pi(x,t)|p^*\rangle + \langle p|\pi(x,t)|q^*\rangle\langle q^*|\pi(x,t)|p^*\rangle]. \tag{5.19}$$

Observing that $|p^*\rangle$ differs from $|p\rangle$ only by surface oscillations we replace in the second term

$$\langle q^*|\pi(x,t)|p^*\rangle \simeq \langle q|\pi(x,t)|p\rangle. \tag{5.20}$$

The expression (5.19) can now be computed by the methods shown in Sec. III and the two terms are found to be equal. In the notation of that section we find

$$\tfrac{1}{2}\langle p|\pi^2(x,t)|p\rangle = \int dy X(y;v,\omega_B)(v\hat{p})[v\hat{p} - \omega(1-v^2)^{1/2}]\Phi(y,v). \tag{5.21}$$

The rest of the Hamiltonian is evaluated in the same way, familiar by now, and we obtain

$$\langle p|\mathcal{H}(x,t)|p^*\rangle = \int dy X(y,v,\omega_B)[(1+v^2)\hat{p}^2 - v\omega(1-v^2)^{1/2}\hat{p} - 2m^2 + 2\lambda\Phi^2(x,v)]\Phi(y,v). \tag{5.22}$$

With the help of (3.6) for $\Phi(x,v)$ this can be rewritten as

$$\langle p|\mathcal{H}(x,t)|p^*\rangle = \int dy X(y;v,\omega_B)v[2v\hat{p} - \omega(1-v^2)^{1/2}]\hat{p}\Phi(y,v). \tag{5.23}$$

But this vanishes as it is precisely the orthogonality relations which follows from the equation of motion (3.30) for X and the one for $(\hat{p}\phi)$ which is just (3.32) with $\omega_B = 0$. In an analogous way it can be shown that the Hamiltonian is diagonal between the soliton and the soliton-plus-one-meson states.

In conclusion, we believe that we have illustrated amply for the example chosen how to build a quantum field theory directly from the field equations and the commutation relations. In further work, we shall show how to extend our concepts to other models and to more complicated parts of Hilbert space.

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[1]G. B. Witham, *Linear and Non-Linear Waves* (Wiley, New York, 1974).
[2]A. Scott, F. Chu, and D. McLaughlin, Proc. IEEE 61, 1443 (1973).
[3]L. D. Faddeev and L. A. Takhtajan, Theor. Math. Phys. 21, 1046 (1975).
[4]R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4114 (1974); 11, 3424 (1975); 12, 2443 (1975).
[5]R. Rajaraman and E. J. Weinberg, Phys. Rev. D 11, 2950 (1975).
[5a]J. M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D 10, 2428 (1974).
[6]P. Vinciarelli, Lett. Nuovo Cimento 4, 905 (1972); CERN Report No. Th. 1993, 1975 (unpublished).
[7]W. A. Bardeen, M. S. Chanowitz, S. D. Drell, M. Weinstein, and T.-M. Yan, Phys. Rev. D 11, 1094 (1975).
[8]K. Cahill, Phys. Lett. 53B, 174 (1974); 56B, 275 (1975).
[9]J. Goldstone and R. Jackiw, Phys. Rev. D 11, 1486 (1975).
[10]R. J. Rivers, Imperial College Report No. ICTP/74/5 (unpublished).
[11]J. L. Gervais and B. Sakita, Phys. Rev. D 11, 2943 (1975); J. L. Gervais, A. Jevicki, and B. Sakita, *ibid.* 12, 1038 (1975).
[12]E. Tomboulis, Phys. Rev. D 12, 1678 (1975); E. Tomboulis and G. Woo, MIT Report No. 489 (unpublished).
[13]N. H. Christ and T. D. Lee, Phys. Rev. D 12, 1606 (1975).
[14]M. Creutz, following paper, Phys. Rev. D 12, 3126 (1975).

[15]T. D. Lee, Columbia Report No. CO-2271-60 (unpublished); N. H. Christ, Columbia Report No. CO-2271-62 (unpublished).

[16]K. Johnson, Phys. Rev. D 6, 1101 (1972); C. M. Bender, J. E. Mandula, and G. S. Guralnik, *ibid.* 11, 409 (1975); 11, 438 (1975).

[17]A. Klein and F. R. Krejs, Univ. of Pennsylvania Report No. UPR-0039T (unpublished).

[18]S. J. Chang, Phys. Rev. D 12, 1071 (1975); S. J. Chang and J. A. Wright, *ibid.* 12, 1595 (1975).

[19]A. Kerman and A. Klein, Phys. Rev. 132, 1326 (1963); A. Klein *Boulder Lectures in Theoretical Physics* (Gordon and Breach, New York, 1969), Vol. XIB, pp. 1–55; A. Klein, Rev. Mex. Fis. 23, 59 (1974).

[20]R. M. Dreizler and A. Klein, Phys. Rev. Lett. 15, 893 (1965); G. Do Dang, G. Dreiss, R. M. Dreizler, A. Klein, and C. S. Wu, Nucl. Phys. A114, 501 (1968).

[21]C. T. Li, A. Klein, and F. Krejs, Phys. Rev. D 12, 2311 (1975).