Magnetic charge and the charge quantization condition*

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Two viewpoints concerning magnetic charge are distinguished: that of Dirac, which is unsymmetrical, and the symmetrical one, which embodies invariance under charge rotation. It is pointed out that the latter is not in conflict with the empirical asymmetry between electric and magnetic charge. The discussion is based on an action principle that uses field strengths and the vector potential A as independent variables; a second vector potential B is defined nonlocally in terms of the field strengths. This nonlocality is described by an arbitrary vector function $f^\nu(y)$, subject only to the restriction $\partial_\nu f^\nu(y) = \delta(y)$ and the additional requirement of oddness, in the symmetrical formulation. The charge quantization conditions for a pair of idealized charges, a and b, are inferred by examining the dependence of the action W on the choice of the arbitrary mathematical function f, and requiring the uniqueness of $\exp[iW]$. For the unsymmetrical viewpoint the half-integer condition of Dirac is obtained, $e_a g_b/4\pi = \frac{1}{2}n$, while the symmetrical formulation requires the integer condition $(e_a g_b - e_b g_a)/4\pi = n$. The Dirac injunction, "a string must never pass through a charged particle," is criticized as unnecessarily restrictive, owing to its origin in a classical action context. As simplified by a restriction to small momentum transfers, permitting the neglect of form-factor and vacuum-polarization effects, the dynamics of a realistic system of two spin-$\frac{1}{2}$ dyons is shown to involve the same interaction structure used in the idealized discussion.

INTRODUCTION

Interest in magnetic charge has revived recently,[1] in contexts that are characterized by such terms as non-Abelian gauge fields, broken symmetry, color, ... It is not my intention to comment on this class of speculations. I wish only to review and attempt to clarify earlier remarks[2] on the troublesome charge quantization condition since they are sometimes misunderstood and because my own views have appreciably altered.

The original work of Dirac[3] pertained to particles with electric charge (e) in the presence of magnetic charge (g) and inferred the charge quantization condition ($\hbar = c = 1$, rational units)

$$eg/4\pi = \tfrac{1}{2}n, \quad n = 0, \pm1, \pm2, \ldots . \tag{1}$$

This statement incorporates the usual concept of an absolute distinction, not subject to continuous variation, between electric and magnetic charge. However, the general form of the Maxwell equations,

$$\begin{aligned} &\partial_\nu F^{\mu\nu} = J_e^\mu, \\ &\partial_\nu {}^*F^{\mu\nu} = J_m^\mu, \\ &{}^*F^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}, \end{aligned} \tag{2}$$

admits the one-parameter rotation group described by

$$\begin{aligned} &J_e \to \cos\phi J_e + \sin\phi J_m, \\ &J_m \to -\sin\phi J_e + \cos\phi J_m, \\ &F \to \cos\phi F + \sin\phi\, {}^*F, \end{aligned} \tag{3}$$

as supplemented by the statement produced with the aid of the iteration property of the dual:

$$^{**}F = -F. \tag{4}$$

The corresponding invariant form of the charge quantization for two particles, a and b, is

$$(e_a g_b - e_b g_a)/4\pi = n, \tag{5}$$

where, for the moment, it suffices to say that n assumes discrete values, including zero, with $n_0 \sim 1$ the magnitude of the smallest nonzero possibility. It is remarkable that this symmetrical viewpoint does *not* conflict with the empirical asymmetry between electric and magnetic charge.

To see this we make an invariant distinction between small charges, which are such that

$$(e_a^2 + g_a^2)/4\pi < n_0, \tag{6}$$

and large charges, which obey

$$(e_a^2 + g_a^2)/4\pi \geq n_0. \tag{7}$$

(The known unit of pure electric charge is comfortably small, $e^2/4\pi = \alpha \ll 1$.) Now apply the charge quantization condition to a pair of small charges, and note that

$$\left|\frac{e_a g_b - e_b g_a}{4\pi}\right| \leq \left(\frac{e_a^2 + g_a^2}{4\pi}\right)^{1/2}\left(\frac{e_b^2 + g_b^2}{4\pi}\right)^{1/2} < n_0, \tag{8}$$

from which it follows that the left-hand side vanishes. Hence, if only small charges are admitted, all possible two-dimensional points with coordinates e_a, g_a must occupy a single line. And, by a conventional choice of coordinate system, this absolute line can be made the axis of pure electric charge, thereby reducing to zero all magnetic charges. In this viewpoint, then, what is special about the world thus far disclosed by experiment is simply that no large charges have yet been produced. Incidentally, the geometrical inequality contained in Eq. (8) informs us that the minimum strength of a large charge, specifically, one that does not lie on the line of small charges, is actually given by

$$(e_a^2+g_a^2)/4\pi > n_0^2/\alpha \gg 1. \qquad (9)$$

In the following discussion we shall consider both the unsymmetrical and the symmetrical viewpoints, although it is the latter, with its possibility of dyon fractional charges and the associated dynamical model of hadrons, that I support.

ACTION

An unsymmetrical, and provisional, but convenient action expression that yields the pair of Maxwell equations is

$$W=\int(dx)[J_e^\mu A_\mu + J_m^\mu B_\mu - \tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}], \qquad (10)$$

where A and F are subject to independent variations and B is defined as

$$B_\mu(x)=\int(dx')\,{}^*F_{\mu\nu}(x')f^\nu(x'-x)+\partial_\mu\lambda_m(x), \qquad (11)$$

in which λ_m is arbitrary and $f^\nu(y)$ obeys

$$\partial_\nu f^\nu(y)=\delta(y). \qquad (12)$$

We shall also introduce

$${}^*f^\nu(y)=-f^\nu(-y), \quad \partial_\nu {}^*f^\nu(y)=\delta(y), \qquad (13)$$

so that (11) is presented alternatively as

$$B_\mu(x)=-\int(dx')\,{}^*f^\nu(x-x')\,{}^*F_{\mu\nu}(x')+\partial_\mu\lambda_m(x). \qquad (14)$$

The stationary requirement on A variations directly yields the first Maxwell set, while F variations produce the relation

$$F_{\mu\nu}(x)=(\partial_\mu A_\nu-\partial_\nu A_\mu)(x) + \int(dx')\,{}^*[f_\mu(x-x')J_{m\nu}(x')-f_\nu(x-x')J_{m\mu}(x')], \qquad (15)$$

the dual of which,

$${}^*F_{\mu\nu}(x)={}^*(\partial_\mu A_\nu-\partial_\nu A_\mu)(x) - \int(dx')[f_\mu(x-x')J_{m\nu}(x')-f_\nu(x-x')J_{m\mu}(x')], \qquad (16)$$

leads to the second Maxwell set. Another consequence of Eq. (15) is the construction

$$A_\mu(x)=-\int(dx')f^\nu(x-x')F_{\mu\nu}(x')+\partial_\mu\lambda_e(x), \qquad (17)$$

the analog of (14).

The analogy between (14) and (17) is not an actual symmetry in the sense of the charge rotation (3) unless

$${}^*f^\nu(y)=f^\nu(y)=-f^\nu(-y). \qquad (18)$$

Then, despite its unsymmetrical appearance, the action expression (10) is invariant under the charge rotation. To verify this, consider the infinitesimal rotation

$$\delta J_e=\delta\phi J_m, \quad \delta J_m=-\delta\phi J_e, \quad \delta F=\delta\phi\,{}^*F, \quad \delta A=\delta\phi B, \qquad (19)$$

which is completed by

$$\delta B_\mu(x)=\delta\phi\int(dx')\,{}^*f^\nu(x-x')F_{\mu\nu}(x') = \delta\phi\left[-A_\mu(x)+\int(dx')({}^*f^\nu-f^\nu)(x-x')F_{\mu\nu}(x')\right], \qquad (20)$$

where gradient terms have been omitted. The responses of the individual pieces of W are

$$\delta\int(dx)(J_eA+J_mB)=\delta\phi\int(dx)(dx')J_m^\mu(x)({}^*f^\nu-f^\nu)(x-x')F_{\mu\nu}(x'), \qquad (21)$$

$$\delta\int(dx)(-\tfrac{1}{2})F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)=-\delta\phi\int(dx)(J_mA+J_eB), \qquad (22)$$

and

$$\delta\int(dx)(\tfrac{1}{4})F^{\mu\nu}F_{\mu\nu}=\delta\phi\left[2\int(dx)J_mA-\int(dx)(dx')J_m^\mu(x)({}^*f^\nu-f^\nu)(x-x')F_{\mu\nu}(x')\right] = \delta\phi\left[2\int(dx)J_eB+\int(dx)(dx')J_e^\mu(x)({}^*f^\nu-f^\nu)(x-x')\,{}^*F_{\mu\nu}(x')\right], \qquad (23)$$

where the last form involves the analog of (15),

$$*F_{\mu\nu}(x)=(\partial_\mu B_\nu-\partial_\nu B_\mu)(x)-\int(dx')\,*[*f_\mu(x-x')J_{e\nu}(x')-*f_\nu(x-x')J_{e\mu}(x')]. \tag{24}$$

The average of the alternatives in Eq. (23) combines with the other contributions to give

$$\delta W=\delta\phi\tfrac{1}{2}\int(dx)(dx')(*f^\nu-f^\nu)(x-x')[J_m^\mu(x)F_{\mu\nu}(x')+J_e^\mu(x)\,*F_{\mu\nu}(x')], \tag{25}$$

which indeed vanishes under the circumstances of Eq. (18).

The action expression (10) was characterized as provisional, for it cannot be applied as it stands to point charges, which constitute the only possibility for a consistent theory of electric and magnetic charges. The *mathematical* difficulty is, of course, the singular nature of the self-action for an individual charge. A simple device to alleviate the problem is based on the following replacement, which is applied to all the terms of (10):

$$\int(dx)G(x)H(x)\to\int(dx)G(x)H(x\pm\lambda)=\int(dx)G(x\pm\lambda)H(x). \tag{26}$$

Here, λ is an arbitrary spacelike displacement that eventually tends to zero, and $\pm\lambda$ indicates the equally weighted average of the two possibilities. The formal action property of $W\to W(\lambda)$ is not affected by this replacement, the field equations being retained intact,[4] and the now finite self-actions of the charges, more precisely, their real parts, can be deleted without ambiguity. (The precise philosophy that underlies this incision need not detain us.) The only consequence in an equation like (25) is conveyed by the replacement $x-x'\to x-x'\pm\lambda$.

CHARGE QUANTIZATION

The basic realization of a function f that obeys Eq. (12) is given by

$$f_\mu(y)=\int_0^\infty d\xi_\mu\delta(y-\xi), \tag{27}$$

where the ξ integration follows some path from the origin to infinity. Different paths produce different functions and we shall indicate specific choices by the superscripts (1) and (2). The difference of two such functions is

$$\begin{aligned}f_\mu^{(1)}(y)-f_\mu^{(2)}(y)&=\left(\int^{(1)}-\int^{(2)}\right)d\xi_\mu\delta(y-\xi)\\&=-\int d\sigma_{\mu\nu}(\partial/\partial\xi_\nu)\delta(y-\xi)\\&=\partial^\nu\int d\sigma_{\mu\nu}\delta(y-\xi),\end{aligned} \tag{28}$$

where the surface integral extends over the area bounded by the two paths. The analogous statements for functions that obey the symmetry restriction of (18) are

$$f_\mu(y)=\int_0^\infty d\xi_\mu\tfrac{1}{2}[\delta(y-\xi)-\delta(y+\xi)], \tag{29}$$

and

$$f_\mu^{(1)}(y)-f_\mu^{(2)}(y)=\partial^\nu\int d\sigma_{\mu\nu}\tfrac{1}{2}[\delta(y-\xi)+\delta(y+\xi)]. \tag{30}$$

In the following, differences of f functions are written as

$$\delta f_\mu(y)=\partial^\nu m_{\mu\nu}(y), \tag{31}$$

which, despite the use of the δ symbol, is not limited to small changes; the surface integral constructions of the antisymmetrical tensor $m_{\mu\nu}(y)$ are exhibited in Eqs. (28) and (30).

The explicit dependence of $W(\lambda)$ on the f function is contained in the $J_m B$ term. A change in the choice of f induces

$$\delta W(\lambda)=-\int(dx)(dx')\,*F_{\mu\nu}(x)\delta f^\mu(x-x'\pm\lambda)J_m^\nu(x'), \tag{32}$$

which is not restricted to small changes since all f dependence is concentrated in δf. The introduction of (31) converts $\delta W(\lambda)$ into

$$\begin{aligned}\delta W(\lambda)&=\tfrac{1}{2}\int(dx)(dx')(\partial_\mu *F_{\nu\kappa}+\partial_\nu *F_{\kappa\mu}+\partial_\kappa *F_{\mu\nu})(x)\\&\quad\times m^{\mu\kappa}(x-x'\pm\lambda)J_m^\nu(x')\\&=-\tfrac{1}{2}\int(dx)(dx')\epsilon_{\mu\nu\kappa\sigma}J_e^\sigma(x)m^{\mu\kappa}(x-x'\pm\lambda)J_m^\nu(x')\\&=-\int(dx)(dx')J_e^\mu(x)\,*m_{\mu\nu}(x-x'\pm\lambda)J_m^\nu(x').\end{aligned} \tag{33}$$

We now insert a point-charge realization of these currents

$$\begin{aligned}J_e^\mu(x)&=\sum_a e_a\int_{-\infty}^\infty ds\,\frac{dx_a^\mu(s)}{ds}\delta(x-x_a(s)),\\J_m^\nu(x')&=\sum_b g_b\int_{-\infty}^\infty ds'\,\frac{dx_b^\nu(s')}{ds'}\delta(x-x_b(s')),\end{aligned} \tag{34}$$

and get

$$\delta W(\lambda) = -\sum_{a,b} e_a g_b \int dx_a^\mu dx_b^\nu \, {}^*m_{\mu\nu}(x_a - x_b \pm \lambda)$$
$$= -\sum_{a,b} e_a g_b \int \tfrac{1}{2}\, {}^*d\sigma_{ab}^{\mu\nu} m_{\mu\nu}(x_a - x_b \pm \lambda), \tag{35}$$

where

$$d\sigma_{ab}^{\mu\nu} = dx_a^\mu dx_b^\nu - dx_a^\nu dx_b^\mu. \tag{36}$$

In the symmetrical formulation, for which $m_{\mu\nu}(-y) = m_{\mu\nu}(y)$, $\delta W(\lambda)$ can be written in terms of the rotationally invariant combination of charges:

$$\delta W(\lambda) = -\tfrac{1}{2}\sum_{a,b} (e_a g_b - e_b g_a) \int \tfrac{1}{2}\, {}^*d\sigma_{ab}^{\mu\nu} m_{\mu\nu}(x_a - x_b \pm \lambda). \tag{37}$$

We first examine the unsymmetrical viewpoint, where (28) is appropriate,

$$\delta W(\lambda) = -\sum_{a,b} e_a g_b \int \tfrac{1}{2}\, {}^*d\sigma_{ab}^{\mu\nu} d\sigma_{\mu\nu} \delta(x_a - x_b \pm \lambda - \xi). \tag{38}$$

The product of orthogonal area elements defines a four-dimensional element of volume for the variable $x_a - x_b \pm \lambda - \xi$, where we now make a particular choice of $+\lambda$ or $-\lambda$. That variable may, or may not, pass through the origin in the course of the integration, correspondingly yielding 1, or 0, as the integral of the δ function. Consider, for definiteness, this situation. The two paths being compared are straight lines in the 12 plane, deviating by the angle θ. The excursions of the variable ξ in this plane are measured by the area element $d\sigma_{12}$. Perpendicular to the latter is ${}^*d\sigma_{ab}^{12} = d\sigma_{ab}^{30}$, and this integration over the appropriate δ-function factors picks out the situation(s) where $x_a^0 - x_b^0 = 0$, $x_a^3 - x_b^3 = 0$, assuming, for simplicity only, that λ lies in the 12 plane. At each occurrence of this situation, should the two-dimensional vector $(x_a - x_b \pm \lambda)_{1,2}$ lie within the triangle of vertex angle θ, the complete integral would equal unity; otherwise, it would equal zero. As for $\pm\lambda$, one must also recognize the possibility that, for one of these choices, the integral equals unity while the other yields zero. Thus, the basic values of the four-dimensional integral are $1, \frac{1}{2}, 0$, with the possibility of $\frac{1}{2}$ dependent on the finiteness of the λ displacement. On singling out a particular ab pair, the uniqueness of $W(\lambda)$, mod2π, corresponding to the physical significance of $\exp[iW]$, requires that

$$e_a g_b (1, \tfrac{1}{2}) = 2\pi n, \tag{39}$$

with n an integer. If the special situation producing $\frac{1}{2}$ is taken seriously, the quantization condition reads

$$e_a g_b / 4\pi = n; \tag{40}$$

if this special situation is disregarded, we get

$$e_a g_b / 4\pi = \tfrac{1}{2} n. \tag{41}$$

In the past, I have insisted that no anomalies should be tolerated during the limiting process $\lambda \to 0$, an attitude which results in the integer quantization condition (40). Yet, it was always apparent that this injunction might be unnecessarily strict, since the theory can be completely free of nonphysical elements only when λ attains its null limiting value. Thus, it would suffice that the circumstances for which the λ displacement is significant be a set of measure approaching zero in that limit. In this more permissive view, to which, in keeping with the times, I now subscribe, the Dirac quantization condition (1) is correct for the unsymmetrical formulation.

In the symmetrical formulation, where $\delta W(\lambda)$ is given by (37) and $m_{\mu\nu}(y)$ by (30), the presence of two disjoint areas, each with the weight factor $\frac{1}{2}$, replaces (39), for $\lambda = 0$, with

$$\tfrac{1}{2}(e_a g_b - e_b g_a) = 2\pi n \tag{42}$$

or

$$(e_a g_b - e_b g_a)/4\pi = n. \tag{43}$$

This is (5) with n restricted to integer values, so that $n_0 = 1$. (The use of the strict construction of the limit $\lambda \to 0$, as described in the preceding paragraph, would further narrow n to the even integers.)

STRINGS

The original Dirac formulation has been prominent in recent developments, which makes it desirable to point out, again, the unnecessarily restrictive nature of the dictum "a string must never pass through a charged particle." To set the scene for this criticism, consider the part of the action (10) that depends explicitly on the coordinates of the point charges, as contained in the current constructions of Eq. (34). The response to a variation of these coordinates is

$$\delta W = \sum_a \delta \int dx_a^\mu [e_a A_\mu(x_a) + g_a B_\mu(x_a)]$$
$$= \sum_a \int \tfrac{1}{2} d\sigma_a^{\mu\nu} [e_a(\partial_\mu A_\nu - \partial_\nu A_\mu)(x_a) + g_a(\partial_\mu B_\nu - \partial_\nu B_\mu)(x_a)], \tag{44}$$

where

$$d\sigma_a^{\mu\nu} = \delta x_a^\mu dx_a^\nu - \delta x_a^\nu dx_a^\mu. \tag{45}$$

On introducing (15) and (24), this becomes

$$\delta W=\sum_a \int \tfrac{1}{2}d\sigma_a^{\mu\nu}[e_a F_{\mu\nu}(x_a)+g_a\, {}^*F_{\mu\nu}(x_a)] - \sum_{a,b} e_a g_b \int d\sigma^\mu_{ab} f_\mu(x_a-x_b) + \sum_{ab} g_a e_b \int d\sigma^\mu_{ab}\, {}^*f_\mu(x_a-x_b), \tag{46}$$

and here

$$d\sigma^\mu_{ab} = {}^*d\sigma_a^{\mu\nu} dx_{b\nu} \tag{47}$$

is a directed three-dimensional element.

These variations are relevant to the derivation of equations of motion for the charged particles, where only field strengths should intervene. To achieve this, Dirac insisted that the additional f-dependent terms in (46) must vanish. Thus, if no point ξ along the integration contour of (27) is allowed to coincide with a value assumed by x_a-x_b, these unwanted terms will disappear. But suppose such a coincidence does occur? Consider first the unsymmetrical situation in which particle a is electrically charged, and we examine this contribution to δW of a magnetically charged particle b:

$$-e_a g_b \int d\sigma^\mu_{ab} d\xi_\mu \delta(x_a - x_b - \xi). \tag{48}$$

The integration element for the variable $x_a - x_b - \xi$, $d\sigma^\mu_{ab} d\xi_\mu$, is an infinitesimal four-dimensional volume. If, in the course of the integration, $x_a - x_b$ and ξ become equal, the change in W will be finite, of magnitude $e_a g_b = 2\pi n$, according to (41), and $\exp[iW]$ remains unaffected. In the symmetrical formulation, where ${}^*f_\mu = f_\mu$, the last terms of (46) read

$$-\sum_{a,b}(e_a g_b - e_b g_a)\int d\sigma^\mu_{ab} f_\mu(x_a-x_b), \tag{49}$$

and the nonzero contribution associated with a particular pair of particles is similarly measured by $\frac{1}{2}(e_a g_b - e_b g_a) = 2\pi n$, as given in Eq. (42). Evidently, the same charge quantization conditions, expressing the uniqueness of $\exp[iW]$, should be and are encountered whether one examines the change of the f function for fixed-particle trajectories or varies the trajectories for a given f function.

REALISTIC SYSTEM

The discussion thus far given is an idealized one, abstracted from the special dynamical features of particular types of particles. Accordingly, it would be instructive to indicate, at least, how the charge quantization condition enforces the consistency of the theory for a realistic system. To that end, but minimizing the difficulty of a fully general discussion, we consider the interaction of two spin-$\frac{1}{2}$ dyons, under conditions of sufficiently small momentum transfer that physical effects associated with vacuum polarization and charge form factors can be omitted. The motion of the individual dyons in a given electromagnetic field is described by Green's functions that obey

$$\left[\gamma^\mu\left(\frac{1}{i}\partial'_\mu - eA_\mu(x') - gB_\mu(x')\right)+m\right]G_+^{A,B}(x',x'') = \delta(x'-x''), \tag{50}$$

where a charge matrix of eigenvalues ± 1 is implicit in e and g. In a matrix notation, this equation is written

$$HG_+^{A,B}=1 \tag{51}$$

or

$$G_+^{A,B} = i\int_0^\infty ds\, e^{-isH}, \tag{52}$$

with

$$H=\gamma(p-eA-gB)+m. \tag{53}$$

The analogy between H and a Hamiltonian is used in deriving the proper-time equation of motion

$$\frac{dx^\mu}{ds}=\frac{1}{i}[x^\mu, H]=\gamma^\mu. \tag{54}$$

We also exploit the significance of the matrix of $\exp(-is_1H)$ as a transformation function,

$$\langle x'|e^{-is_1H}|x''\rangle = \langle x's_1|x''0\rangle, \tag{55}$$

and apply the differential action principle[5]

$$\delta\langle x's_1|x''0\rangle = -i\left\langle x's_1\right|\int_0^{s_1} ds\,\delta H\left|x''0\right\rangle \tag{56}$$

to changes of the vector potentials,

$$\delta\langle x's_1|x''0\rangle = i\left\langle x's_1\right|\int_0^{s_1} ds\left(e\frac{dx}{ds}\delta A + g\frac{dx}{ds}\delta B\right)\left|x''0\right\rangle. \tag{57}$$

The latter is also presented in functional notation as

$$\frac{1}{i}\frac{\delta}{\delta A_\mu(\xi)}\langle x's_1|x''0\rangle = \left\langle x's_1\right|\int_0^{s_1} ds\, e\frac{dx^\mu}{ds}\,\delta(\xi - x(s))\left|x''0\right\rangle \tag{58}$$

and

$$\frac{1}{i}\frac{\delta}{\delta B_\mu(\xi)}\langle x's_1|x''0\rangle = \left\langle x's_1\right|\int_0^{s_1} ds\, g\frac{dx^\mu}{ds}\,\delta(\xi - x(s))\left|x''0\right\rangle. \tag{59}$$

The current expressions appearing here, which resemble (34), are incomplete, however, for they are not conserved. Thus

$$\frac{\partial}{\partial \xi^\mu} \int_0^{s_1} ds\, e \frac{dx^\mu}{ds} \delta(\xi - x(s)) = -e[\delta(\xi - x(s_1)) - \delta(\xi - x(0))] \quad (60)$$

and

$$\frac{\partial}{\partial \xi^\mu} \frac{1}{i} \frac{\delta}{\delta A_\mu(\xi)} \langle x' s_1 | x'' 0 \rangle = -e[\delta(\xi - x') - \delta(\xi - x'')] \langle x' s_1 | x'' 0 \rangle, \quad (61)$$

which relates the nonconservation to the gauge variance of the transformation function, and of the Green's function. What is missing is an electromagnetic model of the source, describing the history of the conserved charges before they apparently appear at x'', and after they seem to disappear at x'. This is effectively introduced by considering the gauge-invariant combination

$$\langle x' s_1 | x'' 0 \rangle_{\text{inv}} = \exp[i\phi(x')] \langle x' s_1 | x'' 0 \rangle \exp[-i\phi(x'')], \quad (62)$$

where

$$\phi(x') = \int (d\xi)[eA_\mu(\xi) + gB_\mu(\xi)] f^\mu(\xi - x'), \quad (63)$$

and analogously for x''. The f function appearing here obeys (12) and can be exhibited, similarly to (27) but with notational changes, as

$$f^\mu(\xi - x') = \int_{s_1}^{\infty} ds \frac{dx^\mu}{ds} \delta(\xi - x(s)), \quad x(s_1) = x', \quad (64)$$

along with[6]

$$-f^\mu(\xi - x'') = \int_{-\infty}^{0} ds \frac{dx^\mu}{ds} \delta(\xi - x(s)), \quad x(0) = x''; \quad (65)$$

in these expressions $x(s)$ is numerically valued and the two paths are arbitrary. When the gauge-invariant transformation function is used, Eqs. (58) and (59) retain their form, but the proper time parameter now ranges from $-\infty$ to $+\infty$. The following remarks refer to the gauge-invariant Green's functions that are constructed as

$$G_+^{A,B}(x', x'')_{\text{inv}} = i \int_0^\infty ds_1 \langle x' s_1 | x'' 0 \rangle_{\text{inv}}. \quad (66)$$

The Green's function that describes the situation of interacting dyons a and b is presented symbolically in[7]

$$G_{ab} = \exp[iW_{ab}] G_{+,\text{inv}}^{A,B}|_a G_{+,\text{inv}}^{A,B}|_b, \quad (67)$$

where W_{ab} is the part of the action expression of the preceding sections associated with two distinct current distributions, $J_{e,m}|_a$ and $J_{e,m}|_b$, and

$$J_e(\xi) \to \frac{1}{i} \frac{\delta}{\delta A(\xi)}, \quad J_m(\xi) \to \frac{1}{i} \frac{\delta}{\delta B(\xi)}; \quad (68)$$

the fields $A(\xi)$, $B(\xi)$ are set equal to zero after the differentiations. The effect of these differentiations is to introduce $\exp[iW_{ab}]$ into the over-all matrix element as an ordered operator, with the functional derivatives replaced by particle operators, in accordance with the gauge-invariant extensions of Eqs. (58) and (59). The current expressions thus obtained have just the form of (34), and their operator aspect can be removed through the evaluation of the matrix element as an infinite product of infinitesimal transformation matrix elements, yielding a functional integral form. The result is an interaction factor having precisely the structure used in the preceding discussion of the charge quantization condition. Of course, one would want to supplement this highly formal discussion with an explicit verification, which inevitably involves an approximation scheme. The most immediate one is a high-energy eikonal approximation[8] to the individual Green's functions in (67), which effectively results in the arbitrary particle paths of the functional integration being replaced by straight-line motion. The details are left to the reader.

Added note. At long last there is experimental evidence for the existence of magnetic charge. P. B. Price, E. K. Shirk, W. Z. Osborne, and L. S. Pinsky [Phys. Rev. Lett. 35, 487 (1975)] have detected a very heavily ionizing particle that has all the characteristics of a particle with magnetic charge $g = 137e$, or in rationalized units, $eg/4\pi = 1$. That is the smallest magnetic charge permissible in the symmetrical formulation, and twice the magnetic unit of the unsymmetrical Dirac version. While this does not prove the validity of the symmetrical viewpoint, as the discovery of magnetic charge $g = \frac{1}{2}(137e)$ would have disproved it, it does lend considerable support to the symmetrical formulation and encourages the serious study of the dyon model of hadronic phenomena. Since the smallest magnetic charge resides on a dyon, the observed particle should also carry a fractional electric charge, $\frac{1}{3}e$ or $\frac{2}{3}e$ in magnitude. That cannot be verified from the present data, but might be tested when such a particle is observed coming to rest.

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[1]For example, Y. Nambu, Phys. Rev. D 10, 4262 (1974); G.'t Hooft, Nucl. Phys. B79, 276 (1974).

[2]J. Schwinger, Phys. Rev. 144, 1087 (1966); 173, 1536 (1968).

[3]P. A. M. Dirac, Phys. Rev. 74, 817 (1948); Proc. R. Soc. Lond. A133, 60 (1931).

[4]Compare Ref. 3.

[5]The concepts used here are fully developed, for simple physical contexts, in J. Schwinger, *Quantum Kinematics and Dynamics* (Benjamin, Reading, Mass., 1970).

[6]The alternative procedure, in which a compensating charge runs from x'' to x', is less convenient here.

[7]This is the immediate generalization of Eq. (5-2.12) in J. Schwinger, *Particles, Sources and Fields* (Addison-Wesley, Reading, Mass., 1973), Vol. II. See also Chap. 7 of Ref. 5.

[8]Such techniques are discussed by H. M. Fried, *Functional Methods and Models in Quantum Field Theory* (MIT Press, Cambridge, Mass., 1972).