Semiclassical quantization of the relativistic Kepler problem

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Semiclassical quantization conditions are applied to a variety of relativistic formulations of the problem of two spinless particles bound by a vector or scalar force. The WKB quantization conditions are found to yield the correct spectra whenever the equations of motion are separable. Miller's new quantum condition, making use of periodic orbits, is applied to almost circular orbits. It provides an approximate method of quantization for almost circular orbits in the action-at-a-distance formulation of relativistic charged particles interacting electromagnetically.

I. QUANTUM CONDITIONS

Semiclassical quantization conditions for a particle bound by a potential have been undergoing improvement from their invention by Bohr until today. Three versions are given as illustrations. The Sommerfeld-Wilson quantization conditions were assumed without proof:

$$\oint p_r dr = n_r h\,, \quad n_r = 0,1,2,\ldots$$
$$\int_0^{2\pi} p_\varphi d\varphi = n_\varphi h\,, \quad n_\varphi = 1,2,3,\ldots\,. \tag{1}$$

The integration over r proceeds from a minimum to a maximum and back to a minimum. The restriction $n_\varphi \neq 0$ is imposed *ad hoc* in order to avoid a divergent ground state for the hydrogen spectrum. After the advent of quantum mechanics, an improved version was derived from the Schrödinger equation in the WKB approximation:

$$\oint p_r dr = (n_r + \tfrac{1}{2})h\,, \quad n_r = 0,1,2,\ldots$$
$$J = (l+\tfrac{1}{2})\hbar\,, \quad l = 0,1,2,\ldots\,. \tag{2}$$

This set is correct for separable problems and gives an angular momentum J which is close to the correct value $[l(l+1)]^{1/2}\hbar$. Modern developments by many authors[1] have led to a variety of quantization conditions for nonseparable systems. One which is found useful below is due to Miller.[2] It makes use of periodic orbits. For N degrees of freedom it becomes

$$\Phi(E) = 2\pi\left(n + \frac{\lambda}{4}\right) + \sum_{i=1}^{N-1} (m_i + \tfrac{1}{2})\omega_i(E)T(E)\,, \tag{3}$$
$$n = 0,1,2,\ldots;\quad m_i = 0,1,2,\ldots\,.$$

where

$$\Phi(E) \equiv \hbar^{-1}\int_0^{T(E)} \vec{p}\cdot d\vec{q}$$

is the action integral over one periodic orbit, $T(E)$ is the period, $\omega_i(E)$ are the stability frequencies, and λ is the number of turning points in the periodic orbit. (The product $v_i = \omega_i T$ is known as stability angle, stability index, or stability parameter.) Equation (3) can be approximately inverted to yield a particularly transparent formula for the energy E of the bound state:

$$E = \Phi^{-1}\left[2\pi\left(n + \frac{\lambda}{4}\right)\right] + \sum_{i=1}^{N-1} (m_i + \tfrac{1}{2})\hbar\omega_i(E)\,. \tag{4}$$

Miller has put this equation in words: "The stability frequencies $\{\omega_i\}$ are the normal-mode frequencies for harmonic perturbation about the periodic orbit. The total energy E, therefore, is a sum of contributions: The first term in (4) is the energy of n quanta in motion *along* the periodic orbit, and the ith term in the sum of $N-1$ terms is the energy of m_i quanta in the ith normal mode of deviation about the periodic orbit." Condition (3) is known to be correct for separable systems provided the motion perpendicular to the chosen periodic orbit is harmonic.

II. THE ELECTROMAGNETIC INTERACTION

The electromagnetic interaction is mediated by a vector field. This fact manifests itself in the form of the classical Hamiltonian of a particle in a Coulomb field

$$H = (p^2 + m^2)^{1/2} - e^2/r\,, \tag{5}$$

where $c = 1$. The potential, like the energy, transforms as the fourth component of a vector. The Hamiltonian leads in classical mechanics to precessing ellipses. Sommerfeld quantized these, using Eq. (1), and derived energy levels which were later found to agree with the Dirac spectrum.[3] This agreement is spurious and results from two mistakes—neglect of spin and wrong quantization. If Eq. (2) is used instead, the spectrum is the correct one for spinless particles in a Coulomb potential. It is, in fact, the Klein-Gordon spectrum.[3]

TABLE I. Summary of energy spectra. In each case the spectrum derived from the semiclassical theory with quantum conditions (2) coincides with the exact quantum-mechanical spectrum.

Classical theory	Quantum theory	Energy spectrum ($n = n_r + l + 1; n_r = 0, 1, 2, \ldots; l = 0, 1, 2, \ldots$)
Sommerfeld (closed ellipses)	Schrödinger equation	$E = -\frac{m\alpha^2}{2n^2}$
Relativistic Sommerfeld (precessing ellipses)	Klein-Gordon equation with vector interaction	$E = m\left\{1 + \frac{\alpha^2}{[n - l - \frac{1}{2} + ((l+\frac{1}{2})^2 - \alpha^2)^{1/2}]^2}\right\}^{-1/2}$ $\approx m\left[1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\left(\frac{n}{l+\frac{1}{2}} - \frac{3}{4}\right)\right]$
Darwin (inclusion of nuclear motion to order v^2/c^2)	Breit equation neglecting Darwin term	$E = m_1 + m_2 - \mu\left(\frac{\alpha^2}{2n^2}\right) - \frac{\mu\alpha^4}{2n^4}\left(\frac{n}{l+\frac{1}{2}} - \frac{3}{4} + \frac{\mu}{4(m_1+m_2)}\right)$ $\mu \equiv m_1 m_2/(m_1+m_2)$
	Quasipotential theory with vector interaction	$E^2 = m_1^2 + m_2^2 + 2m_1m_2\left\{1 + \frac{\alpha^2}{[n - l - \frac{1}{2} + ((l+\frac{1}{2})^2 - \alpha^2)^{1/2}]^2}\right\}^{-1/2}$
Fronsdal, Andersen and von Baeyer (closed ellipses)	Klein-Gordon equation with Yukawa interaction	$E = m\left(1 - \frac{\alpha^2}{n^2}\right)^{1/2}$
Bergmann (precessing ellipses)	Klein-Gordon equation with minimal scalar interaction	$E = m\left\{1 - \frac{\alpha^2}{[n - l - \frac{1}{2} + ((l+\frac{1}{2})^2 + \alpha^2)^{1/2}]^2}\right\}^{1/2}$
Vector-scalar potential theory (closed ellipses)	Klein-Gordon equation with vector-scalar interaction	$E = m(1 - \alpha^2/4n^2)(1 + \alpha^2/4n^2)^{-1}$
	Quasipotential theory with Yukawa interaction	$E^2 = m_1^2 + m_2^2 + 2m_1m_2(1 - \alpha^2/n^2)^{1/2}$

Its fine structure is related to the fact that the ellipses precess.

In Table I the classical equations are listed by name, together with their quantum-mechanical counterparts. In each case the spectra agree if quantization conditions (2) are used.

In Sommerfeld's treatment nuclear motion is not included. He simply replaced the mass m by the reduced mass μ. This procedure, while correct for nonrelativistic theory, is wrong for a relativistic calculation.[4] Nuclear motion was correctly included to order $(v/c)^4$ by Darwin[5] who found, in addition to the replacement of m by μ, a term of order α^4 in the energy. Darwin then quantized his theory using Eq. (1). If Eq. (2) is used instead, the spectrum agrees with that of the Breit equation which is, in fact, the quantum-mechanical version of Darwin's theory. There is one term in Breit's result which has no classical limit, the so-called "Darwin term." It has the value $\mu^2\alpha^4\delta_{l0}/(m_1+m_2)n^3$ and affects only the S wave.

Todorov[6] has derived a quasipotential two-body equation from field theory. It has a structure similar to one-body theory and hence spectra are easy to find once the Schrödinger equation is solved. For spinless particles interacting electromagnetically the spectrum agrees to order α^4 with Breit's result. The Darwin term arises from the commutator of $\vec{p}$ with $\vec{A}$ and is treated as a perturbation. The quasipotential equation can be used to find a classical equation by interpreting operators as c numbers, but as such it does not have much physical relevance. Nevertheless, if this classical equation is quantized semiclassically, the resulting spectrum is correct.

The spectra which result from Todorov's quasipotential theory are identical to those of the relativistic eikonal approximation.[7] This fact allows interpretation of the results in terms of approximate summation of "crossed-ladder" Feynman graphs.

III. OTHER INTERACTIONS

Nonrelativistically all interactions mediated by massless bosons lead to a $1/r$ potential. Relativistically they lead to different equations of motion and different potentials.

A classical relativistic theory corresponding to spinless particles with an interaction mediated by massless scalars can be formulated in a variety of ways. This point was emphasized recently by a number of authors.[8-10] The difference between the two most common versions can best be seen at the field-theoretic level.[9] If ϕ is the field corresponding to charged spinless particles of mass m, and A the massless field which mediates the interaction, then two simple forms of the interaction are the Yukawa interaction

$$2mA(x)\phi^*(x)\phi(x)$$

and the minimal scalar interaction

$$[m+A(x)]^2\phi^*(x)\phi(x)\,.$$

The term "minimal scalar interaction" is used in analogy to the minimal electromagnetic interaction. The minimal couplings are induced by the substitutions $m \to m+A$ and $p^\mu \to p^\mu + A^\mu$ in the free-particle theory. (We avoid the ambiguous term scalar electrodynamics.)

The Yukawa interaction yields, to lowest order, no fine structure.[11] Its classical limit displays no perihelion precession. The minimal scalar interaction, on the other hand, leads to fine structure and perihelion precession as a consequence of its quadratic interaction term.

The classical one-body potential theory corresponding to a massless Yukawa interaction has the Hamiltonian[8]

$$H=(p^2+m^2-2me^2/r)^{1/2}\,. \tag{6}$$

It was independendtly derived from quantum field theory[12] and from a relativistic classical theory of particles interacting with fields.[8] The fact that the ellipses remain closed in spite of the relativistic nature of the calculation was at first surprising. The equation of motion and the conserved Runge-Lenz vector are given in Ref. 8. When quantized semiclassically the theory yields the simple spectrum

$$E=m(1-\alpha^2/n^2)^{1/2}$$

which also follows from the Klein-Gordon equation

$$(p^2+m^2-2me^2/r)\psi=E^2\psi \tag{7}$$

as is easily verified by rewriting

$$\left(\frac{p^2}{2m}-\frac{e^2}{r}\right)\psi=\left(\frac{E^2-m^2}{2m}\right)\psi \tag{8}$$

and comparing with the Schrödinger equation.

The classical one-body potential theory corresponding to minimal scalar coupling is of ancient vintage. It was considered as a component of gravity and is discussed in detail by Bergmann.[13] The Hamiltonian[8] is given by

$$H=[p^2+(m-e^2/r)^2]^{1/2}\,. \tag{9}$$

The perihelion moves through $\frac{1}{6}$ the general relativistic amount and in the wrong direction. The energy can be found as a function of action integrals in a manner analogous to Sommerfeld's method.[10] If condiditon (2) is used for quantization, the spectrum coincides with that obtained from the Klein-Gordon equation with minimal scalar cou-

pling:

$$[p^2+(m-e^2/r)^2]\psi=E^2\psi\,. \tag{10}$$

The spectrum differs only in three signs from the formula for minimal vector coupling.

A rather artificial interaction, combining minimal vector and scalar couplings, can be adjusted to give a conserved Runge-Lenz vector and hence no perihelion motion in the one-particle case. The only theoretical advantage of this example is that it permits, unlike the Yukawa case, an action-at-a-distance formulation[14] while also exhibiting dynamical symmetry. The classical Hamiltonian is

$$H=[p^2+(m+\alpha V)^2]^{1/2}+\beta V\,, \tag{11}$$

where V is the potential and α and β are constants. The correct Newtonian limit requires $\alpha+\beta=1$. A conserved Runge-Lenz vector exists if $\alpha=\beta$. The resulting Hamiltonian is

$$H=\left[p^2+\left(m-\frac{e^2}{2r}\right)^2\right]^{1/2}-\frac{e^2}{2r}\,. \tag{12}$$

The corresponding Klein-Gordon equation is

$$(E-m)\psi=[p^2/(E+m)-e^2/r]\,\psi\,. \tag{13}$$

The spectrum can be read off by comparison with the Schrödinger equation and is listed in Table I.

Two-body equations, both classical and quantum-mechanical, can be constructed as generalizations of all the one-body theories discussed. In Table I we list only the quasipotential equation with a Yukawa interaction[6] which has a simple spectrum without fine structure. Other two-body equations and spectra have been considered by Crater and Naft[10] and by Fronsdal.[11]

IV. CIRCULAR ORBITS

Circular orbits play a very special role in mechanics because of their simplicity. In action-at-a-distance theory, for example, they are among the very few orbits which have been found.[15] Unfortunately, circular orbits are not allowed semiclassically because, according to Eq. (2), there must be zero-point oscillation in the radial variable. Miller's quantization condition (3), however, is applicable to those orbits which are nearly circular, i.e., which differ from the circular (periodic) orbits only by small perturbations. The perturbations are by definition harmonic. In this section we examine the semiclassical quantization of almost circular orbits. Only the electromagnetic interaction is considered.

Nearly circular orbits are characteristic of high angular momenta. At the semiclassical level this means $l\gg n_r$. In order to test the validity of the semiclassical quantization we first expand the correct nonrelativistic hydrogen spectrum in powers of $n_r+\frac{1}{2}$. The formula derived from the Schrödinger equation is

$$\frac{E}{m}=\frac{-\alpha^2}{2(n_r+l+1)^2}=\frac{-\alpha^2}{2(n_r+\frac{1}{2}+l+\frac{1}{2})^2}\,. \tag{14}$$

To first order in $n_r+\frac{1}{2}$ this becomes

$$\frac{E}{m}=\frac{-\alpha^2}{2(l+\frac{1}{2})^2}+\frac{\alpha^2(n_r+\frac{1}{2})}{(l+\frac{1}{2})^3}\,. \tag{15}$$

We now apply Eq. (4) to circular orbits. The first step consists of evaluating the action integral to find the first term in (4). The condition is

$$\Phi(E)=\hbar^{-1}\int_0^T \vec{\mathrm{p}}\cdot d\vec{\mathrm{r}}=2\pi(n+\lambda/4)\,. \tag{16}$$

[The value of λ is found by identifying (2) and (3), a procedure appropriate to a separable system. Using (2) we find

$$\Phi(E)=\hbar^{-1}\oint (p_\varphi d\varphi+p_r dr)$$

$$=2\pi(l+\tfrac{1}{2})+2\pi(n_r+\tfrac{1}{2})\,. \tag{17}$$

This can be identified with (3) if $n\equiv l$, $m_1\equiv n_r$, and $\lambda=2$. We shall see presently that $\omega_r T=2\pi$ to complete the identification.] With $\lambda=2$, we proceed exactly as in the original Bohr theory. Since $\int\vec{\mathrm{p}}\cdot d\vec{\mathrm{r}}=2\pi J$, the energy of circular orbits, labeled with subscript 0, becomes

$$\frac{E_0}{m}=\frac{-\alpha^2}{2(l+\frac{1}{2})^2}\,. \tag{18}$$

The radial oscillations carry additional energy E_r given by

$$E_r=(n_r+\tfrac{1}{2})\hbar\omega_r\,, \tag{19}$$

where ω_r is the stability frequency. In the case of circular orbits in a $1/r$ potential it is an elementary exercise[16] to show that the stability frequency equals the orbital frequency ω_0. This condition results in closed ellipses for nonrelativistic Kepler motion and justifies the assumption $\omega_r T=2\pi$ which we made above. Thus $\omega_r=2\pi T^{-1}=v/r_0=me^4/J^3$. Using the previous result $J=(l+\frac{1}{2})\hbar$ we find from (19)

$$\frac{E_r}{m}=\frac{(n_r+\frac{1}{2})\alpha^2}{(l+\frac{1}{2})^3}\,. \tag{20}$$

Now $E=E_0+E_r$, with the two terms given by (18) and (20), agrees with the approximate expression (15). This is correct through order l^{-3} and allows interpretation of the energy in the sense of Miller.

In the relativistic case the correct energy is given by the Klein-Gordon spectrum which becomes, expanded in powers of $n_r+\frac{1}{2}$,

$$\frac{E}{m}=\left(1-\frac{\alpha^2}{(l+\frac{1}{2})^2}\right)^{1/2}+\frac{(n_r+\frac{1}{2})\alpha^2}{(l+\frac{1}{2})^3}. \tag{21}$$

Note that to all orders in α the second term is identical to the nonrelativistic result. In order to perform semiclassical quantization we need to recall some facts about relativistic Kepler motion.[8] For circular motion

$$\begin{aligned} &E_0=m(1-v^2)^{1/2},\\ &J_0=e^2/v,\\ &r_0=e^2(1-v^2)^{1/2}/mv^2,\\ &v=r_0\omega_0. \end{aligned} \tag{22}$$

Furthermore, the stability frequency is $\omega_r=\omega_0(1-v^2)^{1/2}$, resulting in a perihelion advance. From (22) and $J_0=(l+\frac{1}{2})\hbar$ we find

$$\frac{E_0}{m}=\left(1-\frac{\alpha^2}{(l+\frac{1}{2})^2}\right)^{1/2}. \tag{23}$$

Finally, the radial energy is

$$\frac{E_r}{m}=\frac{(n_r+\frac{1}{2})\hbar\omega_r}{m}=\frac{(n_r+\frac{1}{2})v^3}{\alpha}=\frac{\alpha^2(n_r+\frac{1}{2})}{(l+\frac{1}{2})^3}$$

just as in the nonrelativistic case. The relativistic corrections in r_0 and ω_r cancel so that the radial motion does not contribute to higher order in α, and Eq. (21) is recovered.

Since the quasipotential theory has a structure like the Klein-Gordon equation, a similar analysis can be performed for it.

V. ACTION AT A DISTANCE

An old and frustrating problem is the quantization of classical action-at-a-distance electrodynamics[17] (Wheeler-Feynman theory). To date, the only thing which has been achieved is the quantization of circular orbits by means of Eq. (1), i.e., Bohr quantization.[15] Recently, almost circular orbits have been worked out in the classical theory.[18] We can now go one step further: We can apply Eq. (4) to perturbed circular orbits. Since the action integral underlying Wheeler-Feynman theory, the Fokker action, is not of the Hamiltonian type we must first identify Φ. In a Hamiltonian system

$$\int_0^T \vec{p}\cdot\dot{\vec{q}}\,dt=\int_0^T E\,dt+\int_0^T L\,dt. \tag{24}$$

If the energy is conserved, this becomes

$$\hbar\Phi(E)=ET+\int_0^T L\,dt. \tag{25}$$

We now replace the last term by the Fokker action, so that $\Phi(E)$ becomes

$$\hbar\Phi(E)=ET-m_p\int_0^T d\tau_p-m_e\int_0^T d\tau_e + e^2\int\int \delta(z_{pe}^2)\dot{z}_p^\mu\dot{z}_{e\mu}\,d\tau_p\,d\tau_e. \tag{26}$$

The Fokker action describes two particles with masses m_p and m_e and opposite charges interacting via $\frac{1}{2}$ the sum of retarded and advanced Liénard-Wiechert potentials. In (26) τ is proper time, $\dot{z}^\mu=dz^\mu/d\tau$, $z_{pe}=z_p^\mu-z_e^\mu$, and $z^\mu(\tau)$ label the world points of the particles. The limits of the double integral must be carefully adjusted so that a full period is covered in spite of the restriction on the range of integration imposed by the δ function. For circular orbits the total energy is[15]

$$E=m_p(1-v_p^2)^{1/2}+m_e(1-v_e^2)^{1/2}. \tag{27}$$

This simple result is a consequence of the relativistic virial theorem.[19] If (26) is evaluated for circular orbits, the first three terms cancel. The last term becomes exactly $2\pi J$, where J is given by Schild.[15] This result, which is not trivial, means that quantization of $\Phi(E)$ is equivalent to quantization of J, just as in the Hamiltonian case. It lends credence to the naive Bohr quantization of circular orbits.

In the limit $m_p/m_e\to\infty$ the relativistic Sommerfeld theory is recovered. In the limit $v\ll c$ the Darwin equation is obtained. These two limits serve as useful checks on the complicated problem of almost circular orbits in the full theory. In Sec. II we saw that semiclassical quantization is correct for both.

We have now assembled the tools for applying Eq. (4) to almost circular orbits. Consider the case $m_p=m_e=m$. Equation (27) becomes

$$\frac{E}{m}=2(1-v^2)^{1/2}. \tag{28}$$

The binding energy is defined by

$$\mathcal{E}=\frac{E-m_p-m_e}{\mu}=4(1-v^2)^{1/2}-4. \tag{29}$$

Now we assume

$$J=(l+\tfrac{1}{2})\hbar \tag{30}$$

and read off the energy levels for circular orbits from a plot of J vs $\mathcal{E}$ for circular orbits.[20]

In order to find the stability frequency it is useful to find the parameter $4v^2$ from (29). A plot of ω_r/ω_0 vs $4v^2$ provides ω_r as a function of the unperturbed orbital frequency ω_0.[18] This quantity, finally, can be found from $r_0=v/\omega_0$, also plotted as a function of $4v^2$. The resulting value for ω_r can now be inserted in Eq. (4) to complete the calculation.

VI. SUMMARY

The spectra for the nonrelativistic spinless Kepler or Coulomb problem, and for the separable formulation of the relativistic problem, are correctly reproduced by semiclassical quantization with the WKB quantum conditions (2). Since the quantum field theory is fairly well understood in these cases, this fact will be useful in the establishment of a rigorous transition from field theory through quantum mechanics to classical mechanics. Miller's new quantum condition (3), applicable to both separable and nonseparable problems, makes use of periodic orbits. Applied to the separable relativistic equations and circular orbits, the correct spectra are found only in the limit of small radial oscillations, i.e., in the limit of almost circular orbits. The leading term and first correction in an expansion in powers of $n_r \equiv n - l - 1$ are obtained in the semiclassical treatment.

The action-at-a-distance formulation of the relativistic two-body problem is not separable and not even Hamiltonian. Nevertheless, Miller's quantum condition, together with the classical determination of almost circular orbits, provides a method of quantization. In the case of circular orbits, without radial oscillations, the new quantum condition agrees with the naive Bohr quantization, provided, of course, Eq. (2) is used instead of Eq. (1). This circumstance strengthens the speculation[15] that Bohr quantization can be used for very tightly bound systems and provides a method of estimating the first correction to it. In the elucidation of the meaning of quantized action-at-a-distance theory an important question will be the interpretation of multiple stable modes of radial oscillations which appear at high energies.[18]

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