Statistical black-hole thermodynamics*

Jacob D. Bekenstein
Department of Physics, Ben Gurion University of the Negev, Beer Sheva 84120, Israel
(Received 12 May 1975)

Traditional methods from statistical thermodynamics, with appropriate modifications, are used to study several problems in black-hole thermodynamics. Jaynes's maximum-uncertainty method for computing probabilities is used to show that the earlier-formulated generalized second law is respected in statistically averaged form in the process of spontaneous radiation by a Kerr black hole discovered by Hawking, and also in the case of a Schwarzschild hole immersed in a bath of black-body radiation, however cold. The generalized second law is used to motivate a maximum-entropy principle for determining the equilibrium probability distribution for a system containing a black hole. As an application we derive the distribution for the radiation in equilibrium with a Kerr hole (it is found to agree with what would be expected from Hawking's results) and the form of the associated distribution among Kerr black-hole solution states of definite mass. The same results are shown to follow from a statistical interpretation of the concept of black-hole entropy as the natural logarithm of the number of possible interior configurations that are compatible with the given exterior black-hole state. We also formulate a Jaynes-type maximum-uncertainty principle for black holes, and apply it to obtain the probability distribution among Kerr solution states for an isolated radiating Kerr hole.

I. INTRODUCTION

The many *formal* analogies between black-hole physics and thermodynamics,[1-4] the most striking of which is that between Hawking's classical theorem[5] that "black-hole surface area cannot decrease" and the second law of thermodynamics that "entropy cannot decrease," make the attempt at a *physical* synthesis of the two disciplines worthwhile. Essential to the success of such a program is a version of the second law relevant to systems containing both matter and black holes. Earlier[1,3,6] we conjectured that this generalized second law (GSL) must take the form "black-hole entropy S_{bh} plus common (ordinary) entropy exterior to black holes S_c never decreases,"

$$\Delta S_g \equiv \Delta S_{bh} + \Delta S_c \geq 0 , \tag{1}$$

where S_{bh} is given in terms of black-hole surface area A by

$$S_{bh} = \eta \hbar^{-1} A \tag{2}$$

(units with $G=c=k=1$ are assumed). Here $\hbar$ is the Planck area (2.6×10^{-66} cm^2) and η is a numerical constant of order unity to be determined by independent arguments (see Sec. II).

The GSL is designed to replace the ordinary second law, which is transcended in the presence of a black hole because the common entropy interior to the hole is unobservable by exterior observers.[3,7] (Note that it is not claimed that S_{bh} equals the interior entropy.) On the other hand, the GSL makes a stronger statement than the area theorem. The theorem requires only that A not decrease. The GSL demands that if exterior entropy is lost into black holes, A increase sufficiently for the associated increase in S_{bh} to, at least, compensate for the decrease in S_c. That this actually happens has been verified for the case of the infall of a small entropy package into a generic stationary black hole.[7]

The area theorem, based as it is on a classical energy condition,[5] is expected to be violated by some quantum processes.[8] By contrast the GSL, being an intrinsically quantum law,[1,7] could be expected to fare better. And indeed in the astonishing quantum process of spontaneous radiation by a Kerr black hole[9] discovered by Hawking,[10] the area theorem is flagrantly violated, but the increase in exterior entropy due to the radiation is expected to suffice to uphold the GSL.[10] This confirmation of the validity of the GSL for a process not even dreamt of at the time of its inception is striking evidence of the versatility of the law and of the physical meaningfulness of the concept of black-hole entropy which underpins it. In view of these developments, the time appears ripe for considering black-hole thermodynamics from a statistical viewpoint with the ultimate purpose of clarifying the statistical significance of black-hole entropy.

We make use here of traditional methods of statistical thermodynamics to treat several interconnected problems in black-hole thermodymaics. Using Jaynes's information-theory approach for computing entropy,[11] we show explicitly in Sec. II that the GSL (in a statistically averaged form) is respected by the process of spontaneous radiation by an isolated Kerr black hole, as suggested by Hawking.[10] By the same approach we show in Sec. III that the GSL in averaged form is also respected in the case of a Schwarzschild hole immersed in a blackbody radiation bath regardless of the latter's temperature.

In Sec. IV we generalize the principle of maximum entropy to embrace black-hole entropy. As an application we determine the probability distribution for radiation in thermodynamic equilibrium with a Kerr hole and the associated probability distribution for the various Kerr solution states of definite mass. In Sec. V we show that the same results follow directly from a statistical interpretation of black-hole entropy as the natural logarithm of the number of possible interior configurations of the black hole which are compatible with the exterior state in question.[1,3] Finally in Sec. VI we formulate the black-hole analog of Jaynes's maximum-uncertainty principle for computing the best probability distribution, and apply it to the determination of the probability distribution over Kerr solution states of an isolated radiating Kerr hole.

II. THE GSL FOR A BLACK HOLE RADIATING IN ISOLATION

The increment in black-hole entropy of a Kerr black-hole solution is related to the increments in its mass M, angular momentum along the symmetry axis L, and charge Q by[1]

$$\Delta S_{\rm bh} = (\Delta M - \Omega\Delta L - \Phi\Delta Q)/T_{\rm bh} \tag{3}$$

where Ω and Φ are the rotational frequency and electric potential of the hole, respectively,[12,13] and

$$T_{\rm bh} = \frac{\hbar(M^2 - L^2/M^2 - Q^2)^{1/2}}{32\pi\eta M[M - \frac{1}{2}Q^2/M + (M^2 - L^2/M^2 - Q^2)^{1/2}]} \,. \tag{4}$$

Interpreting $S_{\rm bh}$ seriously as entropy means interpreting $T_{\rm bh}$ as black-hole temperature.[1,3] The very literal character of this temperature is nowhere clearer than in Hawking's process of spontaneous emission of thermal radiation by a Kerr hole.[10]

Hawking has shown that for a Kerr hole formed by collapse, quantum theory indicates that at late times the hole emits all types of quanta to infinity at a uniform rate. Into each mode of radiation specified by frequency ω, azimuthal angular momentum quantum number m, and charge ϵ, the *mean* number of quanta emitted is

$$\langle n\rangle = \Gamma(e^x \mp 1)^{-1} \,, \tag{5}$$

where Γ is the absorptivity of the hole for radiation in the given mode (the fraction of radiation it absorbs),

$$x = (\hbar\omega - \hbar m\Omega - \epsilon\Phi)(4\eta T_{\rm bh})^{-1} \,, \tag{6}$$

and the upper (lower) sign corresponds to a boson (fermion) mode. As stressed by Hawking, the spectrum (5) is just the one expected for an ordinary body with absorptivities Γ at temperature $4\eta T_{\rm bh}$. The thermodynamic[1,3] and quantum[10] approaches thus agree in ascribing to a Kerr black hole a temperature of the same form, an agreement which justifies the choice $\eta = \frac{1}{4}$ in (2). This unambiguous way to fix η is to be contrasted with the earlier heuristic approach,[3] which arrives at a value an order of magnitude smaller.

During the radiation process the area of the hole can decrease so that the area theorem can be violated. However, the radiation entropy increases. Given the analogy between black hole and hot body, one can expect that just as for the latter the body's entropy plus the radiation entropy increases. An argument like this is implicit in Hawking's suggestion[10] that the GSL is respected in the black-hole radiation process. Here we shall prove this result explicitly, not only for its intrinsic interest but also to introduce some concepts for later use.

By conservation of energy the mean change in mass of the black hole in a certain time interval is

$$\Delta M = -\sum \langle n\rangle \hbar\omega \,, \tag{7}$$

where the sum extends over all modes outgoing from the hole in the given time interval and n is given by (5). Analogous expressions can be given for ΔL and ΔQ. From (3) it then follows that the associated mean change in black-hole entropy is

$$\langle \Delta S_{\rm bh}\rangle = -\sum x\,\Gamma(e^x \mp 1)^{-1} \,. \tag{8}$$

We now proceed to calculate the radiation entropy generated in the same time interval. Each radiation state is uniquely specified by the set of occupation numbers of the various modes $\{n\}$; it will occur with some probability $p_{\{n\}}$ which is subject to the normalization condition

$$\sum_{\{n\}} p_{\{n\}} = 1 \,, \tag{9}$$

where the sum runs over all distinct sets $\{n\}$ which are relevant. The entropy of the radiation can be written in the familiar form[11]

$$S = -\sum_{\{n\}} p_{\{n\}} \ln p_{\{n\}} \,. \tag{10}$$

Unfortunately, the original quantum treatment of the problem[10] does not determine the $p_{\{n\}}$, but only the mean occupation number for each mode

$$\sum_{\{n\}} n' p_{\{n\}} = \langle n'\rangle \,, \tag{11}$$

where the prime indicates the particular mode in question and $\langle n' \rangle$ is again given by (5). According to the principles of information theory, as applied by Jaynes[11] to situations like ours (even nonequilibrium ones), the "best" (or least biased) probability distribution one can choose is that which maximizes S subject to whatever information is known. Here this information is given by (9) and (11).

Thus our probability distribution is determined by the variational principle

$$\delta\Big[-\sum_{\{n\}} p_{\{n\}}\ln p_{\{n\}} - \sum_{\text{mod}} \beta \sum_{\{n\}} n' p_{\{n\}} - (\alpha-1)\sum_{\{n\}} p_{\{n\}}\Big] = 0\,, \quad (12)$$

where $\alpha - 1$ and the β (one for each outgoing mode) are Lagrange multipliers. Varying all the $p_{\{n\}}$ independently we get

$$p_{\{n\}} = e^{-\alpha}\prod e^{-\beta n}\,, \quad (13)$$

an expression which agrees with that which follows from a recent full quantum treatment of the problem (see Appendix for details). Substituting this into (10) and using (9) and (11) we get for the radiation entropy

$$S = \alpha + \sum \beta \langle n \rangle \quad (14)$$

where, as before, the sum covers all the modes outgoing from the black hole in the time interval in question. Combining this result with (8) and using (5) we get

$$\langle \Delta S_g \rangle = \langle \Delta S_{\text{bh}} \rangle + \Delta S_c$$
$$= \alpha + \sum \Gamma(\beta - x)(e^x \mp 1)^{-1}\,. \quad (15)$$

We shall now show that every term in this expression is positive; this will establish the validity of the GSL in averaged form for the process in question.

First, we determine e^α by substituting (13) into the normalization condition (9) and remembering that n can take values from 0 to ∞ for boson modes, and from 0 to 1 for fermion modes. Thus

$$e^\alpha = \prod_{\text{bos}} (1 - e^{-\beta})^{-1} \prod_{\text{fer}} (1 + e^{-\beta})\,. \quad (16)$$

Similarly, substituting (13) into the mean occupation number condition (11) and using (5) and (16) gives the implicit expression for the β

$$(e^\beta \mp 1)^{-1} = \Gamma(e^x \mp 1)^{-1}\,. \quad (17)$$

We must now make a detour and consider the separate problem of scattering of a radiation mode off the hole. For a boson mode with $x \geq 0$ the hole absorbs some (or all) of the incident quanta, i.e., $0 < \Gamma \leq 1$; it follows from (17) that $\beta \geq x \geq 0$. By contrast, a boson mode with $x < 0$ is superradiant,[13,8] i.e., it is amplified by the hole upon scattering (stimulated emission) so that $\Gamma < 0$. Since the right-hand side of (17) is positive, $\beta > 0$ again. Thus, for all boson modes $\beta > 0$. This is crucial; otherwise the sums (9) and (11) would clearly diverge, as they would be sums of terms increasing without bound as the n increase. Also, we can now see from (16) that α is a sum of positive terms, one for each mode.

Fermion modes cannot superradiate; the Pauli exclusion principle forbids stimulated emission. Hence for all such ($0 < \Gamma < 1$) it follows from (17) that $\beta > x$. We thus find that for *all* types of modes $\beta - x > 0$. In view of this, it is now clear that each term in the sum in (15) is positive since Γ and $e^x \mp 1$ always have the same sign.

We have shown that

$$\langle \Delta S_g \rangle \equiv \langle \Delta S_{\text{bh}} \rangle + \Delta S_c > 0\,, \quad (18)$$

where the average is over the distribution $p_{\{n\}}$. Thus we find that in the Hawking process the GSL is respected on the average; the radiation entropy generated exceeds the mean decrease in black-hole entropy. This is the most one can show, for the GSL is a statistical law susceptible to violation by statistical fluctuations.[7]

III. THE GSL FOR A BLACK HOLE IN A RADIATON BATH

We noted earlier[7] that if a Kerr hole is placed in a blackbody cavity of temperature T sufficiently low in relation to T_{bh}, then the flow of radiation from the cavity into the hole (lower to higher temperature) will violate the GSL unless some process generates entropy outside the hole. We suggested a complicated mechanism to accomplish this. Now that we know about spontaneous black-hole radiation, a simpler resolution of the difficulty becomes possible, as mentioned by Hawking[10]: The GSL may be upheld by the generation of radiation entropy by the hole itself. We shall now show that indeed the change in radiation entropy plus the mean change in black-hole entropy is non-negative for all values of T.

The following proof applies only to the case that the hole is of the Schwarzschild type ($\Phi = \Omega = 0$), and that the cavity is uncharged and nonrotating. As is well known, the mean occupation number for a mode of frequency ω of the blackbody radiation is

$$\langle n \rangle = (e^y \mp 1)^{-1}\,, \quad (19)$$

where

$$y = \hbar\omega/T\,. \tag{20}$$

When the mode scatters on the hole, the mean number of quanta absorbed is a fraction Γ of the $\langle n\rangle$ given by (19). The mean number of outgoing quanta is a fraction $1-\Gamma$ of that $\langle n\rangle$ (scattered) plus the number given by (5) (spontaneously emitted).[14] From (3) and (7) we can write the mean change in black-hole entropy over a certain time interval as

$$\langle \Delta S_{\rm bh}\rangle = \sum x\Gamma[(e^y \mp 1)^{-1} - (e^x \mp 1)^{-1}]\,, \tag{21}$$

where the sum runs over all modes that scatter off the hole in the given time interval. Expression (21) takes absorption and spontaneous emission into account.[14]

The associated change in exterior common entropy is the entropy of the radiation outgoing at some large distance (scattered and emitted) minus the entropy of the blackbody radiation incoming at that distance. The latter entropy can again be computed by the method used in Sec. II. It is given by an expression like (14) together with (16), with the β's determined by a condition like (11) together with (19). Not surprisingly, they turn out to be just the y's. Therefore, the incoming entropy is

$$\Delta S_{\rm i} = \sum [\mp \ln(1 \mp e^{-y}) + y(e^y \mp 1)^{-1}]\,. \tag{22}$$

We follow the same procedure for computing the outgoing radiation entropy. The parameters analogous to the β's, which we call γ's, are determined by the following generalization of (17):

$$(e^\gamma \mp 1)^{-1} = (1-\Gamma)(e^y \mp 1)^{-1} + \Gamma(e^x \mp 1)^{-1}\,. \tag{23}$$

The first term on the right-hand side is the contribution of scattering, and the second is that of spontaneous emission.[14] The outgoing entropy is given by

$$\Delta S_{\rm o} = \sum [\mp \ln(1 \mp e^{-\gamma}) + \gamma(e^\gamma \mp 1)^{-1}]\,. \tag{24}$$

We will now show that each mode's contribution to $\langle \Delta S_g\rangle \equiv \langle \Delta S_{\rm bh}\rangle + \Delta S_{\rm o} - \Delta S_{\rm i}$ is non-negative because it has a unique extremum, a minimum, with respect to T at $T = T_{\rm bh}$ which vanishes. For this purpose we need several intermediate results. First, from the definitions of x, y, and γ, (6), (20), and (23) respectively, it is clear that for each mode

$$\gamma = y = x \quad \text{at } T = T_{\rm bh} \tag{25}$$

and nowhere else. For the Schwarzschild case $x \geqslant 0$ always, so that superradiance is absent; therefore, for every mode $0 < \Gamma < 1$. It then follows from (23) that γ has a value intermediate between x and y. Differentiation of the relation (23) with respect to T (which we denote by a prime) followed by replacement of $(e^\gamma \mp 1)^{-1}$ from (23) itself shows that

$$\gamma' = (1-\Gamma)y' \quad \text{at } T = T_{\rm bh}\,. \tag{26}$$

Now we return to our main problem. Constructing $\langle \Delta S_g\rangle$ from (21), (22), and (24) we see by virtue of (25) that at $T = T_{\rm bh}$ this quantity vanishes mode by mode. Taking the derivative of $\langle \Delta S_g\rangle$ with respect to T we get after some cancellations

$$\langle \Delta S_g\rangle' = \sum \{\gamma[(e^\gamma \mp 1)^{-1}]' - (y - \Gamma x)[(e^y \mp 1)^{-1}]'\}\,. \tag{27}$$

This can be simplified by means of the derivative of (23) to yield

$$\langle \Delta S_g\rangle' = -\sum (\gamma - \Gamma\gamma - y + \Gamma x)y' e^y (e^y \mp 1)^{-2}\,. \tag{28}$$

Let us focus on a single mode's contribution to this sum. It can vanish only where $\gamma = (y - \Gamma x)\times(1-\Gamma)^{-1}$, or, equivalently, where $y - \gamma = \Gamma(x - y)\times(1-\Gamma)^{-1}$. But γ must lie between x and y. Hence, the two sides of this relation have opposite signs and cannot be equal unless they both vanish, which is true only when $x = y = \gamma$, i.e., only when $T = T_{\rm bh}$ according to (25). Thus each mode's contribution to $\langle \Delta S_g\rangle$ has a *unique* extremum with respect to T at $T = T_{\rm bh}$. Taking one more derivative of $\langle S_{\rm bh}\rangle$, and evaluating it at $T = T_{\rm bh}$ with the help of (25) and (26), we get

$$\langle \Delta S_g\rangle'' = \sum [1 - (1-\Gamma)^2](y')^2 (e^y \mp 1)^{-2} e^y \quad \text{at } T = T_{\rm bh}\,. \tag{29}$$

Since $0 < \Gamma < 1$, it is clear that each mode contributes positively to $\langle \Delta S_g\rangle''$.

Our conclusion, then, is that each mode's contribution to $\langle \Delta S_g\rangle$ has a unique extremum with respect to T at $T = T_{\rm bh}$, always a minimum, which vanishes. Hence each mode contributes positively to $\langle \Delta S_g\rangle$ for all T, except at $T = T_{\rm bh}$, where $\langle \Delta S_g\rangle = 0$. Thus the GSL in averaged form is obeyed: The mean change in black-hole entropy plus the change in exterior entropy is non-negative for all T and vanishes only when the black hole has the same temperature as the cavity. Note that the limiting case $T \to 0$ corresponds to an isolated black hole, a case we dealt with in more generality in Sec. II.

IV. GENERALIZED MAXIMUM-ENTROPY PRINCIPLE FOR EQUILIBRIUM SYSTEMS

The GSL in averaged form has been validated for a variety of situations (Secs. II, III and Ref. 7), so we may well rely on it in general. It states that the mean change in generalized entropy S_g (the mean change in black-hole entropy plus the change in exterior common entropy) is non-negative. The mean of ΔS_{bh} implied here is to be calculated with the probability distribution for the exterior matter or radiation states as this relates to the state of the black hole through the conservation laws [see, for example, the derivation leading to (8)]. Alternatively, one could define the associated probability distribution for black-hole states and use it to calculate the average $\langle \Delta S_{bh}\rangle$ and the average $\langle \Delta S_g\rangle$. The fact that $\langle \Delta S_g\rangle$ is non-negative indicates that, for a system in thermodynamic equilibrium containing a black hole, $\langle S_g\rangle$ should be a maximum for the given total energy, angular momentum, and charge. This observation serves to motivate the following generalized maximum-entropy principle (GMEP): The probability distribution for a system in thermodynamic equilibrium containing a black hole is that which maximizes $\langle S_g\rangle \equiv \langle S_{bh}\rangle + S_c$ subject to the relevant constraints.

Let us illustrate this principle by considering the problem of a Kerr hole formed by collapse which is enclosed in a container with walls that perfectly reflect all the relevant radiations. The hole will radiate spontaneously an average of $\Gamma(e^x \mp 1)^{-1}$ quanta into each outgoing mode according to (5). The quanta go out, are reflected by the walls, return, and scatter off the hole. The hole returns a fraction $(1-\Gamma)$ of them into the outgoing mode together with an average of $\Gamma(e^x\mp 1)^{-1}$ freshly emitted quanta. The process repeats itself. In the limit the mean number of quanta in each mode is given by

$$\langle n\rangle = (e^x \mp 1)^{-1}[\Gamma + (1-\Gamma)\Gamma + (1-\Gamma)^2\Gamma + \cdots]. \tag{30}$$

For a nonsuperradiant mode $0<\Gamma<1$, so the series converges to unity and the limiting $\langle n\rangle$ is just the standard Bose (Fermi) one for temperature T_{bh}. For superradiant modes $\Gamma<0$, so the series diverges. This is not surprising. Those modes are amplified at each scattering and grow indefinitely so long as the superradiance condition is satisfied. We reach the conclusion that the black hole eventually reaches thermodynamic equilibrium only with the nonsuperradiant modes in the container.

Let us now see what account the GMEP gives of the equilibrium subsystem black hole plus nonsuperradiant modes by itself. Each possible state of the radiation in these modes is specified by a set of occupation numbers $\{n\}$ and occurs with some probability $p_{\{n\}}$ subject to the constraint (9). Owing to drainage into the superradiant modes, the total energy of our equilibrium subsystem decreases secularly in time. But we can still think of this energy as well defined at any given instant. The energy is shared between black hole and radiation in accordance with the probability distribution $p_{\{n\}}$. Similar statements can be made about angular momentum and charge. The black hole, then, cannot be regarded as having definite M, L, or Q. Rather, it can be in a number of different Kerr black-hole solution states[9] of definite M, L, and Q, each one occurring with some probability P_{MLQ}. For convenience we shall think of the spectrum of M, L, and Q as totally discrete. There are indeed some quantum reasons for this viewpoint.[15]

The entropy of the radiation is given as usual by (10). The mean black-hole entropy is clearly given by

$$\langle S_{bh}\rangle = \sum_{MLQ} P_{MLQ} S_{bh}(M,L,Q), \tag{31}$$

where $S_{bh}(M,L,Q)$ is the black-hole entropy of the Kerr black-hole solution with parameters M, L, and Q. The sum in (31) runs over the entire allowed spectrum of M, L, and Q, but only for $M \le M_0$, where M_0 is the time-dependent total energy of black hole plus radiation in nonsuperradiant modes. It is clear that each P_{MLQ} equals the sum of the $p_{\{n\}}$ for all the radiation states which can coexist with the given Kerr solution state, i.e., radiation states with occupation numbers satisfying

$$M = M_0 - \sum n\hbar\omega \ge 0, \tag{32}$$

$$L = L_0 - \sum n\hbar m, \tag{33}$$

$$Q = Q_0 - \sum n\epsilon, \tag{34}$$

where the sums go over all the nonsuperradiant modes, and L_0 and Q_0 are, respectively, the (time-dependent) total angular momentum and charge shared by the black hole and the radiation in these modes. We can thus write the GMEP in the particular form $\delta(\langle S_{bh}\rangle + S_c) = 0$, or

$$\delta \sum_{\{n\}} p_{\{n\}}[-\ln p_{\{n\}} + S_{bh}(M,L,Q) - (\alpha - 1)] = 0, \tag{35}$$

where M, L, and Q are given by (32)–(34), and $\alpha - 1$ is a Lagrange multiplier.

Independent variation of the $p_{\{n\}}$ gives

$$p_{\{n\}} = e^{-\alpha}\exp[S_{bh}(M,L,Q)], \tag{36}$$

where α is to be determined by the normalization condition (9). For radiation states with $\sum n\hbar\omega \ll M_0$, $|\sum n\hbar m| \ll |L_0|$, and $|\sum n\epsilon| \ll |Q_0|$ we can expand the argument of the exponential in (36) about M_0, L_0, and Q_0. Using (3) to compute the necessary derivatives we get

$$p_{\{n\}} \propto \exp\left(-\sum nx\right), \tag{37}$$

where the T_{bh}, Ω, and Φ that come into the x are computed with the instantaneous values of M_0, L_0, and Q_0. The distribution (37) has the Boltzmann form for temperature T_{bh}. The mean occupation numbers that follow from (37) are identical to those given by (30) which were inferred from Hawking's result. This agreement lends support to our formulation of the GMEP.

Let us now compute the P_{MLQ}. Each one is the sum of the $p_{\{n\}}$ for all radiation states that can coexist with the given Kerr solution state. According to (36) and (32)–(34), all these probabilities are identical. Let g_{MLQ} be the number of radiation states in question. Then

$$P_{MLQ}=g_{MLQ}e^{-\alpha}\exp[S_{bh}(M,L,Q)]\,, \tag{38}$$

where it is understood that $M \leq M_0$.

Although we shall not attempt to calculate the g_{MLQ} here (we do this in Sec. VI), it is physically evident that this factor should increase rapidly as M decreases, i.e., as more energy is available for the radiation. By contrast the exponential in (38) increases rapidly with M. The P_{MLQ}, being the product of a factor which increases rapidly with M and one that decreases rapidly with M, will have a sharp peak at some M below M_0. Thus, although a Kerr black hole in equilibrium with radiation does not, rigorously speaking, have a precise mass, it does have a fairly sharply peaked distribution of possible masses.

V. INTERIOR CONFIGURATIONS OF A BLACK HOLE

Statistically, thermal entropy can be regarded as the natural logarithm of the number of possible distinct microscopic configurations of the system compatible with the macrostate in question.[16] In introducing the black-hole entropy,[1,3] we suggested by analogy that it may be interpreted as the natural logarithm of the number of possible distinct interior configurations of a black hole compatible with the exterior black-hole state (for example, a Kerr solution state) in question. We shall now find support for this conjecture in that it leads very directly to the results we obtained in the preceding section.

For this purpose we again consider a Kerr hole in equilibrium with radiation in nonsuperradiant modes within a container. Each distinct "microstate" of the system consists of a radiation state specified by occupation numbers and an interior black-hole configuration. We make the traditional postulate that all microstates of a system in equilibrium are equally probable (equal *a priori* probabilities).[16] It follows that the probability of a radiation state regardless of which interior configuration it goes with must be proportional to the number of configurations compatible with it, or, equivalently, compatible with the Kerr solution state associated with that radiation state. By our conjecture this number is just the exponential of the black-hole entropy of the Kerr solution state. We thus recover our previous expression (36) for $p_{\{n\}}$. The results (37) and (38) that follow from it are obtained just as before. We see that indeed a black hole in a given Kerr solution state behaves as if it can be in any of a number of equally probable interior configurations, with the logarithm of this number giving the black-hole entropy. An interesting way to look at these interior configurations in a particular model of collapse has been given by Gerlach.[17]

It is interesting to contrast the approaches of Sec. IV and the present section, which both lead to the same results. In Sec. IV we use a principle motivated by the GSL. It treats the black hole as a black box, and does not attempt to give an interpretation to S_{bh}, but simply takes it as a property of the exterior state of the hole. In the present section the GSL does not come in in any way, but S_{bh} is given a statistical interpretation in terms of the black-hole interior. No two approaches could be more dissimilar, and the agreement between their results speaks for the self-consistency of the ideas we have made use of.

VI. JAYNES'S MAXIMUM-UNCERTAINTY PRINCIPLE FOR BLACK HOLES

The GMEP is of use only for equilibrium systems. Something else is needed for nonequilibrium situations involving black holes; for example, one problem is determining the distribution P_{MLQ} for a Kerr hole radiating in isolation. We may here be guided by Jaynes's maximum-uncertainty principle, already applied in Secs. II and III, which states that the best probability distribution one can assign to the states i of a system is the one which maximizes the information theoretical uncertainty (or entropy),

$$S=-\sum_i p_i \ln p_i\,, \tag{39}$$

subject to the known information. This principle is supposed to apply to nonequilibrium systems, as well as to equilibrium ones. For the latter it is identical to the usual maximum-entropy principle. Here we would like to formulate a Jaynes-type principle for a Kerr hole by itself.

The states i of which the principle speaks are the elementary states of the system. For a black hole these would be the interior configurations and not the Kerr solution states, which are classes of interior configurations. Nevertheless, since only the Kerr solution states are externally observable, it is advantageous to reexpress S in terms of the P_{MLQ}. It seems reasonable to assume that all interior configurations for a given Kerr solution state are equally probable—they all share the same dynamical quantities M, L, and Q and consequently are totally equivalent to an exterior observer. Indeed this view is consistent with the results of Sec. V. If so, for given M, L, and Q

$$p_i = P_{MLQ}/N_{MLQ}, \tag{40}$$

where N_{MLQ} is the number of interior configurations corresponding to the given Kerr solution state. Substituting this into (39), remembering that $\ln N_{MLQ} = S_{bh}(M,L,Q)$, and summing over the N_{MLQ} identical terms for each Kerr solution state, we get

$$S = -\sum_{MLQ} P_{MLQ} \ln P_{MLQ} + \sum_{MLQ} P_{MLQ} S_{bh}(M,L,Q). \tag{41}$$

The first contribution is the uncertainty as to which Kerr solution state the black hole is in; the second represents the uncertainty intrinsic to the Kerr solution states themselves. It is important to stress that the uncertainty to which Jaynes's principle refers is something more than $\langle S_{bh}\rangle$ which appears in the GSL or the GMEP.

Let us apply the Jaynes principle to the determination of the distribution P_{MLQ} for a Kerr hole radiating in isolation. That the hole cannot have definite M, L, and Q follows from the fact that the energy, angular momentum, and charge radiated are subject to a probability law (13). However, since Hawking's result (5) allows the determination of the mean energy, angular momentum, and charge radiated, we can regard $\langle M\rangle$, $\langle L\rangle$, and $\langle Q\rangle$ as known information. The problem then reduces to determining the distribution P_{MLQ} which maximizes

$$S + \sum_{MLQ} P_{MLQ}(1 - \alpha - \mu M + \nu L + \sigma Q), \tag{42}$$

where $1-\alpha$, $-\mu$, ν, and σ are Lagrange multipliers used to introduce all known information as constraints. Variation of each P_{MLQ} independently gives

$$P_{MLQ} = e^{-\alpha-\mu M+\nu L+\sigma Q} \exp[S_{bh}(M,L,Q)], \tag{43}$$

where the α is to be determined from the normalization condition, and the μ, ν, and σ from the condition that the mean values of M, L, and Q be as prescribed. This all can be done rigorously only if the spectrum of M, L, and Q is known.

Instead of taking this approach we shall adopt a simpler one based on the physical expectation that the probability distribution should have a fairly sharp peak at some values $M = M_0$, $L = L_0$, and $Q = Q_0$. This will guarantee that the black hole has rather well-defined mass, angular momentum, and charge. Expanding S_{bh} about M_0, L_0, and Q_0 with the help of (3), and requiring that terms linear in M, L, and Q in the exponent be absent (so that P_{MLQ} can have a peak), we get the conditions

$$\mu = T_{bh}^{-1}, \tag{44}$$

$$\nu = \Omega T_{bh}^{-1}, \tag{45}$$

$$\sigma = \Phi T_{bh}^{-1}, \tag{46}$$

where the quantities on the right-hand sides are evaluated at M_0, L_0, and Q_0, which should be close to $\langle M\rangle$, $\langle L\rangle$, and $\langle Q\rangle$, respectively. It follows that the true parameters of a Kerr hole in isolation are not M, L, and Q, as is usually said,[9] since they are random variables, but rather T_{bh}, Ω, Φ, which are associated with $\langle M\rangle$, $\langle L\rangle$, and $\langle Q\rangle$.

The quadratic terms of the expansion make P_{MLQ} into a trinormal (tri-Gaussian) distribution in the variables $M - M_0$, $L - L_0$, and $Q - Q_0$. The dispersions of these quantities are found to be

$$\Delta M = (\partial T_{bh}^{-1}/\partial M)^{-1/2} \sim \hbar^{1/2} \sim 10^{-5}\text{ g}, \tag{47}$$

$$\Delta L = (\partial \Omega T_{bh}^{-1}/\partial L)^{-1/2} \sim \hbar^{1/2}\langle M\rangle \sim (\langle M\rangle/10^{-5}\text{ g})\hbar, \tag{48}$$

$$\Delta Q = (\partial \Phi T_{bh}^{-1}/\partial Q)^{-1/2} \sim \hbar^{1/2} \sim 12e. \tag{49}$$

It is curious that ΔM and ΔQ are the same for black holes of all sizes. Both ΔM and ΔL exceed by the large factor $M/10^{-5}$ g the quantum-level spacing of the corresponding quantities obtained by one line of reasoning,[15] whereas ΔQ is always several elementary charges. Thus, the discrete nature of the quantum spectrum of Kerr solution states cannot manifest itself directly for a Kerr hole in a "natural" state. Nevertheless, the width of the distributions of M, L, and Q is tiny in relation to the $\langle M\rangle$, $\langle L\rangle$, and $\langle Q\rangle$ corresponding to a macroscopic black hole.

The parameters T_{bh}, Ω, and Φ clearly vary in time for a radiating black hole. Their evolution can be determined from the rates of change of $\langle M\rangle$, $\langle L\rangle$, and $\langle Q\rangle$ which one can calculate from Hawking's results.

We recall that the probability p_i of an interior configuration is the fraction $\exp(-S_{bh})$ of the P_{MLQ} for the corresponding Kerr solution state. From (43) it follows that

$$p_i = e^{-\alpha} \exp[-(M - \Omega L - \Phi Q)/T_{bh}]. \tag{50}$$

We see that this probability distribution is of the (canonical) Boltzmann type for temperature T_{bh}. Probabilities of those configurations with high masses, or with angular momenta or charge of opposite sign to the angular velocity or electric potential, respectively, are strongly suppressed. This is physically appealing; for example, a Kerr hole with positive electric potential should contain negatively charged interior configurations only with fantastically small probabilities.

Although we have developed the previous ideas for an isolated hole, they should be valid also for a hole in a container in equilibrium with its own radiation. In this latter case the mean mass, angular momentum, and charge of the hole can be regarded as known—they are just the total mass, angular momentum, and charge of the black hole and radiation in nonsuperradiant modes minus the mean values of these quantities for the radiation itself. Since the latter depend on the size of the container, T_{bh}, Ω, and Φ should also depend on it. Comparing (43) with (38) we can see that

$$g_{MLQ} \propto \exp[-(M - \Omega L - \Phi Q)/T_{bh}]. \tag{51}$$

This gives the number of radiation states that can be associated with the Kerr solution state described by M, L, Q.

ACKNOWLEDGMENTS

The author thanks Professor J. A. Wheeler, Dr. D. A. Owen, and Dr. R. Wald for helpful comments.

APPENDIX

Wald[18] has implicitly given an expression obtained by a full quantum treatment for the probability that a Schwarzschild hole emits n bosons into a given mode:

$$p_n \propto \sum_{m=m}^{\infty} e^{-m\hbar\omega/T} \Gamma^n (1-\Gamma)^{m-n} \frac{m!}{n!\,(m-n)!}. \tag{A1}$$

We shall show that this formula is equivalent to that given in Sec. II of the present paper.

From (13) it follows after summation over all modes except that in question that the probability that the black hole emits n quanta into that mode is

$$p_n \propto e^{-\beta n}, \tag{A2}$$

where according to (17) for bosons

$$e^{-\beta} = \Gamma e^{-x}[1 - (1-\Gamma)e^{-x}]^{-1}. \tag{A3}$$

We shall make use of the expansion (good for $|z|<1$)

$$(1-z)^{-n-1} = \sum_{m=n}^{\infty} \frac{m!}{n!\,(m-n)!} z^{m-n}, \tag{A4}$$

which follows from n-fold differentiation of the expansion

$$(1-z)^{-1} = \sum_{m=0}^{\infty} z^m. \tag{A5}$$

Taking $z = (1-\Gamma)e^{-x}$ in (A4) and using this in conjunction with (A3) we get

$$e^{-\beta(n+1)} = \Gamma^{n+1} e^{-x(n+1)} \times \sum_{m=n} (1-\Gamma)^{m-n} e^{-(m-n)x} \frac{m!}{n!\,(m-n)!}. \tag{A6}$$

Therefore,

$$p_n \propto e^{-\beta n} = e^{\beta - x}\Gamma \times \sum_{m=n}^{\infty} e^{-mx} \Gamma^n (1-\Gamma)^{m-n} \frac{m!}{n!\,(m-n)!}. \tag{A7}$$

With the Schwarzschild value $x = \hbar\omega/T$ this is just the expression given by Wald, since the factor $e^{\beta - x}\Gamma$ is independent of n, and is eventually absorbed in the normalization factor.

Thus the information theory approach of Sec. II yields, already in closed form, the same probability distribution that is obtained from a full quantum treatment of the problem.

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*Work supported in part by a grant from the Israel Academy of Sciences.

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