Black holes and magnetic fields

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(Received 10 March 1975; revised manuscript received 16 June 1975)

The possible stationary axisymmetric electromagnetic fields in a vacuum cavity between an initially neutral black hole and a surrounding plasma shell are investigated. It is shown that such fields must be near "uniform" (in a sense defined in the paper) and that the flux of the magnetic field across one half of the surface of a neutral hole (of fixed mass) decreases as the angular momentum of the hole increases.

I. INTRODUCTION

In this paper we shall study stationary axisymmetric fields surrounding a rotating neutral black hole. The kind of situation we have in mind is one in which a supernova explosion leaves a collapsing core surrounded by a vacuum region which is in turn surrounded by a highly conducting supernova remnant. Such a collapsing core might possess a magnetic moment and be unable to radiate it away to infinity because of the presence of the plasma, as has been pointed out by Hájiček.[1] In any realistic situation the effect of both the magnetic field and the plasma on the space-time geometry of the collapsing core will be negligible, so one would expect that the geometry would eventually settle down to that of a Kerr black hole, while the magnetic field could be treated as a test field on the Kerr background. One might hope that this field would attain a stationary state after a time, and it is natural to assume that it must then be axisymmetric. Thus the object under study will be a Kerr black hole which *does* possess (stationary axisymmetric) "hair"—this hair being precisely those parts of the magnetic field normally discarded[2,3] because they do not die away quickly enough at infinity. Here of course a plasma cloud shields these fields from a distant observer. We shall see that because of selective charge accretion an initially neutral hole has a tendency to acquire a nonzero over-all charge. This charge is, however, small enough to be safely treated as a test field in the Kerr background. (For holes with charges large enough to affect the geometry Gibbons[4] and Carter[5] have shown that quantum effects will be important unless $M_{\text{hole}} \gtrsim 10^5 M_\odot$.)

Section II describes the geometrical formalism used in the paper. We introduce here an orthonormal tetrad which is very useful for calculations on or near the horizon of the hole. Section III gives the complex electromagnetic field components and the invariants. Section IV discusses the fields and the magnetic flux across the horizon of the hole, while Sec. V is the Conclusion.

II. GEOMETRICAL FORMALISM

We shall be interested in calculating quantities such as the flux of magnetic field across constant-time sections of the future horizon of the black hole, so it is important to use coordinates which cover this region in a regular way. We therefore give the metric in the Kerr form

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right)dv^2 - 2dr\,dv - \Sigma\, d\theta^2 - \frac{A}{\Sigma}\sin^2\theta\, d\phi^2 + 2a\sin^2\theta\, dr\,d\phi + \frac{4Mar}{\Sigma}\sin^2\theta\, dr\,d\phi\,, \tag{2.1}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta\,,$$
$$A = (r^2+a^2)^2 - \Delta a^2\sin^2\theta\,,$$
$$\Delta = r^2 - 2Mr + a^2$$

and M and a are constants giving the mass and specific angular momentum of the hole as seen from infinity, with $M > |a|$. The future horizon of the hole is the null hypersurface

$$r = r_+ = M + (M^2 - a^2)^{1/2}\,,$$

so that $\Delta(r_+) = 0$.

To calculate the flux of electric and magnetic fields across portions of the "surface" of the black hole, one needs to choose first constant-time sections of the horizon, that is, spacelike 2-surfaces H which lie in the null hypersurface $r = r_+$. One then constructs the normal bivector $N^{[\mu}M^{\nu]}$ to H out of two normals to H, where N^μ, M^ν are unit timelike and spacelike vectors, respectively, and square brackets denote antisymmetrization. Given the area element dS on H, the required fluxes follow on integrating expressions like

$$F_{\mu\nu}N^{[\mu}M^{\nu]}dS \text{ and } F^*_{\mu\nu}N^{[\mu}M^{\nu]}dS\,,$$

where $F_{\mu\nu}$ is the electromagnetic field tensor and

$F^*_{\mu\nu}$ its dual defined by

$$F^*_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\lambda\tau}F^{\lambda\tau}.$$

In the present case it is clear that the 2-surfaces H should be just the invariantly defined surfaces

$$\{r=r_+,\ v=\text{constant}\}.$$

A simple way to construct the bivector $N^{[\mu}M^{\nu]}$ is to find an orthonormal basis $\{\vec{E}_a\}$ of vectors such that $\vec{E}_2, \vec{E}_3$ lie in the 2-surfaces $\{r=\text{constant},\ v=\text{constant}\}$ at every point while $\vec{E}_0, \vec{E}_1$ are, respectively, unit timelike and spacelike orthogonal vectors. A possible candidate for such a basis is the ZAMO (zero-angular-momentum observer) basis $\{\vec{e}_{(a)}\}$ of Bardeen[6]; but this basis becomes singular precisely where we want to use it, namely on the horizon $r=r_+$. This is because the timelike ZAMO vector $\vec{e}_{(t)}$ is defined to be orthogonal to the hypersurfaces $\{t=\text{constant}\}$ (where t is the Boyer-Lindquist time coordinate) and these hypersurfaces become null at the horizon. However, we can overcome this if we apply to the ZAMO vectors $\vec{e}_{(t)}, \vec{e}_{(r)}$ a radial boost which becomes infinite on the horizon; a suitably chosen boost leads to the basis

$$\begin{aligned}
\vec{E}_0 &= \left[\frac{A}{\Delta(\Sigma+2Mr)}\right]^{1/2}\left[\vec{e}_{(t)} - \frac{2Mr}{A^{1/2}}\vec{e}_{(r)}\right],\\
\vec{E}_1 &= \left[\frac{A}{\Delta(\Sigma+2Mr)}\right]^{1/2}\left[\frac{-2Mr}{A^{1/2}}\vec{e}_{(t)} + \vec{e}_{(r)}\right],\\
\vec{E}_2 &= \vec{e}_{(\theta)},\\
\vec{E}_3 &= \vec{e}_{(\phi)},
\end{aligned} \tag{2.2}$$

where use should be made of the identity

$$A = 4M^2r^2 + \Delta(\Sigma+2Mr)$$

in verifying that $\vec{E}_0, \vec{E}_1$ are unit vectors. The boost defining (2.2) has been chosen so that $\vec{E}_0$ is the normal to the spacelike[7] hypersurfaces $\{v-r=\text{constant}\}$ and $\vec{E}_1$ is the projection of the normal to $\{r=\text{constant}\}$ orthogonal to $\vec{E}_0$. The simplest way to see this is to use the transformation

$$\begin{aligned}
dv &= dt + \frac{r^2+a^2}{\Delta}\,dr,\\
d\phi &= d\tilde{\phi} + \frac{a}{\Delta}\,dr
\end{aligned} \tag{2.3}$$

relating the Boyer-Lindquist coordinates $t, \tilde{\phi}$ to the Kerr coordinates v, ϕ in order to transform the ZAMO *one-forms* $\vec{e}^{(t)}, \vec{e}^{(r)}$ given in Ref. 6. One then finds

$$\vec{E}_{0\mu}dx^\mu = \left(\frac{\Sigma}{\Sigma+2Mr}\right)^{1/2}(dv-dr),$$

showing that $\vec{E}_0$ is orthogonal to $\{v-r=\text{constant}\}$. From the fact that $\vec{e}^{(r)} = (\Sigma/\Delta)^{1/2}dr$ one sees that the unit normal to $\{r=\text{constant}\}$ is just $\vec{e}^{(r)}$, so the required projection orthogonal to $\vec{E}_0$ must be $\vec{E}_1$ as written and we can use $\vec{E}_0, \vec{E}_1$ as $\vec{N}, \vec{M}$ above. For convenience we give the Kerr coordinate components of the $\vec{E}_a$:

$$\begin{aligned}
\vec{E}_0 &= \left(\frac{\Sigma}{\Sigma+2Mr}\right)^{1/2}\left(\frac{\partial}{\partial v} - \frac{2Mr}{\Sigma}\frac{\partial}{\partial r}\right),\\
\vec{E}_1 &= \left[\frac{1}{\Sigma A(\Sigma+2Mr)}\right]^{1/2}\\
&\quad\times\left[A\frac{\partial}{\partial v} + A\frac{\partial}{\partial r} + a(\Sigma+2Mr)\frac{\partial}{\partial\phi}\right],\\
\vec{E}_2 &= \Sigma^{-1/2}\frac{\partial}{\partial\theta},\\
\vec{E}_3 &= \frac{\Sigma^{1/2}}{A^{1/2}\sin\theta}\frac{\partial}{\partial\phi}.
\end{aligned} \tag{2.4}$$

The basis $\{\vec{E}_a\}$ shares the following properties with the ZAMO basis:

(1) It is group-invariant (i.e., the $\vec{E}_a$ commute with the timelike and axial Killing vectors $\vec{K}_t, \vec{K}_\phi$).

(2) Its timelike vector is hypersurface-orthogonal; it also forms a zero angular momentum congruence (in the sense $\vec{E}_0 \cdot \vec{K}_\phi = 0$).

(3) It tends to the orthogonal Minkowski frame

$$\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right\} \text{ as } r\to\infty.$$

In addition the frame $\{\vec{E}_a\}$ has the following property not shared by the ZAMO frame:

(4) It is defined and nonsingular throughout the range of Kerr coordinates (v, r, θ, ϕ) except at the curvature singularity $\Sigma=0$.

[To see (3) note that the boost giving $\vec{E}_a$ from the ZAMO's has relative velocity $(-2Mr/A^{1/2})$, while (4) can be read off from (2.4).] The main drawbacks of the $\{\vec{E}_a\}$ are the slight extra algebraic complexity of their coordinate components and the fact that they are not so elegantly defined as the ZAMO's. [The integral curves of $\vec{e}_{(t)}$ are group trajectories, which is not true of $\vec{E}_0$; further, the $\{\vec{E}_a\}$ is not the only frame with properties (1)–(4), even under the restriction that they are obtained from the ZAMO frame by a radial boost.]

III. THE COMPLEX FIELD COMPONENTS AND INVARIANTS

As is usually done,[3,2] we shall use spinor and complex null tetrad techniques for calculating the electromagnetic fields and shall assume the standard results of spinor calculus described for example by Pirani.[8] Taking a spinor basis λ^A, ν^A with $\lambda^A\nu_A = 1$, the associated complex null tetrad is defined by the spinor equivalents

$$l^\mu \leftrightarrow \lambda^A \bar{\lambda}^{\dot{W}}, \quad n^\mu \leftrightarrow \nu^A \bar{\nu}^{\dot{W}}, \qquad m^\mu \leftrightarrow \lambda^A \bar{\nu}^{\dot{W}}, \quad \bar{m}^\mu \leftrightarrow \nu^A \bar{\lambda}^{\dot{W}}. \tag{3.1}$$

Writing the spinor equivalent of the electromagnetic field tensor $F_{\mu\nu}$ as

$$F_{\mu\nu} \leftrightarrow \epsilon_{AB}\bar{\phi}_{\dot{W}\dot{X}} + \phi_{AB}\epsilon_{\dot{W}\dot{X}}, \tag{3.2}$$

where $\phi_{AB} = \phi_{(AB)}$ since $F_{\mu\nu} = F_{[\mu\nu]}$ (round brackets denote symmetrization), one finds the following relations between the null tetrad components φ_0, ϕ_1, ϕ_2 and the spinor components ϕ_{AB} of $F_{\mu\nu}$:

$$\begin{aligned} \phi_0 &= F_{\mu\nu} m^\mu l^\nu = \phi_{11}, \\ \phi_1 &= \tfrac{1}{2} F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu) = \phi_{12} \\ \phi_2 &= F_{\mu\nu}\bar{m}^\mu n^\nu = \phi_{22}. \end{aligned} \tag{3.3}$$

The spinor equivalent of the dual $F^*_{\mu\nu}$ is

$$F^*_{\mu\nu} \leftrightarrow i(\epsilon_{AB}\bar{\phi}_{\dot{W}\dot{X}} - \phi_{AB}\epsilon_{\dot{W}\dot{X}}), \tag{3.4}$$

so using (3.2), (3.4), and (3.3) we get the electromagnetic field invariants as

$$\begin{aligned} \tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} &= \mathcal{B}^2 - \mathcal{E}^2 = 2(\phi_0\phi_2 + \bar{\phi}_0\bar{\phi}_2 - \phi_1{}^2 - \bar{\phi}_1{}^2), \\ \tfrac{1}{2}F_{\mu\nu}F^{*\mu\nu} &= -2\mathcal{E}\cdot\mathcal{B} = 2i(\bar{\phi}_0\bar{\phi}_2 - \phi_0\phi_2 - \phi_1{}^2 + \bar{\phi}_1{}^2). \end{aligned} \tag{3.5}$$

Instead of writing directly the source-free Maxwell equations for ϕ_0, ϕ_1, and ϕ_2 it is more convenient to follow Fackerell and Ipser[2] and define quantities Φ_0, Φ_1, Φ_2 by the equations

$$\phi_0 = \frac{\rho\Phi_0}{\chi}, \quad \phi_1 = \rho^2\Phi_1, \; \phi_2 = \frac{\rho\Phi_2}{\sin\theta}, \tag{3.6}$$

where

$$\rho = -(r - ia\cos\theta)^{-1} \text{ and } \chi = \Delta\rho^2\sin\theta. \tag{3.7}$$

Then the source-free Maxwell equations which apply in the vacuum region between the hole and the inner edge of the plasma cloud take the forms

$$\begin{aligned} D\Phi_1 &= \frac{1}{\rho\chi}\bar{\delta}\Phi_0, \\ \Delta\Phi_0 &= \rho\chi\delta\Phi_1, \\ D\Phi_2 &= \rho\sin\theta\bar{\delta}\Phi_1, \\ \rho\sin\theta\,\Delta\Phi_1 &= \delta\Phi_2, \end{aligned} \tag{3.8}$$

where

$$D = l^\mu\frac{\partial}{\partial x^\mu}, \quad \Delta = n^\mu\frac{\partial}{\partial x^\mu},$$

$$\delta = m^\mu\frac{\partial}{\partial x^\mu} \text{ and } \bar{\delta} = \bar{m}^\mu\frac{\partial}{\partial x^\mu}.$$

We shall use the Kinnersley null tetrad $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$ which in Kerr coordinates is

$$\begin{aligned} l^\mu\frac{\partial}{\partial x^\mu} &= \frac{\partial}{\partial r}, \\ n^\mu\frac{\partial}{\partial x^\mu} &= -\frac{r^2+a^2}{\Sigma}\frac{\partial}{\partial v} - \frac{\Delta}{2\Sigma}\frac{\partial}{\partial r} - \frac{a}{\Sigma}\frac{\partial}{\partial\phi}, \\ m^\mu\frac{\partial}{\partial x^\mu} &= -\frac{\sqrt{2}\bar{\rho}}{2}\left(ia\sin\theta\frac{\partial}{\partial v} + \frac{\partial}{\partial\theta} + i\csc\theta\frac{\partial}{\partial\phi}\right). \end{aligned} \tag{3.9}$$

If we now impose the assumptions of stationarity and axisymmetry of the fields, Eqs. (3.8) become greatly simplified as all quantities are now independent of the "ignorable" coordinates v, ϕ. From the first and last equations (3.8) one finds

$$\frac{\partial\Phi_0}{\partial\theta} = -2\frac{\partial\Phi_2}{\partial\theta},$$

while the middle pair imply

$$\frac{\partial\Phi_0}{\partial r} = -2\frac{\partial\Phi_2}{\partial r}.$$

Thus $\Phi_0 = -2\Phi_2 + \text{constant}$, and using (3.6) and assuming that both ϕ_0, ϕ_2 are finite for $\theta = 0$ we get the relation

$$\phi_0 = -\frac{2\phi_2}{\Delta\rho^2}. \tag{3.10}$$

A separable equation involving only ϕ_0 has been constructed by Teukolsky[3]; performing the separation in the form

$$\phi_0 = R(r)S(\theta) \tag{3.11}$$

the equation splits into the two ordinary differential equations

$$\frac{1}{\Delta}\frac{d}{dr}\left(\Delta^2\frac{dR}{dr}\right) - (l-1)(l+2)R = 0, \tag{3.12}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + [(l-1)(l+2) + 1 - \cot^2\theta]S = 0, \tag{3.13}$$

where the separation constant l must be a positive integer. We shall return to the integration of the radial equation (3.12) later, but let us for the moment note[3] that the solutions of (3.13) are the spin-one spherical harmonics, which for the axisymmetric case reduce to $S_l(\theta) = (d/d\theta)P_l(\cos\theta)$. Thus from (3.10) and (3.11)

$$\begin{aligned} \phi_0^{(l)} &= R_l(r)\frac{d}{d\theta}P_l(\cos\theta), \\ \phi_2^{(l)} &= -\tfrac{1}{2}\Delta\rho^2 R_l(r)\frac{d}{d\theta}P_l(\cos\theta). \end{aligned} \tag{3.14}$$

It is now straightforward though rather tedious to verify from (3.8) that the solution for $\phi_1^{(l)}$ is

$$\Phi_1^{(l)} = \frac{\phi_1^{(l)}}{\rho^2}$$
$$= \left[\Delta R_l - r\frac{d}{dr}(\Delta R_l) \right] \frac{P_l(\cos\theta)}{\sqrt{2}}$$
$$+ \frac{ia}{(2l+1)\sqrt{2}} \frac{d}{dr}(\Delta R_l)[P_{l-1}(\cos\theta) - P_{l+1}(\cos\theta) + (2l+1)\cos\theta P_l(\cos\theta)], \tag{3.15}$$

where we have excluded for the time being the possibility of adding a constant to Φ_1, which would correspond to an electric or magnetic monopole. Returning to the radial equation (3.12) we note first that it can be immediately solved in the case $l=1$, the solution being

$$R_1 = A_1 \int \frac{dr}{\Delta^2} - \frac{iB_1}{\sqrt{2}},$$

with A_1, B_1 (possibly complex) constants, the factor $-i/\sqrt{2}$ being inserted for later convenience. Since $\Delta(r_+)=0$ the A_1 term has a logarithmic divergence at the horizon and using (3.14) and (3.15) one can see that the field invariants (3.5) become infinite there. We must therefore discard this solution, taking

$$R_1 = -\frac{iB_1}{\sqrt{2}}. \tag{3.16}$$

Thus from the $l=1$-pole solution we must discard the "dipole" contribution ($\propto A_1$) leaving only the "uniform" contribution ($\propto B_1$). This is in agreement with the electromagnetic "no hair" theorems,[2,3] where it is shown that the only field dying away at infinity and regular on the horizon is the monopole field we have already discarded. The "dipole" field is zero at infinity and so must diverge at the horizon. Hence the $l=1$ field is the "uniform" field previously found by an elegant method by Wald[9] and also obtained for the Schwarzschild case by Hájiček.[1]

The $R_l(r)$ for $l>1$ are known[3] to be given in terms of hypergeometric functions. In fact, using the variable

$$x \equiv -\frac{r-r_+}{2(M^2-a^2)^{1/2}},$$

which is negative everywhere outside the horizon we find that

$$R_l(r) = F(\alpha,\beta,2,x) \times \left[A_l \int \frac{dx}{[x(1-x)F(\alpha,\beta,2,x)]^2} + B_l \right],$$

where A_l, B_l are arbitrary constants, $\alpha+\beta=3$, $\alpha\beta=-(l-1)(l+2)$, and $F(\alpha,\beta,\gamma,x)$ is the standard hypergeometric function.[10] Since the A_l functions are easily seen to diverge at the horizon ($x=0$) they must be discarded. The contribution of the higher multipoles (i.e., B_l with $l>1$) will be small in the vicinity of the hole. This is because the function $R_l(r)$ increases with r asymptotically as r^{l-1} so that to avoid unrealistically large fields at the plasma cloud surrounding the hole we must take B_l small, and from the properties of the hypergeometric function we have

$$R_l(r) \sim B_l + O(r-r_+)$$

near the horizon. Thus to a good approximation we can consider the field to be simply the "uniform" field (3.16) near the hole. The higher multipoles are only important near the plasma shell where they are responsible for bending the field lines to satisfy the boundary conditions there. We shall discuss in a later paper how a collapsing core with a dipole magnetic field can develop into a black hole with a magnetic field which settles down to an almost uniform configuration.

IV. FIELDS AND FLUXES

We are now in a position to construct electric and magnetic fields for the black hole and to calculate their fluxes through parts of the horizon. With a given $F_{\mu\nu}$, an observer moving with 4-velocity t^ν will see electric and magnetic fields given by

$$\mathcal{E}_\mu = F_{\mu\nu}t^\nu, \quad \mathcal{B}_\mu = F^*_{\mu\nu}t^\nu,$$

respectively. From the expression[2]

$$F_{\mu\nu} = 2(\phi_1+\bar\phi_1)n_{[\mu}l_{\nu]} + 2\phi_2 l_{[\mu}m_{\nu]} + 2\bar\phi_2 l_{[\mu}\bar m_{\nu]} + 2\phi_0 \bar m_{[\mu}n_{\nu]} + 2\bar\phi_0 m_{[\mu}n_{\nu]} + 2(\phi_1-\bar\phi_1)m_{[\mu}\bar m_{\nu]}$$

and one for $F^*_{\mu\nu}$ (given by replacing ϕ_0, ϕ_1, ϕ_2 by minus themselves and multiplying the resulting tensor by i) we could calculate the $\{\vec{E}_a\}$ components of $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ using (2.4), but the fields take a slightly simpler form in the ZAMO frame, which is better adapted to the symmetry of the situation. We get

$$\mathcal{B}_{(t)}=\mathcal{B}_{(\phi)}=0\,,$$
$$\mathcal{B}_{(r)}=F^*_{(r)(t)}$$
$$=\frac{\sqrt{2}}{A^{1/2}\Sigma}[\sqrt{2}(r^2+a^2)\Sigma\,\mathrm{Im}\phi_1-a^2\Delta\,\mathrm{Im}\phi_0\sin\theta\cos\theta],$$
$$\mathcal{B}_{(\theta)}=F^*_{(\theta)(t)}$$
$$=\frac{\sqrt{2}\,\Delta^{1/2}}{A^{1/2}\Sigma}[\sqrt{2}a\Sigma\,\mathrm{Re}\phi_1\sin\theta-(r^2+a^2)r\,\mathrm{Im}\phi_0]\,,$$
$$\mathcal{E}_{(t)}=\mathcal{E}_{(\phi)}=0\,, \tag{4.1}$$
$$\mathcal{E}_{(r)}=\frac{\sqrt{2}}{A^{1/2}\Sigma}[\sqrt{2}(r^2+a^2)\Sigma\,\mathrm{Re}\phi_1-ar\Delta\,\mathrm{Im}\phi_0\sin\theta]\,,$$
$$\mathcal{E}_{(\theta)}=-\frac{\sqrt{2}\,\Delta^{1/2}a}{A^{1/2}\Sigma}[\sqrt{2}\Sigma\,\mathrm{Im}\phi_1\sin\theta-(r^2+a^2)\mathrm{Im}\phi_0\cos\theta]\,,$$

where we have used (3.10), and chosen ϕ_0 purely imaginary in order to give a purely magnetic field at infinity [using (3.5)]. The fields in the $\{\vec{E}_a\}$ frame can be obtained by the appropriate radial boost, which leaves $\mathcal{B}_{(r)}=\mathcal{B}_1$ unchanged—a fact of vital importance in calculating the flux—but introduces a $\mathcal{B}_3$ component. In view of the remarks at the end of the last section we shall only calculate $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ for the "uniform" case (3.16) ($l=1$). To obtain a field which is purely magnetic at large r we must choose B_1 real in (3.16). Dropping the suffix from B_1 we get

$$\mathcal{B}_{(r)}=\frac{B}{A^{1/2}\Sigma^2}\{(r^2+a^2)[(r^2-a^2)(r^2-a^2\cos^2\theta)+2a^2r(r-M)(1+\cos^2\theta)]-a^2\Delta\Sigma\sin^2\theta\}\cos\theta\,,$$
$$\mathcal{B}_{(\theta)}=\frac{-B\Delta^{1/2}}{A^{1/2}\Sigma^2}\{a^2[2r(r^2-a^2)\cos^2\theta-(r-M)(r^2-a^2\cos^2\theta)(1+\cos^2\theta)]+(r^2+a^2)\Sigma r\}\sin\theta\,,$$
$$\mathcal{E}_{(r)}=\frac{-Ba}{A^{1/2}\Sigma^2}\{(r^2+a^2)[2r(r^2-a^2)\cos^2\theta-(r-M)(r^2-a^2\cos^2\theta)(1+\cos^2\theta)]+r\Delta\Sigma\sin^2\theta\}\,, \tag{4.2}$$
$$\mathcal{E}_{(\theta)}=\frac{-Ba\Delta^{1/2}}{A^{1/2}\Sigma^2}[(r^2-a^2)(r^2-a^2\cos^2\theta)+2a^2r(r-M)(1+\cos^2\theta)-(r^2+a^2)\Sigma]\cos\theta\sin\theta\,.$$

This field corresponds to that given by Wald's[9] four-potential

$$\vec{A}=\tfrac{1}{2}B(\vec{K}_\phi+2a\vec{K}_t).$$

One notes that in the Schwarzschild case ($a=0$) the field is purely magnetic, with $\mathcal{B}_{(r)}=B\cos\theta$, $\mathcal{B}_{(\theta)}=-B\sin\theta$, the result obtained by Hájiček.[1]

At the horizon the ZAMO frame becomes singular, and we should transform to the $\{\vec{E}_a\}$ frame to obtain meaningful results. This transformation is easy to perform near the horizon, and we obtain at the horizon fields with only an $\vec{E}_1$ component, these components being precisely equal to the $\mathcal{E}_{(r)}, \mathcal{B}_{(r)}$ components above, for $\vec{\mathcal{E}}, \vec{\mathcal{B}}$ remain unchanged by a boost parallel to themselves, as is well known. Thus $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are purely "radial" at the horizon. (Note that Carter's[11] boundary conditions for the energy-momentum tensor on the horizon are automatically satisfied as $\vec{\mathcal{E}}, \vec{\mathcal{B}}$ are test fields: The result here is completely independent.)

Because of the skew-symmetry of the fields (4.2) under the reflection $\theta\to\pi-\theta$, the total flux of $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ across H is zero (as it must be, from the source-free property). However, it is interesting to calculate the flux $\mathfrak{F}$ of $\vec{\mathcal{B}}$ through just the upper half ($0\le\theta\le\pi/2$) of H. A simple calculation gives the result

$$\mathfrak{F}=\int_{\text{upper half of }H}F^*_{\mu\nu}E_1^{[\mu}E_0^{\nu]}dS$$
$$=\pi r_+{}^2B(1-a^4/r_+{}^4)\,. \tag{4.3}$$

In the Schwarzschild case, we get $\mathfrak{F}=\pi r_+{}^2B=4\pi M^2B=$ (cross sectional area)$\times B$. In the extreme Kerr case $|a|=M$ we find $\mathfrak{F}=0$. In fact $\mathfrak{F}$ decreases monotonically from $4\pi M^2B$ to zero as we increase a holding M fixed. ($\partial\mathfrak{F}/\partial a|_M$ can be shown to be strictly negative except at $a=0$.) This is not a consequence of the decreasing area of H, as one can see by calculating $\partial\mathfrak{F}/\partial a|_{r_+}$—this, too, is always negative. The lines of force of $\vec{\mathcal{B}}$ seem to experience a centrifugal repulsion as the hole is spun up. For a charge Q the corresponding magnetic flux is $2\pi Qa/M$, which *increases* with a.

Another quantity of interest is the invariant $\vec{\mathcal{E}}\cdot\vec{\mathcal{B}}$. We find

$$\vec{\mathcal{E}}\cdot\vec{\mathcal{B}}=\frac{B^2a\cos\theta}{\Sigma^4}\{\Delta r\Sigma^2\sin^2\theta+[(r^2-a^2)(r^2-a^2\cos^2\theta)+2a^2r(r-M)(1+\cos^2\theta)]$$
$$\times[2r(r^2-a^2)\cos^2\theta-(r-M)(r^2-a^2\cos^2\theta)(1+\cos^2\theta)]\}\,. \tag{4.4}$$

The fact that $\vec{\mathcal{E}}\cdot\vec{\mathfrak{B}}$ is not zero means that charged particles will be pulled into the hole along the lines of force of $\vec{\mathfrak{B}}$. To see which sign of charge gets accreted we need to find the relative sign of $\vec{\mathcal{E}}\cdot\vec{\mathfrak{B}}$ and $\vec{\mathfrak{B}}$. For our purposes it is sufficient to determine this as a function of θ on the surface H of the hole. With a positive we find that there is accretion of *positively* charged particles near the poles $\theta=0,\pi$ and accretion of negatively charged particles near the equator; the crossover occurs at the zeros of the second factor in square brackets in (4.4), which are given by the roots of the quadratic (for $|a|\neq M$)

$$a^2\lambda^2+2r_+(r_++M)\lambda-r_+{}^2=0\,,$$

with $\lambda\equiv\cos^2\theta$. One root of this equation is negative; the physical root is

$$\cos^2\theta=\frac{r_\pm}{a}\left\{\left[\left(\frac{r_++M}{a}\right)^2+1\right]^{1/2}-\frac{r_++M}{a}\right\}$$

$$\approx\frac{r_+}{2(r_++M)}\,.$$

It is easy to show that this root decreases monotonically from $\frac{1}{3}$ (when $a=0$) to a minimum value of $\sqrt{5}-2$ (when $|a|=M$) as we increase a keeping M fixed; thus the band around the equator where negative charges are attracted varies in half-latitude between 35°15′ ($a\approx0$) and 29° ($|a|=M$) approximately. This accretion process has been studied by Wald[9]: For quasistatic accretion an injection-energy argument shows that the hole will charge up to the value $2BaM$, when the quasistatic accretion stops.

The invariant $\vec{\mathcal{E}}\cdot\vec{\mathfrak{B}}$ does *not* vanish in the limit $M=0$ as one might at first expect—this is because it is singular at $\Sigma=0$, and although the Kerr metric is flat for $M=0$ one can no longer treat $\vec{\mathcal{E}},\vec{\mathfrak{B}}$ as test fields because they become unboundedly large at the coordinate singularity $\Sigma=0$. (A similar phenomenon occurs with the linearized electromagnetic field of a charged Kerr-Newman black hole[12]: If we treat this as a test field on a Kerr background, the invariant $\vec{\mathcal{E}}\cdot\vec{\mathfrak{B}}$ does not tend to zero as $M\to0$.)

V. CONCLUSION

In this paper we have considered electromagnetic fields in a vacuum cavity between a black hole and a surrounding plasma shell. Under the assumption that the field settles down to a stationary and axisymmetric state we have derived the following main results:

(a) The field must be to a good approximation the "uniform" magnetic field (4.2) except near the plasma shell where higher multipoles bend the field lines.

(b) The flux of magnetic field across one "hemisphere" of a neutral black hole (of fixed mass) decreases monotonically from a maximum value $\pi r_+{}^2B$ at $a=0$ to zero at the extreme limit $|a|=M$.

(c) The hole will tend to accrete charged particles *selectively*: for $a>0$, positively charged particles over the upper and lower parts of its surface, and negatively charged particles in a comparatively narrow band around the equator.

We shall investigate possible astrophysical consequences in a forthcoming paper.

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*Work supported by the Deutsche Forschungsgemeinschaft.

†On leave from the Institute of Astronomy, Polish Academy of Sciences, Warsaw, Poland.

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