Internal structure of multicomponent static spherical gravitating fluids

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The Maxwell-Einstein equations for a fluid comprised of more than one type of particle are not a determinate system even if an equation of state is added. The problem of what the charge distribution is in such fluids is therefore also not determinate. To complete the definition of the problem, more equations are needed. We obtain these for the simple case of a static spherically symmetric multicomponent system (imbedded in a Minkowskian background) by minimizing the energy of the fluid with respect to variations in the number densities of the constituents, with the side conditions that the total number of each constituent is constant during the variations. This procedure results in a determinate set of hydrostatic equilibrium equations, the sum of which is the familiar Tolman-Oppenheimer-Volkoff equation. Some general conclusions can be drawn. For example, the necessary and sufficient condition for charge neutrality is that the mass-energy density be some (arbitrary) function of some (arbitrary) *linear* combination of the number densities. Thus since it is well known that the electrons in a white dwarf star at absolute zero form a degenerate gas, there *must* be a charge imbalance throughout such a star. This imbalance can then be computed self-consistently. An over-all physical interpretation of the new equations is that in equilibrium at any point in the fluid the sum of the *non*gravitational forces per unit energy is the same for constituent 1 as for constituent 2 and so on. This is the analog of the corresponding (Galilean) statement for gravitational forces that is valid even without equilibrium.

I. INTRODUCTION

The Einstein equations of general relativity determine uniquely the internal structure of a perfect fluid of particles of a single type if the pressure and mass-energy density of the fluid are defined in terms of the particle number density. Defining the pressure and mass-energy density in terms of the number density is equivalent to providing an equation of state relating the two quantities. If the particles are allowed to possess a characteristic unit charge, the Maxwell equations and the Einstein equations still determine the structure uniquely. In this case, the charge density is just the unit charge times the particle number density.

However, one is frequently interested in a distribution of more than one type of particle. At the least, one might desire a distribution of two types of particles. It is not difficult to show that the structure of such a system is not determined uniquely by the above procedure whether or not the particles are charged. Let us assume that the pressure $p(n_1,n_2)$ and the mass-energy density $\rho_m(n_1,n_2)$ are given in terms of the number densities. Also let ν be an index running from 0 to 3. Then, if the particles are uncharged, there are 16 unknown functions[1]:

$$\begin{aligned} &n_1, \text{ number density of particles of type one (1);}\\ &n_2, \text{ number density of particles of type two (1);}\\ &u^\nu, \text{ four-velocity of fluid (4);}\\ &g^{\mu\nu}, \text{ metric tensor components (10).}\end{aligned} \tag{1.1}$$

However, there are only 15 equations:

$$\begin{aligned} &u^\nu u_\nu=1, \quad \text{metric restriction (1);}\\ &C_\nu(g)=0, \quad \text{coordinate conditions (4);}\\ &Z_{\mu\nu}=0, \quad \text{Einstein field equations (10).}\end{aligned} \tag{1.2}$$

Now let either type of particle assume a characteristic unit charge q_i $(i=1,2)$. We define the charge density, ρ_e, as $\rho_e=q_1n_1+q_2n_2$. Then there are 24 unknown functions, n_1, n_2, u^ν, $g^{\mu\nu}$, ϕ^ν, and J^ν, where ϕ^ν are the field potentials and J^ν is the charge current 4-vector; however, there are only 23 equations:

$$\begin{aligned} &u^\nu u_\nu=1, \quad \text{metric restriction (1);}\\ &J^\nu=\rho_e u^\nu, \quad \text{current definition (4);}\\ &Z_{\mu\nu}=0, \quad \text{Einstein field equations (10);}\\ &C_\mu=0, \quad \text{coordinate conditions (4);}\\ &g^{\alpha\beta}\phi_{u;\alpha\beta}-g^{\alpha\beta}\phi_{\alpha;u\beta}+J_u=0, \quad \text{Maxwell equations (4).}\end{aligned} \tag{1.3}$$

The argument can be extended to any number of different particle types.

Thus the following question arises: How can the system of equations be completed so as to make the internal structure problem determinate? In the literature,[2] the usual way of avoiding the difficulty is to assume that the charge density is zero (neutral system) or that the ratio of charge to matter densities is a constant (single-component system), or to treat the system in terms of some average molecular weight (an effective single-component system). If there are several reacting constituents in a system in equilibrium, then chemical equilibrium equations reduce the number of independent unknown densities. But in a multicomponent charged system the number of unknown functions is at least one higher than the number of equations.

The purpose of this paper is to present a method for obtaining equations beyond those of Maxwell and Einstein that will make determinate the spherically symmetric problem involving more than one component. In fact the method is not new, as it is simply an extension of the energy-minimization principle (or the entropy-maximization principle, which is equivalent to it in an equilibrium situation) to the case at hand. Energy minimization has been cast into minimization of the externally observed mass of a star by Harrison, Thorne, Wakano, and Wheeler[3] (HTWW in what follows) and applied by them to an uncharged single-component system, and it has been applied to a single-component charged system by Omote and Sato[4] for the purpose of studying stability. The plan of this paper is to extend the principle to multicomponent systems with or without charge, and by so doing overcome the indeterminacy difficulty discussed above. Applications of the method to particular systems will be presented in a later paper.

There is a long-standing problem in the description of white dwarf stars, to which our equations will give a quantitative resolution.[5] The standard model for the white dwarf star is that of a gas mixture of electrons (constituent 1) and heavy charged nuclei (constituent 2); the electrons form a degenerate gas. The partial pressure for the nuclei is regarded as negligible, but that for the electrons is of course significant. Finally the system is treated as electrically neutral. But the congregation of these assertions leads to an inconsistency of sorts: If the system is neutral, the nuclei must be supported everywhere in the star to the same extent as are the electrons. But what supports them if their pressure is zero? The standard answer[6] is to suppose that there is in reality a small electric field in the star, caused by a small charge imbalance, and that this field is present to just such an extent as to hold up the nuclei. In a sense it transmits the pressure from the electrons to the nuclei. The electric field is supposed to be large enough to support the nuclei yet small enough not to matter otherwise in a realistic discussion of the internal structure of the star. Several authors have argued that this can well be the case. However, the following question remains: How can one describe the system self-consistently, taking into account rigorously (at least in principle) all the various effects? We shall arrive in Sec. V at equations that can be applied to this white dwarf problem and can give a consistent picture to all the effects involved.

The paper is constructed as follows. In Sec. II the Einstein-Maxwell equations for the spherically symmetric fluid are presented. In Sec. III the variational principle is described. In Sec. IV the variational equations for two constituents are discussed, and the condition for charge neutrality given in a quite general form. In Sec. V structure equations are substituted into the variational equations and the consequences are discussed. Finally the actual calculation of the variational equations is presented in the Appendix.

II. THE BASIC PROBLEM

In this section the relevant Einstein-Maxwell equations are written down and the problem associated with the presence of more unknown functions than independent equations is described.

The gravitational metric tensor for the static, spherically symmetric case is given by[7]

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad (2.1)$$

where e^{ν} and e^{λ} are functions of r only. For a static, spherically symmetric perfect fluid the Einstein-Maxwell equations[7] reduce to

$$\frac{8\pi G}{c^4} T_0{}^0 = -\frac{G\mathcal{E}^2}{c^4 r^4} - \frac{8\pi G}{c^4}\rho_m c^2 = e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right)\frac{1}{r^2}, \qquad (2.2)$$

$$\frac{8\pi G}{c^4} T_1{}^1 = -\frac{G\mathcal{E}^2}{c^4 r^4} + \frac{8\pi G}{c^4} p = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \qquad (2.3)$$

$$\frac{8\pi G}{c^4} T_2{}^2 = \frac{G\mathcal{E}^2}{c^4 r^4} + \frac{8\pi G}{c^4} p = \frac{8\pi G}{c^4} T_3{}^3 = e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{(\nu')^2}{4} + \frac{\nu' + \lambda'}{2r}\right). \qquad (2.4)$$

The charge density ρ_e, the mass-energy density ρ_m, and the pressure p are regarded as known functions of the number densities $n_1, n_2, \ldots$ of the various constituents of the fluid:

$$\rho_e = \rho_e(n_1 n_2 \cdots), \quad \rho_m = \rho_m(n_1 n_2 \cdots), \quad p = p(n_1 n_2 \cdots). \qquad (2.5)$$

These equations constitute the information of an equation of state.

In Eqs. (2.2)–(2.4) the charge integral $\mathcal{E}$ is defined as

$$\mathcal{E}(r) = 4\pi \int_0^r \rho_e e^{\lambda/2} r^2 dr . \tag{2.6}$$

This represents the total charge below the radius r.

Equations (2.2)–(2.4) plus Eq. (2.5) constitute all that the Einstein-Maxwell equations plus an equation of state tell us. The divergence equation of electrodynamics is satisfied identically, and the divergence equations of general relativity reduce to just one equation of substance:

$$e^{-\lambda} p' = (\rho_m c^2 + p)\left(-\frac{m}{r^2} - \frac{4\pi G r p}{c^4} + \frac{G\mathcal{E}^2}{2c^4 r^3}\right) + \frac{\mathcal{E}\mathcal{E}' e^{-\lambda}}{4\pi r^4} . \tag{2.7}$$

This is known as the Tolman-Oppenheimer-Volkoff equation; we shall refer to it as the TOV equation. It is not independent, but can be derived from Eqs. (2.2)–(2.4). The function $m(r)$ is defined below in Eq. (3.4).

Thus we have essentially three equations, Eq. (2.2)–(2.4), and *at least* three unknowns: λ, ν, and the number densities $n_1, n_2, \ldots$. If there is one constituent, then the number densities reduce to just one, n_1, and the system is determinate, as discussed in the Introduction. But if there are two constituents there are two densities n_1, n_2, whence there are four unknowns, but only three equations. And so on. The problem is to find new equations from which all the unknown functions can be determined.

III. THE ENERGY MINIMIZATION PRINCIPLE

In this section an outline is given for the variational principle that is used to supplement the Einstein-Maxwell equations.

The procedure minimizes the energy U of an isolated fluid (radius R) subject to the side conditions that the total number of each type of constituent particle $N_i = \lim_{r\to\infty} N_i(r)$,

$$N_i(r) = 4\pi \int_0^r n_i(r) e^{\lambda/2} r^2 dr , \tag{3.1}$$

is kept constant during the variations. Just what the numbers N_i are could be asserted at the outset as the definition of the object under consideration, but can equivalently be determined by boundary conditions.

In general relativity, the energy density is frequently not a uniquely definable quantity. However, the total energy U of an isolated system imbedded in a Minkowskian background is known to be uniquely defined. One suitable expression is in terms of the Landau-Lifshitz pseudotensor t_{uv}:

$$U = -\int_0^\infty g(\tau^{00} + t^{00}) dx dy dz . \tag{3.2}$$

Another way to specify the energy of such a system is in terms of its mass. The first integral of Eq. (2.2) is

$$e^{-\lambda(r)} = 1 - \frac{2m(r)}{r} = 1 - \frac{2\Lambda(r)}{r} + \frac{G\mathcal{E}^2(r)}{r^2 c^4} , \tag{3.3}$$

where

$$m(r) = \frac{4\pi G}{c^2} \int_0^r \left(\rho_m + \frac{\mathcal{E}^2}{8\pi c^2 r^4}\right) r^2 dr , \tag{3.4}$$

$$\Lambda(r) = \frac{4\pi G}{c^2} \int_0^r \left(\rho_m + \frac{\mathcal{E}\mathcal{E}'}{4\pi c^2 r^3}\right) r^2 dr . \tag{3.5}$$

At large distances from the object, the $\Lambda(r)$ function is the observed mass. But $\Lambda(r)$ reaches its asymptotic value at the surface R of the object. It can be shown that the energy U of Eq. (3.2) is proportional to $\Lambda(R)$. Thus, following HTWW we can minimize $\Lambda(R)$ directly without going through the bother of evaluating Eq. (3.2).

The minimization of Λ can be made with respect to variations in the n_i functions directly. This was in fact the first way we solved the problem. However, it simplifies the equations considerably to use as independent functions the $N_i(r)$ of Eq. (3.1) which represent the numbers of particles below r. An added advantage to using the $N_i(r)$'s is that, instead of introducing and then eliminating Lagrange multipliers for the side conditions, one needs only to restrict the variations to be such that $\delta N_i(0) = \delta N_i(\infty) = 0$. However, ρ_m is conceived as depending on the $n_i(r)$'s directly, not on the N's, so there is the problem of converting from n_i's to N_i's.

The variational calculation is carried out in the Appendix.

IV. THE VARIATIONAL EQUATIONS FOR TWO CONSTITUENTS–GENERAL DISCUSSION

In the remainder of this article we discuss the problem of two constituents, 1 and 2. The variational equations are then two in number, and from Appendix A are

$$\left(\frac{\partial \rho_m}{\partial n_1}\right)' = \frac{q_1 \mathcal{E} e^{\lambda/2}}{c^2 r^2} + \frac{\partial \rho_m}{\partial n_1}\left[\tfrac{1}{2}\lambda' - \frac{4\pi r G}{c^2} e^{\lambda}\left(n_2 \frac{\partial \rho_m}{\partial n_2} + n_1 \frac{\partial \rho_m}{\partial n_1}\right)\right], \tag{4.1}$$

$$\left(\frac{\partial \rho_m}{\partial n_2}\right)' = \frac{q_2 \mathcal{E} e^{\lambda/2}}{c^2 r^2} + \frac{\partial \rho_m}{\partial n_2}\left[\tfrac{1}{2}\lambda' - \frac{4\pi r G}{c^2} e^{\lambda}\left(n_2 \frac{\partial \rho_m}{\partial n_1} + n_1 \frac{\partial \rho_m}{\partial n_1}\right)\right]. \tag{4.2}$$

These equations are fairly general, as the equation of state has not been specified: Only general relations of the type in Eq. (2.5) are assumed. In this section we discuss the interpretation of these equations from a general viewpoint, and in the next section we discuss a special case.

First of all, the sum of these two equations is just a generalized version of the TOV equation, although it is perhaps hard to recognize without making some detailed equations of state. The sum is then the "old" result. The new equation comes from taking the difference of Eqs. (4.1) and (4.2). One way to write this difference is simply to solve for the charge integral:

$$\mathcal{E}(r)=r^2e^{-\lambda/2}\,\frac{\left(\frac{\partial\rho_m}{\partial n_1}\right)'\Big/\frac{\partial\rho_m}{\partial n_1}-\left(\frac{\partial\rho_m}{\partial n_2}\right)'\Big/\frac{\partial\rho_m}{\partial n_2}}{q_1\left(\frac{\partial\rho_m}{\partial n_1}\right)^{-1}-q_2\left(\frac{\partial\rho_m}{\partial n_2}\right)^{-1}}. \quad (4.3)$$

This is the new equation that makes the Einstein-Maxwell system determinate.

To get some clue as to the significance of this equation, consider the condition necessary for charge neutrality. Even without a particular equation of state, Eq. (4.3) shows this condition to be

$$\frac{1}{\alpha_1}\frac{\partial\rho_m}{\partial n_1}+\frac{1}{\alpha_2}\frac{\partial\rho_m}{\partial n_2}=0, \quad (4.4)$$

where α_1 and α_2 are arbitrary constants. The general solution of Eq. (4.4) is

$$\rho_m=\rho_m(\alpha_1 n_1-\alpha_2 n_2). \quad (4.5)$$

That is, ρ_m must be some (arbitrary) function of some (arbitrary) *linear* sum of the n's. If there are uncharged constituents in the fluid the same result applies, since the terms which vanish when $\mathcal{E}$ vanishes are the terms containing the q_i. An example of a system for which charge neutrality would hold is one where only the zero-point energies of the particles contribute to ρ_m:

$$\rho_m=\sum n_i m_i, \quad (4.6)$$

where m_i is the mass of particles of type i. (If there are three types of particles, the same scheme of equations and solutions appear.)

An example of a system where charge neutrality would *not* hold is one where at least one type of particle is degenerate, for then ρ_m is a complicated function of the n's, or where interactions are taken into account, for then typically n^2 terms appear. Thus the theory developed in this paper would seem to have relevance to a white dwarf star, or to a neutron star.

Now Eq. (4.4) looks very much like an equation of chemical equilibrium,[8] since $\partial\rho_m/\partial n_1$ is what is defined to be the chemical potential μ_1 of the particles of type 1. A typical equation of chemical equilibrium $(A+B\rightarrow C)$ is

$$\sum \nu_i\mu_i=0\,, \quad (4.7)$$

where the ν_i's are the numbers of molecules of type i that enter into the chemical reaction equation. Thus, Eq. (4.7) is similar to Eq. (4.4) provided we identify the α_i (which are so far constants of integration) with the ν_i, and provided we agree to imagine a kind of "chemical" equilibrium to exist for noninteracting particles, which we have assumed so far.

By this general approach, we can perhaps interpret the basic new equation, Eq. (4.3) as a kind of equilibrium-of-forces equation for noninteracting particles with charge. [Since it is the derivative of Eq. (4.4), Eq. (4.3) must be associated with forces.] In the next section, we shall see that when more specific equations of state are introduced the various terms do have the look of identifiable forces.

V. THE VARIATIONAL EQUATIONS FOR TWO CONSTITUENTS–A PARTICULAR CASE

In order to apply Eqs. (4.1) and (4.2) to some concrete cases, let us suppose that the matter densities and pressure are additive,

$$\begin{aligned}\rho_m&=\rho_{m_1}(n_1)+\rho_{m_2}(n_2),\\ p&=p_1(n_1)+p_2(n_2),\end{aligned} \quad (5.1)$$

and that the equation corresponding to the first law of thermodynamics for one constituent is valid separately for two constituents[9]:

$$\begin{aligned}\frac{\partial\rho_{m_1}}{\partial n_1}&=\frac{\rho_{m_1}+p_1c^{-2}}{n_1}\,,\\ \frac{\partial\rho_{m_2}}{\partial n_2}&=\frac{\rho_{m_2}+p_2c^{-2}}{n_2}\,.\end{aligned} \quad (5.2)$$

Equations such as (5.1) and (5.2) are easily shown to be verified for, say, the constituents of a white-dwarf star as found in the literature.

With Eqs. (5.1) and (5.2), Eqs. (4.1) and (4.2) become

$$e^{-\lambda}p_1'=(\rho_{m_1}c^2+p_1)\left(-\frac{m}{r^2}-\frac{G4\pi rp}{c^4}+\frac{\mathcal{E}^2G}{c^42r^3}\right)+e^{-\lambda/2}r^{-2}\mathcal{E}q_1n_1, \quad (5.3)$$

$$e^{-\lambda}p_2'=(\rho_{m_2}c^2+p_2)\left(-\frac{m}{r^2}-\frac{G4\pi rp}{c^4}+\frac{\mathcal{E}^2G}{c^42r^3}\right)+e^{-\lambda/2}\mathcal{E}q_2n_2r^{-2}. \quad (5.4)$$

It is easy to see that the sum of these equations gives the usual TOV equation, (2.7).

If Eq. (5.4) is subtracted from Eq. (5.3) we get

$$\frac{r^{-2}\mathcal{E}n_2q_2 - e^{-\lambda/2}p_2'}{c^2\rho_{m_2}+p_2} = \frac{r^{-2}\mathcal{E}n_1q_1 - e^{-\lambda/2}p_1'}{c^2\rho_{m_1}+p_1}. \quad (5.5)$$

This is written in a way different from Eq. (4.3) but contains the same information. It is the new equation obtained by the variational procedure.

An interesting interpretation can be found for this equation by appealing to the nonrelativistic limit for a clue, for then the denominators reduce to just the ρ_m terms and the numerators are identifiable forces, either pressure forces or electrostatic forces. The equation then says that in equilibrium the sum of the *non*gravitational forces per unit mass is the same on particles of type 1 as on those of type 2, etc. The classical gravitational forces per unit mass behave this way anyhow (Galileo's discovery), even without equilibrium. The new equation exhibits an analogous relation for the sum of the nongravitational forces in equilibrium. Equation (4.3) can then be interpreted as the relativistic and general version of this. It is this relation that makes the Einstein-Maxwell equations determinate.

As a final point, let us return to the last paragraph of the Introduction, and consider the white-dwarf problem from the point of view of whether or not a charge imbalance exists, and if so how to calculate it self-consistently.

From Eq. (4.5) we see immediately that charge neutrality *cannot* hold if one of the constituents is degenerate. Let the electrons be denoted as constituent 1 and the nuclei as constituent 2. Then Eq. (5.4) is a general-relativistic TOV equation for the nuclei alone, but is coupled to the electron parameters through the functions in the large parentheses. In particular, if, as is usually done, we set $p_2=0$, and $\rho_{m_2}=m_2n_2$, then Eq. (5.4) reduces to

$$\mathcal{E}q_2r^{-2} = m_2\left(\frac{mc^4}{r^2G}+4\pi rp-\frac{\mathcal{E}^2}{2r^3}\right)e^{\lambda/2}\frac{G}{c^2}. \quad (5.6)$$

This special case of the nuclear TOV equations shows that the electrical force on the nuclei (the left-hand side) equals the gravitational force on them (the first term on the right) plus relativistic corrections. (Notice that the *electron* pressure enters the large parentheses on the right in the correction term.) This verifies the argument cited in the Introduction, made by several authors,[6,5] that the nuclei are held up in the star by an electrical force. In fact Eqs. (5.3) and (5.4) are the general-relativistic version of Eqs. (10) of Auluck and Kothari.[6]

Equation (5.5) applied to this case gives

$$\frac{r^{-2}\mathcal{E}q_2}{m_2c^2} = \frac{r^{-2}\mathcal{E}q_1n_1 - e^{-\lambda/2}p_1'}{\rho_{m_1}c^2+p_1}. \quad (5.7)$$

This result shows that the electrical force per mass on the nuclei (i.e., the "electrical acceleration" of the nuclei) is not equal to the "electrical acceleration" of the electrons, or even Z times it, where Z is the charge on the nucleus. Rather it equals the "electrical acceleration" of the electrons plus (or rather minus) the "pressure acceleration"; the latter two are, strictly speaking, normalized relative to $\rho_{m_1}c^2+p_1$. It is this fact that relates the electrical to the mechanical properties of the electrons and nuclei and ultimately ensures self-consistency.

The entire problem for the white dwarf can therefore be handled self-consistently starting from the assumption $p_2=0$, provided p_1 and the ρ's are known in terms of n_1 and n_2. In fact the mathematical problem can be reduced to a single nonlinear, second-order differential equation involving just one unknown function. The solutions of this equation are presently under study.

VI. CONCLUSIONS

In this paper we have outlined a method for making the Einstein-Maxwell equations determinate for a multiconstituent fluid in static equilibrium and having spherical symmetry. We have indicated generally what the condition for charge neutrality is, and we have made some attempts at interpreting the new equations in Secs. IV and V.

In future publications the application of the new equations to some special cases will be made, with an attempt to discover quantitatively how large the effects are in standard star problems, where else one might look for significant effects, and under what conditions stability (an energy minimum rather than a maximum) occurs.

APPENDIX: THE VARIATIONAL PRINCIPLE[3,10,4,11]

The problem is to minimize the quantity

$$\Lambda(r) = \frac{4\pi G}{c^2}\int_0^r\left(\rho_m+\frac{1}{4\pi^2}\frac{\mathcal{E}\mathcal{E}'}{r^3}\right)r^2dr \quad (A1)$$

in the limit $r\to\infty$, subject to the restrictions that the total numbers $N_i(r)$ of each type of particle remain constant during the variations. We shall take as the independent variational functions the $N_i(r)$ defined as

$$N_i(r) = 4\pi\int_0^r n_ie^{\lambda/2}r'^2dr'. \quad (A2)$$

The side conditions are then taken into account by stipulating that at the end points

$$\delta N_i(\infty)=0, \quad \delta N_i(0)=0. \tag{A3}$$

In Eq. (A1) the $\mathcal{E}$ and $\mathcal{E}'$ are immediately functions of the $N_i(r)$ and $N_i'(r)$ through the relation in Eqs. (2.6) and (3.1):

$$\mathcal{E}(r)=\sum_i q_i N_i(r). \tag{A4}$$

However, the ρ_m term in Eq. (1) is treated as a function of the $n_i(r)$ directly:

$$\rho_m=\rho_m(n_i). \tag{A5}$$

From Eq. (A1) we need

$$\delta\Lambda(r)=\frac{4\pi G}{c^2}\int_0^r\left[\delta\rho_m+\frac{1}{4\pi c^2 r^3}\delta(\mathcal{E}\mathcal{E}')\right]r^2 dr. \tag{A6}$$

The second term in the integrand is immediately

$$\delta\mathcal{E}\mathcal{E}'=\mathcal{E}'\sum q_i\delta N_i+\mathcal{E}\sum q_i\delta N_i'. \tag{A7}$$

To proceed with the first term we use

$$\delta\rho_m=\sum_i\frac{\partial\rho_m}{\partial n_i}\delta n_i \tag{A8}$$

and to get the δn_i we take the derivative of Eq. (A2):

$$\delta n_i=\frac{1}{4\pi r^2}(e^{-\lambda/2}\delta N_i'+N_i'\delta e^{-\lambda/2}). \tag{A9}$$

From Eq. (3.3) we get

$$\delta e^{-\lambda/2}=e^{\lambda/2}\left[-r^{-1}\delta\Lambda(r)+\frac{G\mathcal{E}}{c^2r^2}\sum q_i\delta N_i\right]. \tag{A10}$$

We need now to get $\delta\Lambda(r)$. To do this,[4] go back to Eq. (A6) and take an r derivative, using for $\delta\rho_m$ Eqs. (A8)–(A10). After some manipulation, it becomes clear that the resulting equation has the form

$$\frac{d}{dr}(\delta\Lambda)+a\delta\Lambda=b, \tag{A11}$$

where

$$a=\frac{Ge^{\lambda/2}}{c^2r}\sum_i\frac{\partial\rho_m}{\partial n_i}N_i', \tag{A12}$$

$$b=\frac{G}{c^2}e^{-\lambda/2}\sum_i\frac{\partial\rho_m}{\partial n_i}\delta N_i'+\frac{G}{c^4r}\delta(\mathcal{E}\mathcal{E}')+\left(\frac{G}{c^2}\right)^2\frac{\mathcal{E}e^{\lambda/2}}{r^2}\left(\sum_i\frac{\partial\rho_m}{\partial n_i}N_i'\right)\delta\mathcal{E}. \tag{A13}$$

The exact solution of Eq. (A11) with the boundary conditions of Eq. (A3) is

$$\delta\Lambda(r)=e^{-D(r,0)}\int_0^r e^{D(r',0)}b(r')dr', \tag{A14}$$

where

$$D(r,s)=\int_s^r a(r')dr'. \tag{A15}$$

We let $r\to\infty$ in Eq. (A14) and obtain an integrand of the desired type depending only on variations of the N_i and their derivatives [using of course Eq. (A4)]. Integrating by parts in the usual way for the $\delta N_i'$ terms, we end up with the form

$$\delta\Lambda(\infty)=e^{-D(\infty,0)}\int_0^\infty e^{D(r',0)}\sum_i Z_i(r')\delta N_i dr'=0. \tag{A16}$$

For independent variations, the Z's are set equal to zero. Rather than write out the Z's as they appear in Eq. (A16) we rearrange the terms in each equation $Z_i=0$ so as to resemble the usual TOV equation as much as possible:

$$\left(\frac{\partial\rho_m}{\partial n_i}\right)'=\frac{q_i\mathcal{E}e^{\lambda/2}}{c^2r^2}+\frac{\partial\rho_m}{\partial n_i}\left(\frac{\lambda'}{2}-\frac{4\pi G}{c^2}re^{\lambda}\sum_j n_j\frac{\partial\rho_m}{\partial n_j}\right). \tag{A17}$$

This is the final result of the variational procedure. One must also investigate stability, but this type of problem will be deferred to a later paper.

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[1]The Cauchy problem has been discussed by J. L. Synge [*Relativity, the General Theory* (North-Holland, Amsterdam, 1965), Chap. 10] and Y. Bruhat [in *Gravitation*, edited by L. Witten (Wiley, New York, 1962), pp. 150, 151]. Synge is somewhat misleading, as he asserts that there are 21 equations and 21 unknowns as he formulates the problem. But when the Maxwell equations are reduced from field variables to potentials, it is clear that there is an extra unknown function. The discussion in Bruhat's article illustrates this. When spherical symmetry is imposed, as in the present article, there is no question about it.

[2]A partial list of articles dealing with spherical distributions is the following: R. L. Brahamachary, Nuovo Cimento 4, 1216 (1956); 6, 1502 (1959); A. Das, Proc. R. Soc. London A267, 1 (1962); W. B. Bonner, Z. Phys. 160, 59 (1960); Mon. Not. R. Astron. Soc. 129, 443 (1965); U. K. De and A. K. Raychaudhuri, Proc. R. Soc. London A303, 97 (1968); J. Katz and G. Horowitz, Nuovo Cimento 5B, 59 (1971); M. Bailyn and D. Eimerl, Phys. Rev. D 5, 1897 (1972); M. Omote, Lett. Nuovo Cimento 6, 49 (1973).

[3]B. Harrison, K. Thorne, M. Wakano, and J. Wheeler, *Gravitation Theory and Gravitational Collapse* (Univ.

of Chicago Press, Chicago, 1965), Chap. 3.
[4]M. Omote and H. Sato, Gen. Relativ. Gravit. 5, 387 (1974).
[5]See for example E. E. Salpeter, Astrophys. J. 134, 669 (1961). See also p. 84 of Ref. 3.
[6]F. C. Auluck and L. S. Kothari, Proc. Nat. Inst. Sci. India 17, 15 (1951). See also M. Rudkjobing, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. 27, No. 5 (1950). Salpeter (Ref. 5) has called Rudkjobing's conclusions into question.
[7]Equations (2.1)–(2.7) follow from standard material in textbooks, and can be found in most of the articles cited in Ref. 2.
[8]For a discussion of chemical equilibrium in astrophysics see H.-Y. Chiu, *Stellar Physics* (Blaisdell, Waltham, Mass., 1968), Vol. 1, Eq. (3.235).
[9]See Ref. 3, p. 17.
[10]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), p. 304.
[11]R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1953), Vol. 1, Chap. IV.