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# Are torsion theories of gravitation equivalent to metric theories?*†

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An investigation is made of gravitation theories which are based on a Lagrangian constructed from the scalar curvature, in which torsion is allowed and in which the metric compatibility with the connection is not assumed in general. It is shown that because of the invariance of the scalar curvature under a projective transformation of the connection, only that part of the connection that is projectively invariant is determined by the theory. The theory is then reformulated so that the total action functional (gravitation plus source fields) depends only on the projective invariant part of the connection. The resulting theory is shown to be metrizable and equivalent to Einstein's theory with a modified source Lagrangian.

## I. INTRODUCTION AND OUTLINE

In Einstein's standard theory of gravitation, the affine connection coefficients $\Gamma^{\alpha}{}_{\mu\nu}$ (those coefficients from which covariant derivatives are formed as, e.g., $V^{\alpha}{}_{;\mu} = V^{\alpha}{}_{,\mu} + V^{\beta}\Gamma^{\alpha}{}_{\beta\mu}$ in the case of vector fields) are taken to be identical to the Christoffel symbols

$$\left\{ {\alpha \atop \mu\nu} \right\} = \tfrac{1}{2} g^{\alpha\beta}(g_{\mu\beta,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) \qquad (1)$$

in a coordinate basis. One viewpoint on general relativity, first discussed by Palatini, is to consider metric coefficients and symmetric connection coefficients as *a priori* independent dynamical variables in an action integral. Variation of the metric and of the connection generates not only the standard Einstein field equations in a vacuum but also shows the equality of the connection coefficients and the Christoffel symbols.[1,2]

Work on the foundations of general relativity has stimulated interest in several generalizations of Einstein's original theory. Cartan[3] proposed a generalization of the Palatini variational principle to include the antisymmetric part of the connection $(\frac{1}{2}\Gamma^{\mu}{}_{\alpha\beta} - \frac{1}{2}\Gamma^{\mu}{}_{\beta\alpha})$, a tensor which he called the torsion of the spacetime. The antisymmetric part of the connection in Cartan's theory is to be understood as an additional part of the gravitational field, not as a different physical field. There also have been many attempts to use the antisymmetric part of the connection to describe electromagnetism in unified field theories.[4] In this work we only consider Cartan's viewpoint and we do not consider any applications to a unified field theory. Cartan's theory has recently been reconsidered and developed by Hehl,[5,6] Trautman,[7] and their co-workers, although for brevity we will refer to their formulations as "theories with torsion." The field equations of these theories do not determine the connection uniquely; the additional condition $g_{\alpha\beta;\mu} = 0$ has to be added which for an arbitrary source term raises serious doubts as to the consistency of the resulting theories. Although the condition $g_{\alpha\beta;\mu} = 0$ is motivated physically by the desire to induce a reasonable connection on a spinor basis, it is our purpose to consider these theories solely in terms of what comes out of an action principle.

In this paper we consider a theory with torsion that is based on an action functional constructed from the curvature scalar (for the gravitational Lagrangian) and an arbitrary source field Lagrangian, as do the preceding authors. However, we take the full connection and the metric to have no *a priori* relation.

The curvature scalar is invariant under a projective transformation of the connection [cf. Eq. (7) and Ref. 8]. The vacuum field equations derived from it are also projective invariant and hence do not determine the full connection field. The gravitational Lagrangian depends only upon the "projective-invariant" part of the connection. This projective invariance of the gravitational part of the action imposes four algebraic constraints on the source-field part of the action. These constraints impose conditions on the source fields which can lead to inconsistencies. (The physical implications of the projective invariance of the scalar curvature are unclear. We shall regard this invariance as a mathematical accident rather than a spacetime symmetry and leave further speculation to another paper.) This constraint problem for an arbitrary source is resolved in this work by reformulating the source-field Lagrangian in a manner such that it too only depends upon the "projective-invariant" part of the connection. The constraint conditions are then automatically satisfied and only that part of the connection that is determined uniquely is used in the reformulated action. It is then shown that the gravitational Lagrangian and the source-field

Lagrangian (both now projectively invariant) can be regrouped into an equivalent action (i.e., the new action takes on the same values as the old action and hence those values of the fields which extremized the old action functional also extremize the new action functional), which is based on a Riemannian spacetime (connection coefficients equal to the Christoffel symbols) with a modified source-field Lagrangian. This (Lagrangian-based) theory with torsion is thus shown to be equivalent to Einstein's theory.

This paper uses the conventions of Ref. 9. The units are chosen such that $8\pi G = c = 1$ and the coupling constants for the source fields are absorbed by these fields and are not shown explicitly. All tensors are referred to a coordinate basis and a general tensor is written as $\phi_A$ where $A$ is a collective index. (E.g., $\phi_A = \phi_\mu$ for a vector, $\phi_A = \phi_{\mu\nu}$ for a second-rank tensor, etc.) The covariant derivative of $\phi_A$ is written as

$$\phi_{A;\mu} = \phi_{A,\mu} + \phi_B \sigma_A^{B\alpha}{}_\beta \Gamma^\beta{}_{\alpha\mu} , \tag{2}$$

where $\phi_{A,\mu}$ is the ordinary partial derivative, $\Gamma^\beta{}_{\alpha\mu}$ are the connection fields, and $\sigma_A^{\beta\alpha}{}_\beta$ are a set of Kronecker $\delta$'s that handle the bookkeeping for the connection terms; e.g., $\sigma^{(\mu)\alpha}_{(\nu)}{}_\beta = -\delta^\alpha_\nu \delta^\mu_\beta$ for a covariant vector field, $\sigma^{(\mu\nu)\alpha}_{(\rho\tau)}{}_\beta = -\delta^\mu_\beta \delta^\nu_\tau \delta^\alpha_\rho - \delta^\mu_\rho \delta^\nu_\beta \delta^\alpha_\tau$ for a covariant second-rank tensor field, etc. Square brackets on indices denote antisymmetrization and round brackets denote symmetrization, e.g.,

$$A_{[\alpha\beta]} = \tfrac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha})$$

and

$$A_{(\alpha\beta)} = \tfrac{1}{2}(A_{\alpha\beta} + A_{\beta\alpha}) .$$

## II. ACTION PRINCIPLE AND THE FIELD EQUATIONS

We assume there exists a spacetime metric $g_{\alpha\beta}$, a linear connection $\Gamma^\mu{}_{\alpha\beta}$, and a source field $\phi_A$ on a four-dimensional manifold $M$. The notion of "the gravitational field" is generalized to mean the metric and the (*a priori* independent) connection fields on $M$.

In this scheme the Riemann curvature tensor

$$R^\alpha{}_{\beta\mu\nu}(\Gamma) \equiv \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu} \tag{3}$$

and the Ricci curvature tensor

$$R_{\alpha\beta}(\Gamma) = R^\sigma{}_{\alpha\sigma\beta}(\Gamma) \tag{4}$$

depend only on the connection, while the Ricci scalar

$$R(g, \Gamma) = g^{\alpha\beta} R_{\alpha\beta}(\Gamma) \tag{5}$$

is constructed from both the connection and the metric.

The field equations relating the connection, the metric, and the source fields are assumed to come from a variational principle. As in most theories with torsion we take the gravitational action to be

$$S_G[g, \Gamma] = \int_M d^4x \sqrt{-g}\, R(g, \Gamma) , \tag{6}$$

where $g = \mathrm{Det}\,(g_{\alpha\beta})$. We will now see from two different points of view that even in the absence of sources the connection cannot be uniquely determined from the action $S_G$.

(i) The action is invariant under the "projective transformation" of the connection defined by

$$\Gamma^\mu{}_{\alpha\beta} \to \bar{\Gamma}^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\alpha\beta} + \delta^\mu_\alpha \psi_\beta , \tag{7}$$

where $\psi_\beta$ is an arbitrary covariant vector field.[8] Consequently, the variation of the gravitational action generated by a projective transformation ($\delta\Gamma^\mu{}_{\alpha\beta} = \delta^\mu_\alpha \psi_\beta$) will vanish as an identity. Therefore, some of the Euler equations are vacuous, and the projective invariance makes it impossible to determine the connection fields uniquely from the action.

(ii) This can be seen independent of a discussion of the projective invariance. The field equations resulting from a variation of the $\Gamma^\mu{}_{\alpha\beta}$ can be written (in vacuum) in the form[4]

$$g_{\alpha\beta,\mu} - g_{\rho\beta}\, {}^*\Gamma^\rho{}_{\alpha\mu} - g_{\alpha\rho}\, {}^*\Gamma^\rho{}_{\mu\beta} = 0 , \tag{8}$$

where ${}^*\Gamma^\mu{}_{\alpha\beta}$ is the linear combination of $\Gamma^\mu{}_{\alpha\beta}$ given by

$${}^*\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\alpha\beta} - \tfrac{2}{3}\delta^\mu_\alpha \Gamma^\rho_{[\rho\beta]} . \tag{9}$$

The ${}^*\Gamma^\mu{}_{\alpha\beta}$ are determined uniquely by Eq. (8) and turn out to be the Christoffel symbols defined by Eq. (1), i.e., for vacuum ${}^*\Gamma^\mu_{\alpha\beta} = \{^{\,\mu}_{\alpha\beta}\}$. The vector $\Gamma^\rho_{[\rho\beta]}$ is not determined by the variational principle and the combination ${}^*\Gamma^\mu{}_{\alpha\beta}$ contains no information about this vector, since

$${}^*\Gamma^\rho{}_{\rho\beta} - {}^*\Gamma^\rho{}_{\beta\rho} = 0 , \tag{10}$$

as can be easily verified from Eq. (9). Therefore, even in a vacuum spacetime, the variational principle determines the connection only up to a vector field, i.e., up to a projective transformation.

We will now show that these two viewpoints are equivalent. First note that the ${}^*\Gamma^\mu{}_{\alpha\beta}$ constructed from a connection $\Gamma^\mu{}_{\alpha\beta}$ and the ${}^*\Gamma^\mu{}_{\alpha\beta}$ constructed from the projectively transformed connection $\bar{\Gamma}^\mu{}_{\alpha\beta}$ are equal, i.e., that the ${}^*\Gamma^\mu{}_{\alpha\beta}$ is a projective invariant. The ${}^*\Gamma^\mu{}_{\alpha\beta}$ is the projective-invariant part of the connection. If the ${}^*\Gamma^{\mu}{}_{\alpha\beta}$ is now regarded as a connection, then the curvature scalar $R$ constructed from the ${}^*\Gamma^{\mu}{}_{\alpha\beta}$ is equal to the curvature scalar constructed from the $\Gamma^{\mu}{}_{\alpha\beta}$, since the curvature scalar is a projective invariant. When ${}^*\Gamma^{\mu}{}_{\alpha\beta}$ and $\Gamma^{\mu}{}_{\alpha\beta}$ are regarded as connections, they differ only by a projective transformation with $\psi_{\beta}=-\frac{2}{3}\Gamma^{\rho}_{[\rho\beta]}$ in Eq. (7). The action functional for the gravitational field can therefore be written in an obviously projectively invariant form

$$S_G[g,\Gamma]=\int_M d^4x\sqrt{-g}g^{\alpha\beta}R_{\alpha\beta}({}^*\Gamma)\ , \tag{11}$$

where the Ricci tensor is constructed from the ${}^*\Gamma^{\mu}{}_{\alpha\beta}$ in the usual way,

$$R_{\alpha\beta}({}^*\Gamma)={}^*\Gamma^{\mu}{}_{\alpha\beta,\mu}-{}^*\Gamma^{\mu}{}_{\alpha\mu,\beta}+{}^*\Gamma^{\mu}{}_{\rho\mu}\,{}^*\Gamma^{\rho}{}_{\alpha\beta}-{}^*\Gamma^{\mu}{}_{\rho\beta}\,{}^*\Gamma^{\rho}{}_{\alpha\mu} \tag{12}$$

and the action is to be regarded as an explicit functional of $g_{\alpha\beta}$ and ${}^*\Gamma^{\mu}{}_{\alpha\beta}$, and an implicit functional of $\Gamma^{\mu}{}_{\alpha\beta}$ through ${}^*\Gamma^{\mu}{}_{\alpha\beta}$. Since no information about $\Gamma^{\mu}{}_{\alpha\beta}$ other than contained in ${}^*\Gamma^{\mu}{}_{\alpha\beta}$ appears in $R$, the fact that only ${}^*\Gamma^{\mu}{}_{\alpha\beta}$ is determined by the field equations is seen to be a consequence of projective invariance.

The interaction of the gravitational field with another physical field is generated by adding a source action $S_M$ for this field to the gravitational action $S_G$ and applying the variational principle to the total action $S$. (Spinor fields will not be considered explicitly. The generalization to include them requires the addition of a spinor connection in addition to the tensor connection $\Gamma^{\mu}{}_{\alpha\beta}$ as an independent field in the action functional. This analysis will be left to a future paper, although we expect a result similar to the one described in this work, c.f. Ref. 10.) To formulate a Lagrangian density for the source fields in a curved spacetime requires a consideration of "the equivalence principle." For the field theories we are interested in we will use "the equivalence principle" in an operational sense and define the notion of "minimal coupling" as a prescription for the generalization of equations from flat Minkowski spacetime to curved spacetime. Usually "minimal coupling" is taken to mean that (i) all inner products are to be evaluated with the metric tensor explicitly and (ii) all partial derivatives are to be replaced by covariant derivatives (the "comma-goes-to-semicolon rule"). The connection fields then appear only through the covariant derivatives. On spacetimes with torsion the rules for "minimal coupling" can create physical theories with very different properties, e.g., Maxwell's equations are no longer gauge invariant if a covariant derivative with torsion is used in place of the ordinary exterior derivative in the field equations. For the purpose of this work all we require is that if the connection fields appear in the source Lagrangian, then they should only appear through covariant derivative. In general, the Lagrangian for a field on a curved spacetime is given by taking the flat spacetime Lagrangian for that field and applying the above rules for "minimal coupling" to the gravitational field. The Lagrangian density $\mathcal{L}$ for the source is then constructed from the source $\phi_A$, its covariant derivative $\phi_{A;\mu}$, and the metric $g_{\alpha\beta}$. The source action is then given by

$$S_M[\phi,g,\Gamma]=\int_M d^4x\,\mathcal{L}(\phi,\phi_{;\mu},g)\ . \tag{13}$$

In general, a variation of this action generated by a projective transformation will not vanish as an identity as the variation of the gravitational action did. Instead, such a variation generates the four algebraic constraints on the source field

$$\left(\frac{\partial\mathcal{L}}{\partial\Gamma^{\mu}{}_{\alpha\beta}}\right)\delta^{\mu}_{\alpha}=0\ . \tag{14}$$

These constraints must either be satisfied as identities for a particular source Lagrangian or they are extra conditions on the fields. In many cases, these constraints are strong enough to imply, when used together with the field equations for the $\phi_A$, that all the higher derivatives of the $\phi_A$ vanish.

The previous theories with torsion imposed the condition that the connection be metric, i.e.,

$$g_{\alpha\beta;\mu}=g_{\alpha\beta,\mu}-g_{\rho\beta}\Gamma^{\rho}{}_{\alpha\mu}-g_{\alpha\rho}\Gamma^{\rho}{}_{\beta\mu}=0\ , \tag{15}$$

in order to determine the full connection and remove the projective invariance. [Note the order of the indices in the last term and compare this with Eq. (8).] Under a projective transformation of the connection the covariant derivative of the metric tensor transforms as

$$g_{\alpha\beta;\mu}\to g_{\alpha\beta;\mu}-2g_{\alpha\beta}\psi_{\mu}, \tag{16}$$

so putting $g_{\alpha\beta;\mu}=0$ does indeed remove the projective freedom. However, in addition to removing the projective freedom in the connection this condition imposes the constraint that the metric and connection satisfy an equation of the form

$$g_{\alpha\beta;\mu}-2g_{\alpha\beta}\xi_{\mu}=0\ , \tag{17}$$

where $\xi_{\mu}$ is some vector field constructed from the metric and the connection. As will be shown shortly, the field equations for the connection are algebraic and can be solved to give the projective-invariant part of the connection ${}^*\Gamma^{\mu}{}_{\alpha\beta}$ in terms of the Christoffel symbols and the source fields. When this solution for the connection is substituted into these additional conditions, the source fields are further constrained. If one allows completely arbitrary sources, then in general the theory becomes inconsistent with these constraints.

A convenient way of satisfying the algebraic constraints in Eq. (14) follows if we alter the "comma goes to semicolon" rule. The constraints imply that $S_M$ can depend on $\Gamma^{\mu}{}_{\alpha\beta}$ only through ${}^*\Gamma^{\mu}{}_{\alpha\beta}$. Thus, by analogy with the gravitational action, the $\Gamma^{\mu}{}_{\alpha\beta}$ in the $\phi_{A;\mu}$ terms in the source Lagrangian are replaced by ${}^*\Gamma^{\mu}{}_{\alpha\beta}$. Let $\phi_{A\,;\mu}^{\,*}$ signify the covariant derivative of $\phi_A$ with respect to the ${}^*\Gamma^{\mu}{}_{\alpha\beta}$ combination of the $\Gamma^{\mu}{}_{\alpha\beta}$:

$$\phi_{A\,;\mu}^{\,*} = \phi_{A,\mu} + \phi_B \sigma_{A}^{B\alpha}{}_{\beta}\, {}^*\Gamma^{\beta}{}_{\alpha\mu} \,. \tag{18}$$

(The ${}^*\Gamma^{\mu}{}_{\alpha\beta}$ differ from the $\Gamma^{\mu}{}_{\alpha\beta}$ by a tensor. Therefore, ${}^*\Gamma^{\mu}{}_{\alpha\beta}$ is also a connection and the covariant derivative with respect to it is well defined.) With this alteration, the source Lagrangian is also projectively invariant and satisfies Eq. (14) as an identity. The imposition of an additional condition [e.g., Eq. (15)] to remove the projective freedom is no longer necessary since only that part of the connection that is determined uniquely by the field equations is used in the total action.

To treat the source action similarly as the gravitational action in the Palatini form, it is convenient to consider the source Lagrangian in first-order form.[11,18] The generalized covariant canonical momentum densities $\pi^{A\mu}$ are defined by

$$\pi^{A\mu} = \frac{\partial \mathcal{L}}{\partial \phi_{A\,;\mu}^{\,*}} \,, \tag{19}$$

and the first-order Lagrangian density $\Lambda$ is defined by

$$\Lambda(\phi,\pi,g) = \pi^{A\mu}\phi_{A\,;\mu}^{\,*} - \mathcal{L}(\phi,\phi^{*}_{;\mu},g) \,. \tag{20}$$

The total action functional is then given by

$$S[g,\Gamma,\phi,\pi] = \int_M d^4x \left[\sqrt{-g}\, g^{\alpha\beta} R_{\alpha\beta}({}^*\Gamma) + \pi^{A\mu}\phi_{A\,;\mu}^{\,*} - \Lambda(\phi,\pi,g)\right] \,, \tag{21}$$

where the $\Gamma^{\mu}{}_{\alpha\beta}$ appears only in the projectively invariant combination ${}^*\Gamma^{\mu}{}_{\alpha\beta}$. Requiring this action to be stationary for independent variations of $g_{\alpha\beta}$, $\Gamma^{\mu}{}_{\alpha\beta}$, $\phi_A$, and $\pi^{A\mu}$ gives the field equations for the theory. In particular, the variation of the connection $\Gamma^{\mu}{}_{\alpha\beta}$ implies

$$(\sqrt{-g}\, g^{\alpha\beta})^{*}_{;\mu} - \delta^{\beta}_{\mu}(\sqrt{-g}\, g^{\alpha\rho})^{*}_{;\rho} - 2\sqrt{-g}\, g^{\alpha\rho}\, {}^*\Gamma^{\beta}_{[\mu\rho]} = \sqrt{-g}\, J^{\alpha\beta}{}_{\mu}(\phi,\pi,g) \,, \tag{22}$$

where

$$\sqrt{-g}\, J^{\alpha\beta}{}_{\mu}(\phi,\pi,g) = -\pi^{A\beta}\phi_B \sigma_{A}^{B\alpha}{}_{\mu} + \tfrac{2}{3}\phi_B \sigma_{A}^{B\rho}{}_{\rho}\, \pi^{A[\beta}\delta^{\alpha]}_{\mu} \,. \tag{23}$$

Unlike the other fields, the ${}^*\Gamma^{\mu}{}_{\alpha\beta}$ are subject to algebraic rather than to differential equations, and can thus be expressed as

$$\begin{aligned} {}^*\Gamma^{\mu}{}_{\alpha\beta} &= \left\{{}_{\alpha}{}^{\mu}{}_{\beta}\right\} + \tfrac{1}{2} g^{\mu\nu}(L_{\nu\alpha\beta} - L_{\alpha\beta\nu} + L_{\beta\nu\alpha}) \\ &= \left\{{}_{\alpha}{}^{\mu}{}_{\beta}\right\} + H^{\mu}{}_{\alpha\beta}(\phi,\pi,g) \,. \end{aligned} \tag{24}$$

Here,

$$L_{\alpha\beta\mu} = J_{\alpha\beta\nu} - \tfrac{1}{2} g_{\alpha\beta}(J^{\rho}{}_{\rho\nu} - \tfrac{1}{3} J_{\nu}{}^{\rho}{}_{\rho}) - \tfrac{1}{3} g_{\beta\nu} J_{\alpha}{}^{\rho}{}_{\rho} \,. \tag{25}$$

This solution for ${}^*\Gamma^{\mu}{}_{\alpha\beta}$, when substituted back into the action functional in Eq. (21), defines a new action functional $S'$ depending on $\phi$, $\pi$, $g$, and $\left\{{}_{\alpha}{}^{\mu}{}_{\beta}\right\}$:

$$\begin{aligned} S'[g,\{\;\},\phi,\pi] &= S[g,\Gamma=\{\;\}+H,\phi,\pi] + \text{a divergence} \\ &= \int_M d^4x \left[\sqrt{-g}\, g^{\alpha\beta} R_{\alpha\beta}(\{\;\}) + \pi^{A\mu}\phi_{A|\mu} - \bar{\Lambda}(\phi,\pi,g)\right], \end{aligned} \tag{26}$$

where $R_{\alpha\beta}(\{\;\})$ is the Ricci tensor constructed from the Christoffel symbol,

$$\phi_{A|\mu} = \phi_{A,\mu} + \phi_B \sigma_{A}^{B\alpha}{}_{\beta}\left\{{}_{\alpha}{}^{\beta}{}_{\mu}\right\} \tag{27}$$

is the Riemannian covariant derivative, and

$$\begin{aligned} \bar{\Lambda}(\phi,\pi,g) = {} & \Lambda(\phi,\pi,g) + \sqrt{-g}\, g^{\alpha\beta}(H^{\rho}{}_{\mu\beta}H^{\mu}{}_{\alpha\rho} - H^{\rho}{}_{\mu\rho}H^{\mu}{}_{\alpha\beta}) \\ & - \pi^{A\mu}\phi_B \sigma_{A}^{B\alpha}{}_{\beta} H^{\beta}{}_{\alpha\mu} \end{aligned} \tag{28}$$

is the modified first-order Lagrangian for the source field. The new actions $S'$ defines an equivalent variational problem[12] (i.e., produces the same physics).

The action $S'$ is based on a Riemannian space without torsion. Requiring $S'$ to be stationary for independent variations of $g_{\alpha\beta}$, and of $\phi_A$ and $\pi^{A\mu}$ gives the usual Einstein field equations with the field energy tensor constructed from the $\bar{\Lambda}$ in the usual way [see Eq. (A6) in the Appendix], and the Euler-Lagrange equation in a Hamilton-type form for $\phi_A$ and $\pi^{A\mu}$.

## III. DISCUSSION

Our calculations above were quite general. Only the variational principle was used; no extra conditions were imposed. As a consequence, the variational principle did not give *all* the information about the gravitational field, because no information about $\Gamma^{\sigma}_{[\sigma\beta]}$ was given. In most theories with torsion another viewpoint is taken. The condition $g_{\alpha\beta;\mu}=0$ is imposed as a supplementary condition to the action integral and the range of allowed variations is thereby restricted.

By formulating the theory in a projective-invariant manner, we have shown that the resulting theory is metrizable and equivalent to Einstein's theory with a modified source Lagrangian.

One of the major reasons for the renewed interest in theories with torsion is the possibility that these theories do not satisfy some of the recently proven global theorems on singularities and that singularities, inevitable in Einstein's theory, can be avoided in theories with torsion. Indeed, cosmological models with phenomenological sources have been constructed with no initial singularity.[13] This infringement on the signularity theorems[14] has been discussed in Ref. 15 in terms of a violation of the energy conditions which is due to the torsion. From the viewpoint of the present paper the "violation" of the singularity theorems is not attributed to a "pressure" generated by torsion in the spacetime geometry, but rather by a violation of the energy conditions by the summetric "metric energy tensor"(see Appendix), constructed from the modified source Lagrangian in the usual way.[16] In our viewpoint, the avoidance of singularities is due to a strange source Lagrangian in the usual theory rather than the usual source Lagrangian in a strange theory.

In theories with torsion, the "energy tensor" that appears on the right-hand side of the gravitational field equations can be either the canonical energy tensor or the symmetric "metric energy tensor," depending on whether one uses an Einstein tensor formed from an abstract connection or an Einstein tensor formed from the Christoffel symbols. These energy tensors (constructed from the actions $S$ and $S'$) and the conservation laws they obey are discussed in the Appendix and are the generalizations of the Belinfante-Rosenfeld relations[17] between the canonical and symmetric energy tensors to spacetimes with abstract connections. When the explicit solution [Eq. (24)] for the connection is substituted into these general relations, the usual conservation laws for the canonical energy tensor constructed in the usual way from the modified source is recovered.

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## APPENDIX: ENERGY TENSORS IN THE THEORY

The action $S$ and $S'$, defined by Eqs. (21) and (26), respectively, are invariants, and therefore variations of $S$ or $S'$ generated by coordinate transformations vanish as identities (the conservation laws for energy and momentum). As it is usually done, we assume that the source fields satisfy appropriate field equations. If the coordinate transformations are generated by an arbitrary vector field $\xi^\mu$, then the variations of the various fields are the Lie derivatives of these fields with respect to the vector field $\xi^\mu$. For the action $S$, the relevant Lie derivatives are

$$\mathcal{L}_\xi g_{\alpha\beta}=2g_{\rho(\alpha}\xi^{\rho}{}^{*}_{;\beta)}+\xi^\rho g_{\alpha\beta}{}^{*}_{;\rho}+2\xi^\rho(g_{\sigma\alpha}{}^*\Gamma^{\sigma}_{[\beta\rho]}+g_{\sigma\beta}{}^*\Gamma^{\sigma}_{[\alpha\rho]}) \tag{A1}$$

and

$$\mathcal{L}_\xi {}^*\Gamma^{\mu}{}_{\alpha\beta}=\xi^{\mu}{}^{*}_{;\alpha}{}^{*}_{;\beta}+R^{\mu}{}_{\alpha\sigma\beta}({}^*\Gamma)\xi^\sigma-2(\xi^\sigma\,{}^*\Gamma^{\mu}_{[\sigma\alpha]})^{*}_{;\beta}\,, \tag{A2}$$

where $R^{\mu}{}_{\alpha\sigma\beta}({}^*\Gamma)$ is the curvature tensor constructed from the ${}^*\Gamma^{\mu}{}_{\alpha\beta}$ connection in the usual way. Substituting these expressions for the variations of the connections and metric fields into the variation for $S$, and restricting the $\xi^\mu$ to vanish on the boundary of $M$, we set the conservation law

$$-\frac{1}{4}\left(\frac{\partial\theta}{\partial g_{\mu\nu}}\right)g_{\mu\nu}{}^{*}_{;\alpha}+\frac{1}{2}\left(\frac{\partial\theta}{\partial g_{\mu\nu}}g_{\alpha\nu}\right)^{*}_{;\mu}-\left(\frac{\partial\theta}{\partial g_{\mu\nu}}\right){}^*\Gamma_{\mu[\nu\alpha]}+(\sqrt{-g}\,S^{\mu\nu}{}_{\alpha})^{*}_{;\nu}{}^{*}_{;\mu}+\sqrt{-g}\,S^{\mu\nu}{}_{\rho}R^{\rho}{}_{\mu\alpha\nu}({}^*\Gamma)+2(\sqrt{-g}\,S^{\mu\nu}{}_{\rho})^{*}_{;\nu}{}^*\Gamma^{\rho}_{[\alpha\mu]}=0\,, \tag{A3}$$

where $\theta=\theta^{\alpha}{}_{\alpha}$ is the trace of the canonical energy tensor[18] given in first-order form by

$$\theta^{\alpha}{}_{\beta}=\pi^{A\alpha}\phi_A{}^{*}_{;\beta}-\delta^{\alpha}{}_{\beta}\pi^{A\rho}\phi_A{}^{*}_{;\rho}+\delta^{\alpha}{}_{\beta}\Lambda\,. \tag{A4}$$

For the action $S'$ (on a Riemannian manifold) the relevant Lie derivative is

$$\mathcal{L}_\xi g_{\alpha\beta}=\xi_{\alpha|\beta}+\xi_{\beta|\alpha} \tag{A5}$$

and the corresponding conservation law is

$$\left[\frac{\partial}{\partial g_{\mu\nu}}(\pi^{A\alpha}\phi_{A|\alpha}-\Lambda)-\frac{\partial}{\partial x^\rho}\left(\frac{\partial(\pi^{A\alpha}\phi_{A|\alpha})}{\partial g_{\mu\nu,\rho}}\right)\right]_{|\nu}=0\,. \tag{A6}$$

The expression in Eq. (A3) is the generalization of the conservation law for the canonical energy tensor $\theta_{\mu\nu}$ to a spacetime manifold with an unrelated connection and metric. The $S^{\mu\nu}{}_{\alpha}$ terms are analogous to the Rosenfeld "spinterms."[17] The expression in Eq. (A6) is the usual conservation law for the symmetric energy tensor in first-order form (cf. Ref. 16).

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