PHYSICAL REVIEW D VOLUME 12, NUMBER 10 15 NOVEMBER 1975

Cosmology and particle pair production via gravitational spin-spin interaction in the Einstein-Cartan-Sciama-Kibble theory of gravity*

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(Received 28 July 1975)

General relativity with spin and torsion [the Einstein-Cartan-Sciama-Kibble (ECSK) theory] provides a more complete account of local gauge invariance with respect to the Poincaré group than is possible in Einstein's general relativity. The ECSK theory introduces a new gravitational spin-spin contact interaction which is here shown to be repulsive between Dirac particles whose spins are aligned and attractive when spins are opposed. The presence of a Dirac field as source for the torsion causes a positive "effective mass" term in the energy condition of the generalized singularity theorem for this theory. The spin-spin interaction enhances rather than opposes singularity formation, in contrast with previous results for a convective spinning dust distribution. The strength of the interaction depends on number density, and for electron densities in excess of $\bar{\rho} \simeq 10^{47}\ \mathrm{g\,cm^{-3}}$, can cause particle pair production.

THE ECSK THEORY

Cartan's[1] proposal for modifying Einstein's general theory of relativity via the introduction of his torsion tensor, first formulated consistently by Sciama[2] and Kibble,[3] (the ECSK theory[4,5,6]) yields a new very weak, gravitational spin-spin interaction of zero range.[5] This new interaction produces no observable deviation from general relativity in any present or proposed experimental test. However, a number of theoretical considerations argue for the preferred position of the ECSK theory as a relativistic description of spinning matter.[6,7]

Kopczyński,[8] Trautman,[9] and others[6,10,11] have advanced cosmological solutions of the ECSK field equations which exhibit a "bounce" at a small but finite radius parameter and huge density, provided that the shear and rotation are sufficiently small.[6,10] In this paper we argue that such a "bounce" depends on a postulated form for the energy and spin distributions of spinning matter and that this bounce *does not appear* when the Dirac field is taken as the source for the metric and the torsion. We also note the consequences of the ECSK spin-spin interaction for particle pair production in the early universe.

Perhaps the most compelling reason for adopting the ECSK theory derives from its natural relation to the concept of local gauge invariance under the group of motions of Minkowski space, namely the Poincaré group. In the local gauge theory, as detailed by Utiyama,[12] we introduce $4m$ gauge potentials A^b_μ in the gauge-covariant derivative operator $\nabla^G_\mu \psi = \partial_\mu \psi - T_b \psi A^b_\mu$, where the m quantities T_b are infinitesimal generators of the gauge group G. When G is the Poincaré group, it is possible to identify this gauge-covariant derivative with the covariant derivative ∇ with respect to the total connection (including torsion) of a Riemann-Cartan spacetime. In the language of fiber bundle theory, such an identification is seen as an isomorphism between the connection on the principal bundle of the Poincaré group over Minkowski space and the connection on the bundle of affine orthonormal frames over a Riemann-Cartan spacetime. The connections are isomorphic but the bundles themselves are not.[6]

The local gauge approach demands the substitution of the gauge-covariant derivative ∇^G for all partial derivatives in the matter Lagrangian (minimal coupling). The "spinning dust" solutions of the ECSK equations are not derivable from a Lagrangian. Rather, the convective forms for the canonical energy-momentum and spin angular momentum tensors must be postulated.[11] Such postulated forms may not be consistent with the interpretation of ECSK as a local gauge theory.

In contrast to the case of spinning dust, where we must postulate the form of the canonical tensors, the Dirac Lagrangian in ECSK theory is well known[13]:

$$L = +\frac{\hbar c}{2}\left[(\nabla_\alpha \bar{\psi})\gamma^\alpha \psi - \bar{\psi}\gamma^\alpha \nabla_\alpha \psi - \frac{2mc}{\hbar}\bar{\psi}\psi\right]. \qquad (1)$$

(All formulas are expressed in (anholonomic) orthornormal tetrad components, where $g(e_\alpha, e_\beta) = \eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$. The gamma matrices satisfy $\gamma_{(\alpha}\gamma_{\beta)} = \eta_{\alpha\beta}$, $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, and the adjoint spinor $\bar{\psi}$ is given by $\bar{\psi} = -i\psi^\dagger \gamma_0$, where $\psi^\dagger$ is the Hermitian conjugate of ψ. All other conventions follow Jauch and Rohrlich.[14] Symmetrization and antisymmetrization over indices are denoted by ( ) and [ ], respectively.)

The spinor-covariant derivatives are given by

$$\nabla_\alpha \psi = e_\alpha \psi - \tfrac{1}{4}\Gamma_{\alpha\mu\nu}\gamma^{[\mu}\gamma^{\nu]}\psi,$$
$$\nabla_\alpha \bar{\psi} = e_\alpha \bar{\psi} + \tfrac{1}{4}\Gamma_{\alpha\mu\nu}\bar{\psi}\gamma^{[\mu}\gamma^{\nu]}, \qquad (2)$$

where $\Gamma_{\alpha\mu\nu} = \eta_{\nu\lambda}\Gamma^{\lambda}_{\alpha\mu}$. The canonical tensors for the Dirac field are

$$\Sigma_{\alpha\beta} = -\frac{\hbar c}{2}\left[(\nabla_\alpha\bar{\psi})\gamma_\beta\psi - \bar{\psi}\gamma_\beta\nabla_\alpha\psi\right], \tag{3}$$

$$\tau^{\alpha\beta\gamma} = \tau^{[\alpha\beta\gamma]} = +\frac{\hbar c}{4}\bar{\psi}\gamma^{[\alpha}\gamma^{\beta}\gamma^{\gamma]}\psi. \tag{4}$$

From Hamilton's principle, we obtain by variation the Dirac equation in a Riemann-Cartan spacetime[13],

$$\gamma^\alpha\nabla^{\{\}}_\alpha\psi + \tfrac{1}{4}K_{\alpha\beta\lambda}\gamma^{[\alpha}\gamma^{\beta}\gamma^{\lambda]}\psi + \frac{mc}{\hbar}\psi = 0, \tag{5}$$

where $\nabla^{\{\}}_\alpha$ is the covariant derivative with respect to the Riemannian connection and the contortion $K_{\alpha\beta\lambda}$ is the difference between the Riemannian and Riemann-Cartan connections. Solving for the contortion via the gravitational field equations and substituting in Eq. (5) yields the spinor equation

$$\gamma^\alpha\nabla^{\{\}}_\alpha\psi + \tfrac{3}{8}l^2(\bar{\psi}\gamma_5\gamma^\mu\psi)\gamma_5\gamma_\mu\psi + \frac{mc}{\hbar}\psi = 0, \tag{6a}$$

nonlinear in ψ, which generalizes the Dirac equation to ECSK ($l = 8\pi G\hbar/c^3 \simeq 10^{-32}$ cm). The adjoint equation is

$$(\nabla^{\{\}}_\alpha\bar{\psi})\gamma^\alpha - \tfrac{3}{8}l^2(\bar{\psi}\gamma_5\gamma^\mu\psi)\bar{\psi}\gamma_5\gamma_\mu - \frac{mc}{\hbar}\bar{\psi} = 0. \tag{6b}$$

Equations (6) have not been solved exactly,[15] but we can discover the nature of the gravitational spin-spin interaction by solving Eq. (5) for electron plane waves in the approximations of (1) Riemann-flatness and (2) constant, externally applied "background" contortion. For a test electron at rest in a background which simulates an electron distribution of number density n and spin in the $+z$ direction, the energy levels are[7]

$$\omega = m \pm \tfrac{3}{8}l^2 n. \tag{7}$$

The positive sign corresponds to test particle spin in the $+z$ direction, aligned with the background and the negative sign to test particle spin opposed to the background. The same result obtains for positrons which have the same gravitational properties as electrons. From these examples we may conclude that *the gravitational spin-spin interaction which distinguishes ECSK from general relativity is repulsive for Dirac particles with aligned spins and attractive for opposed spins.*

This new spin contact interaction affects the singularity behavior of spacetime. From the canonical tensors (3), (4) one may easily compute the combined stress-energy tensor which appears in the pseudo-Einsteinian form of the field equations $G^{\alpha\beta}(\{\}) = k\tilde{\sigma}^{\alpha\beta}$ ($k = 8\pi G/c^4$) and in the dominant energy condition for singularity theorems in ECSK,[11]

$$W = (\tilde{\sigma}_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}\tilde{\sigma}_\mu{}^\mu)u^\alpha u^\beta \geq 0, \tag{8}$$

where u^α is the four-velocity of the cosmological fluid. For the Dirac field, we obtain the particularly simple form

$$\tilde{\sigma}_{\alpha\beta} = \Sigma_{(\alpha\beta)}(\{\}) + \frac{3}{16k}l^4\eta_{\alpha\beta}(\bar{\psi}\gamma_5\gamma^\mu\psi)(\bar{\psi}\gamma_5\gamma_\mu\psi), \tag{9}$$

and, using (5), we obtain

$$W = \Sigma_{\alpha\beta}(\{\})u^\alpha u^\beta - \tfrac{1}{2}mc^2\bar{\psi}\psi, \tag{10}$$

which is valid in general for Dirac fields in ECSK.

COSMOLOGY

Let us now restrict our attention to solutions of the ECSK-Dirac system wherein the Dirac field ψ is a function of $t = x^0$ alone (as would occur in spatially homogeneous cosmological models). We choose our orthonormal frames e_α and their dual one-forms θ^α ($\langle e_\alpha, \theta^\beta\rangle = \delta^\beta_\alpha$—see Ref. 16) so that $u^\alpha = (1,0,0,0)$, the vector $\vec{u} = e_0$, and the one-form $u = -\theta^0$. With this choice the Dirac equation and its adjoint become (overdots denote $\partial/\partial t$)

$$\dot{\psi} = -\gamma_0\Omega^\beta{}_{\alpha\beta}\gamma^\alpha\psi - \tfrac{1}{4}(\Omega_{\alpha\beta\gamma} + K_{\alpha\beta\gamma})\gamma_0\gamma^{[\alpha}\gamma^\beta\gamma^{\gamma]}\psi - \frac{mc}{\hbar}\gamma_0\psi, \tag{11a}$$

$$\dot{\bar{\psi}} = -\Omega^\beta{}_{\alpha\beta}\bar{\psi}\gamma^\alpha\gamma_0 + \tfrac{1}{4}(\Omega_{\alpha\beta\gamma} + K_{\alpha\beta\gamma})\bar{\psi}\gamma^{[\alpha}\gamma^\beta\gamma^{\gamma]}\gamma_0 + \frac{mc}{\hbar}\bar{\psi}\gamma_0. \tag{11b}$$

The object of anholonimity[17] $\Omega^\gamma{}_{\alpha\beta} = g^{\gamma\delta}\Omega_{\alpha\beta\delta}$ is related to the commutators of the basis vectors e_α and the exterior derivative of the basis one-forms θ^α by

$$[e_\alpha, e_\beta] = 2\Omega^\gamma{}_{\alpha\beta}e_\gamma, \tag{12a}$$

$$d\theta^\gamma = \Omega^\gamma{}_{\alpha\beta}\theta^\alpha\wedge\theta^\beta. \tag{12b}$$

It will be convenient to define the quantities

$$\hat{\Omega}^\alpha \equiv -\frac{1}{3!}\epsilon^{\alpha\beta\gamma\delta}\Omega_{\beta\gamma\delta}, \tag{13a}$$

and

$$\hat{\tau}^\alpha \equiv -\frac{1}{3!}\epsilon^{\alpha\beta\gamma\delta}\tau_{\beta\gamma\delta} \tag{13b}$$

where the totally antisymmetric tensor density $\epsilon^{\alpha\beta\gamma\delta}$ satisfies $\epsilon^{0123} = -1$.

In our chosen frames, we find that Eqs. (11) imply

$$\Sigma_{\alpha\beta}(\{\})u^\alpha u^\beta = \Sigma_{00}(\{\})$$
$$= -\frac{\hbar c}{2}(\dot{\bar{\psi}}\gamma_0\psi - \bar{\psi}\gamma_0\dot{\psi}) - \tau^{0\beta\gamma}\Omega_{0\beta\gamma} \quad (14)$$

and therefore

$$\Sigma_{00}(\{\}) = mc^2\bar{\psi}\psi + 6k\hat{\tau}^\alpha\hat{\tau}_\alpha - 6\hat{\tau}^0\hat{\Omega}_0 - 4\hat{\tau}^i\hat{\Omega}_i\,, \quad (15)$$

where i is summed from 1 to 3. The quantity W which appears in the energy condition (8) thus becomes

$$W = \tfrac{1}{2}mc^2\bar{\psi}\psi + \frac{3l^4}{8k}(\bar{\psi}\gamma_5\gamma^\mu\psi)(\bar{\psi}\gamma_5\gamma_\mu\psi)$$
$$- 6\hat{\tau}^0\hat{\Omega}_0 - 4\hat{\tau}^i\hat{\Omega}_i\,. \quad (16)$$

The last term in Eq. (16) represents an interaction between the spin of the Dirac field and the vorticity of the cosmological fluid, since $\hat{\Omega}^\alpha$ is proportional to the vorticity vector $\omega^\alpha = \epsilon^{\alpha\beta\gamma\delta}u_\beta\partial_\gamma u_\delta$. Such a term also arises in the analogous calculation for semiclassical spinning dust.[11] The term involving $\tau^0\Omega_0$ vanishes for homogeneous cosmological models of Bianchi types I, III, V, and VI,[18] and whenever the pseudoscalar $\bar{\psi}\gamma_5\gamma_0\psi$ vanishes.

For the simple case of a nonrotating Bianchi type I (spatially flat) cosmological model, the energy condition is

$$W = \tfrac{1}{2}mc^2\bar{\psi}\psi + \frac{3}{8k}l^4(\bar{\psi}\gamma_5\gamma^\mu\psi)(\bar{\psi}\gamma_5\gamma_\mu\psi) \geq 0. \quad (17)$$

Both terms in W are positive-definite. Therefore, at least in simple cases, we find that the formation of singularities is enhanced rather than averted by the presence of spin corrections, in contrast to the result for convective spinning dust.

PARTICLE PAIR PRODUCTION

The spin contact interaction also has important consequences for particle pair production in the early universe. At ordinary densities, the number density n_0 is approximately $\lambda\!\!\!^{-3}$, where $\lambda\!\!\!^- \simeq \hbar/m_e c$ is the reduced electron Compton wavelength. The splitting $\Delta\omega/m$ for this density, found in (7), is of the order 10^{-42} and completely negligible. However, for densities of the order of

$$\bar{\rho} = \frac{m_e{}^2c^4}{8\pi G\hbar^2} \simeq 10^{47}\ \mathrm{g\,cm^{-3}}\,, \quad (18)$$

the splitting is of order unity and particle pair production becomes possible. The density $\bar{\rho}$, huge as it is, is less by many orders of magnitude than the density at which particle production via tidal forces takes place. According to Zel'dovich,[19] pairs are created by tidal forces when

$$\int_0^t (\text{tidal forces})dt \simeq 2mc^2\,. \quad (19)$$

That is, most creation takes place when the radius of the universe is of the order of $\lambda\!\!\!^-$, which corresponds to a mass density more than thirty orders of magnitude greater than $\bar{\rho}$.

We conclude that spin effects can dominate mass effects whenever densities greater than $\bar{\rho}$ are attained, and that these effects must therefore be taken into account in any discussion of pair production in the early universe.

ACKNOWLEDGMENTS

The author wishes to thank F. W. Hehl, A. Trautman, J. A. Wheeler, and L. Ford for helpful discussions.

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*Work supported by NSF Grant No. GP30799X to Princeton University.

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