Electromagnetic plane-wave perturbations in Kasner cosmologies*

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Maxwell's equations with no sources are solved for the vector potential of electromagnetic plane waves in a Kasner background spacetime (these fields are generally not null). By a transformation of the time coordinate, the one equation not identically zero is transformed into the Euler-Poisson-Darboux equation. The basic theory for this equation and its plane-wave solutions are presented. The stress-energy tensor due to these waves is computed in the limit as the singularity is approached. For some of the wave modes the stress-energy tensor introduces negligible source terms in the perturbation equations for the gravitational field. However, for other wave modes, the stress-energy terms in Einstein's gravitational field equations grow faster as the singularity is approached than the terms due to the Kasner background spacetime. Hence these wave modes perturb the Kasner background in an unstabilizing manner.

I. INTRODUCTION

A time singularity of the spacetime metric is a property which occurs in general classes of solutions to Einstein's gravitational field equations.[1-8] The Khalatnikov-Lifshitz[1-3] generic time singular spacetimes grew out of the generalization of the Kasner[4,8] spacetimes. In this paper the solutions to Maxwell's equations for electromagnetic plane waves in Kasner spacetimes are obtained and their effect on the background spacetime is determined.

In Sec. II the problem of solving Maxwell's equations for plane waves in a Kasner background metric is reduced to solving the Euler-Poisson-Darboux (EPD) equation.[9-11] The solution A_μ is the vector potential for the Faraday tensor $F_{\mu\nu}$. Then the basic properties of the EPD equation and its solutions are presented. In general, for each Kasner metric, there are six possible values of the parameter k in the EPD; corresponding to each value of k there are one or more types of plane waves (which are not null in general).

In Sec. III the stress-energy tensors due to each wave mode are calculated in the limit as the singularity is approached, i.e., as $t\to 0$. For some wave modes the stress-energy tensor is shown to be a negligible source term in the perturbation equations for the gravitational field. However, other solutions give rise to a stress-energy tensor with terms, including the T_0^0 component, which grow faster as $t\to 0$ than the corresponding terms in Einstein's gravitational field equations due to the Kasner background metric. In Sec. IV the possible effect of this unstabilizing perturbation on the background spacetime is considered.

II. PLANE-WAVE SOLUTIONS TO MAXWELL'S EQUATIONS

The vector potential A_μ for plane waves is found by solving Maxwell's equations with no sources in a curved spacetime.[7] In a coordinate frame, Maxwell's equations become

$$\frac{\partial}{\partial x^\beta}\left[|g|^{1/2}g^{\alpha\mu}g^{\beta\nu}(A_{\nu,\mu}-A_{\mu,\nu})\right]=0. \tag{1}$$

The Kasner metric[1,7,8] is

$$\begin{aligned} ds^2 &= -dt^2+t^{2p_1}dx^2+t^{2p_2}dy^2+t^{2p_3}dz^2 \\ &= g_{\mu\nu}^{(B)}dx^\mu dx^\nu . \end{aligned} \tag{2}$$

This model represents an anisotropic expanding universe. The p_i, $i=1,2,3$ are constants and satisfy

$$p_1+p_2+p_3=p_1{}^2+p_2{}^2+p_3{}^2=1.$$

Following the convention of Belinskii *et al.*,[1] we can arrange the p_i in order $p_1\le p_2\le p_3$, then $-\frac{1}{3}\le p_1\le 0\le p_2\le\frac{2}{3}\le p_3\le 1$, and we can represent p_1,p_2,p_3 in parameter form

$$p_1(u)=\frac{-u}{1+u+u^2}, \quad p_2(u)=\frac{1+u}{1+u+u^2},$$

$$p_3(u)=\frac{u(1+u)}{1+u+u^2}, \text{ where } u\ge 1.$$

The electromagnetic perturbations of the Kasner spacetime satisfy Maxwell's equations (1) with the metric given by (2). (See Sec. III for a derivation of this result.) We are seeking solutions analogous to the plane-wave solutions in Minkowski spacetime. Hence we will look for solutions such that $A_0=0$ and only one of the components A_1,A_2,A_3 is nonzero. This nonzero component A_i will be a function of the time t and only one of the space variables x^j where $j\ne i$. Hence there are six possible modes of solution $A_1(t,y)$, $A_1(t,z)$, $A_2(t,x)$, $A_2(t,z)$, $A_3(t,x)$, and $A_3(t,y)$. For each mode we shall change the coordinates to transform the one Maxwell equation which is not identically equal to zero into the Euler-Poisson-Darboux (EPD) equation.

Case 1. *Solution modes* $A_1(t,z)$ *and* $A_2(t,z)$. We change the coordinates to

$$\tau=\frac{t^q}{q^q}, \quad \bar{x}=q^{p_1}x, \quad \bar{y}=q^{p_2}y, \quad \xi=q^{p_3}z,$$

where $q=1-p_3$ and $p_3\neq 1$. In the new coordinates the line element (2) becomes

$$ds^2=-\tau^a d\tau^2+\tau^b d\bar{x}^2+\tau^c d\bar{y}^2+\tau^a d\xi^2, \tag{3}$$

where $a=2p_3/q$, $b=2p_1/q$, and $c=2p_2/q$;

$$(|g|)^{1/2}=\tau^d,$$

where $d=(1+p_3)/q$.

In these coordinates Maxwell's equations (1) with $A_0=0$ become the following equations (4a)–(4d) for $\alpha=0,1,2,3$ respectively:

$$\tau^{d-a-b}A_{1,01}+\tau^{d-a-c}A_{2,02}+\tau^{d-2a}A_{3,03}=0, \tag{4a}$$

$$(\tau^{d-a-b}A_{1,0})_{,0}-\tau^{d-b-c}(A_{1,2}-A_{2,1})_{,2}-\tau^{d-a-b}(A_{1,3}-A_{3,1})_{,3}=0, \tag{4b}$$

$$(\tau^{d-a-c}A_{2,0})_{,0}-\tau^{d-b-c}(A_{2,1}-A_{1,2})_{,1}-\tau^{d-a-c}(A_{2,3}-A_{3,2})_{,3}=0, \tag{4c}$$

$$(\tau^{d-2a}A_{3,0})_{,0}-\tau^{d-a-c}(A_{3,1}-A_{1,3})_{,1}-\tau^{d-a-c}(A_{3,2}-A_{2,3})_{,2}=0. \tag{4d}$$

For mode $A_1(t,z)$, set $A_2=A_3=A_{1,1}=A_{1,2}=0$. Equations (4a), (4c), and (4d) are identically zero and Eq. (4b) reduces to the EDP equation

$$v^{(k)}_{\tau\tau}-v^{(k)}_{\xi\xi}+\frac{k}{\tau}v^{(k)}_{\tau}=0, \tag{5}$$

with $v^{(k)}=A_1$, $\xi=z$, and $k=d-a-b=1+2u$; $u\geq 1$ so $3\leq k<+\infty$.

For mode $A_2(t,z)$, set $A_1=A_3=A_{2,1}=A_{2,2}=0$. Equations (4a), (4b), and (4d) are identically zero and Eq. (4c) reduces to the EPD equation with $v^{(k)}=A_2$, $\xi=z$, and $\infty<k\leq -3$.

Case 2. *Solution modes* $A_1(t,y)$ *and* $A_3(t,y)$. Now let

$$\tau=\frac{t^r}{r^r}, \quad \bar{x}=r^{p_1}x, \quad \xi=r^{p_2}y, \quad \bar{z}=r^{p_3}z,$$

where $r=1-p_2$ and the value $p_3=1$ is allowed. In the new coordinates the line element (2) becomes

$$ds^2=-\tau^e d\tau^2+\tau^f d\bar{x}^2+\tau^e d\xi^2+\tau^g d\bar{z}^2, \tag{6}$$

where $e=2p_2/r$, $f=2p_1/r$, and $g=2p_3/r$. In these coordinates Maxwell's equations (1) with $A_0=0$ take a form similar to Eqs. (4a)–(4d).

For mode $A_1(t,y)$, set $A_2=A_3=A_{1,1}=A_{1,3}=0$. Then Eqs. (1) with $\alpha=0,2,3$ are identically zero and the $\alpha=1$ equation becomes the EPD equation with $v^{(k)}=A_1$, $\xi=y$, $1\leq k\leq 3$, and $k=1$ for $p_3=1$.

For mode $A_3(t,y)$, set $A_1=A_2=A_{3,1}=A_{3,3}=0$. Then Eqs. (1) with $\alpha=0,1,2$ are identically zero and the $\alpha=3$ equation becomes the EPD equation with $v^{(k)}=A_3$, $\xi=y$, $-3\leq k\leq -1$, and $k=-1$ for $p_3=1$.

Case 3. *Solution modes* $A_2(t,x)$ *and* $A_3(t,x)$. Now let

$$\tau=\frac{t^s}{s^s}, \quad \xi=s^{p_1}x, \quad \bar{y}=s^{p_2}y, \quad \bar{z}=s^{p_3}z,$$

where $s=1-p_1$, and the value $p_3=1$ is allowed. In the new coordinates the line element (2) becomes

$$ds^2=-\tau^j d\tau^2+\tau^j d\xi^2+\tau^i d\bar{y}^2+\tau^m d\bar{z}^2, \tag{7}$$

where $j=2p_1/s$, $i=2p_2/s$, and $m=2p_3/s$. In these coordinates, with $A_1=A_3=0$, calculations similar to the previous cases show that $A_2(t,x)$ satisfies the EPD equation with $v^{(k)}=A_2$, $\xi=x$, $0\leq k\leq 1$, and $k=1$ for $p_3=1$. With $A_1=A_2=0$, mode $A_3(t,x)$ satisfies the EPD equation with $v^{(k)}=A_3$, $\xi=x$, $-1\leq k\leq 0$, and $k=1$ for $p_3=1$.

To complete these considerations, for $p_3=1$, let $\tau=\ln t$. Then using the τ coordinate, $A_1(t,z)$ and $A_2(t,z)$ satisfy the EPD equation with $v^{(k)}=A_1$ and $v^{(k)}=A_2$, respectively, $\xi=z$ and $k=0$.

A. Properties of the Euler-Poisson-Darboux equation

The equations for the various wave modes $A_i(t,x^\alpha)$ all have the form of Eq. (5), the EPD equation[9–11] in one space variable with the range of k being $-\infty<k<+\infty$. The superscript k in $v^{(k)}$ indicates the dependence of $v^{(k)}(t,\xi)$ on k. When $k=0$, eq. (5) reduces to the well-known plane-wave equation. Note, however, that for $k\neq 0$ functions of the form $v(\tau,\xi)=f(t\pm\xi)$ are not solutions to Eq. (5), so there are no traveling plane waves for $k\neq 0$.

We state the solution of Eq. (5) for the Cauchy problem with initial conditions

$$v^{(k)}(0,\xi)=f(\xi), \quad v^{(k)}_{\tau}(0,\xi)=0. \tag{8}$$

For $k>0$,

$$v^{(k)}(\tau,\xi)=\frac{\Omega_k}{\Omega_{k+1}}\int_{-1}^{+1}f(\xi+\alpha\tau)(1-\alpha^2)^{(k-2)/2}d\alpha, \tag{9}$$

where $\Omega_n=2\pi^{n/2}/\Gamma(n/2)$, and $\Gamma(n)$ is the gamma function. For $k<0$, the solution is not unique. In addition the cases where k is an odd negative integer must be considered separately. For $k<0$ and k not an odd integer

$$v^{(k)}(\tau,\xi)=\tau^{1-k}\left(\frac{1}{\tau}\frac{\partial}{\partial\tau}\right)^n(\tau^{k+2n-1}v^{(k+2n)}), \tag{10}$$

where n is a positive integer such that $k+2n>0$, and $v^{(k+2n)}(\tau, \xi)$ in Eq. (10) solves the Cauchy problem with initial conditions

$$v^{(k+2n)}(0, \xi) = f(\xi)/(k+1)(k+3)\cdots(k+2n-1),$$
$$v^{(k+2n)}_{\tau}(0, \xi) = 0. \tag{11}$$

For $k = -1, -3, -5, \ldots$, the exceptional cases of odd negative integers,[10] let $k = -2n - 1$, $n = 0, 1, 2, \ldots$ then

$$v^{(k)}(\tau, \xi) = \sum_{r=1}^{n+1} a_{r,n+1} \frac{\tau^r \partial^r \chi}{\partial \tau^r}, \tag{12}$$

where the $a_{r,n+1}$ coefficients are determined by the recursion formula

$$a_{0,n} = 0, \quad a_{n,n} = 1, \quad a_{r,n+1} = a_{r-1,n} - a_{r,n}(2n - r) \tag{13}$$

and

$$\chi(\tau, \xi) = \int_{-1}^{+1} \phi(\xi + \alpha\tau) \frac{\ln[(\tau/4)(1 - \alpha^2)]}{(1 - \alpha^2)^{1/2}} d\alpha, \tag{14}$$

with

$$f(\xi) = \Pi\left[a_{1,n+1} + \sum_{r=2}^{n+1} a_{r,n+1}(r-1)(-1)^{r+1}\right]\phi(\xi). \tag{15}$$

For $k < 0$, the solution given by Eqs. (10) or (12) is not unique. For example, any function of the form

$$w^{(k)}(\tau, \xi) = \tau^{1-k} v^{(2-k)}(\tau, \xi) \tag{16}$$

can be added to Eqs. (10) or (12), where $v^{(2-k)}(\tau, \xi)$ is a solution to the Cauchy problem for parameter value equal to $2 - k$. Such solutions have zero Cauchy data at $t = 0$,

$$w^{(k)}(0, \xi) = w^{(k)}_{\tau}(0, \xi) = 0.$$

Thus the Kasner case differs from the Minkowski or Friedmann cases. In the latter cases, the vector potential for plane-wave solutions satisfies the wave equation (EPD equation with $k = 0$) and the only solution with zero initial data is identically zero.

In addition to the solutions given above, for $k = 0$ $v^{(0)}(\tau, \xi) = f(\tau - \xi)$ will be considered as a possible solution.

We are interested in examining solutions similar to propagating monochromatic plane-wave solutions of the form

$$v(\tau, \xi) = B \sin\omega(\tau - \xi).$$

However, unless $k = 0$, such functions are not solutions to the EPD equation. So we drop the requirement of propagation and instead examine monochromatic plane-wave solutions by imposing a periodic initial condition in Eq. (8),

$$v^{(k)}(0, \xi) = f(\xi) = B \sin\omega\xi. \tag{17}$$

For $k = 2, 4, 6, \ldots$ the integral in Eq. (9) can be evaluated by integration by parts. Some solutions are

$$v^{(0)} = B \sin\omega\xi \cos\omega\tau,$$
$$v^{(2)} = (B \sin\omega\xi \sin\omega\tau)/\omega\tau,$$
$$v^{(4)} = 3B \sin\omega\xi(\sin\omega\tau - \omega\tau\cos\omega\tau)/(\omega\tau)^3.$$

For these solutions we see that $v^{(k)}$ has the structure of a standing wave which is periodic in time and which decays as

$$\tau^{-k/2} \text{ as } \tau \to \infty.$$

III. EFFECT OF THE ELECTROMAGNETIC PERTURBATION ON THE KASNER BACKGROUND SPACETIME

The perturbing effect of these plane waves on the Kasner background spacetime is determined[1,2] essentially by computing the stress-energy tensor due to these solutions and then comparing each term to the other terms in the Einstein field equation in which it occurs. More precisely, consider the combined Einstein-Maxwell equations for the gravitational and electromagnetic fields with Einstein's field equations in the form

$$R^{\mu}_{\nu} = 8\Pi(T^{\mu}_{\nu} - \tfrac{1}{2}\delta^{\mu}_{\nu} T) \tag{18}$$

and Maxwell's equations given by Eq. (1). Also, the Faraday tensor

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{19}$$

determines the stress-energy tensor

$$4\Pi T^{\mu}_{\nu} = F^{\mu}{}_{\alpha} F_{\nu}{}^{\alpha} - \tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \delta^{\mu}_{\nu}. \tag{20}$$

Let $g^{(B)}_{\mu\nu}$ be given by Eq. (2) and consider an electromagnetic perturbation to $g^{(B)}_{\mu\nu}$ which results in a new spacetime and a Faraday tensor given by

$$g_{\mu\nu} = g^{(B)}_{\mu\nu} + h_{\mu\nu} \text{ and } F_{\mu\nu}, \tag{21}$$

respectively, where $g_{\mu\nu}$, $F_{\mu\nu}$ are solutions to the Einstein-Maxwell equations and $h_{00} = h_{0\mu} = 0$.

Now assume $g_{\mu\mu}$ in Eq. (21) is given as an expansion in powers of t, where $h_{\mu\mu}$ has as a factor a more positive power of t than that of $g^{(B)}_{\mu\mu}$, $\mu \geq 1$. Furthermore, assume that the remaining nonzero $h_{\mu\nu}$ have as a factor a more positive power of t than that of all the $g_{\mu\mu}$, $\mu \geq 1$. Then

$$|g| = |g^{(B)}| + \text{terms in more positive powers of } t.$$

Hence near $t = 0$, we can make the approximation $|g| \approx |g^{(B)}|$ in Eqs. (1) to obtain the perturbation equations for A_{μ}, where now the raising and lowering of indices is done by $g^{(B)}_{\alpha\beta}$.

In Sec. II plane-wave solutions to Eqs. (1) were found. Now we must verify that the assumptions made on the relative smallness of $h_{\mu\nu}$ are satisfied. The perturbation equations for $h_{\mu\nu}$ are given in

TABLE I. Asymptotic form of the vector potential A_μ and the stress-energy tensor T^μ_ν.

Nonzero component of A_μ	Asymptotic form of A_μ	Asymptotic form of $8\pi T^\mu_\nu$ in Kasner coordinates, nonzero components
$A_2(t,x); 0 \le k \le 1$	$B \sin\omega\xi\left(1-\frac{\omega^2\tau^2}{2(k+1)}\right)$	$T^0_0=-\left(\frac{t}{s}\right)^{-2(p_1+p_2)}B^2\omega^2\cos^2(\omega s^{p_1}x)=-T^1_1=-T^2_2=T^3_3$ $T^0_1=-T^0_0 t\omega\frac{s^{p_1}}{p_3}\tan(\omega s^{p_1}x)$, $T^1_0=-t^{-2p_1}T^0_1$
$A_1(t,y); 1 \le k \le 3$	given by $A_2(t,x)$ case with indices 1 and 2 interchanged, r replacing s and y replacing x	
$A_1(t,z); 3 \le k < +\infty$	given by $A_1(t,y)$ case with indices 2 and 3 interchanged, q replacing r and z replacing y	
	Type II	
$A_3(t,x); -1 < k < 0$	given by $A_2(t,x)$ case with indices 2 and 3 interchanged	
$A_3(t,y); -3 \le k < -1$	given by $A_1(t,y)$ case with indices 1 and 3 interchanged	
$A_2(t,z); -\infty < k \le -3$	given by $A_1(t,z)$ case with indices 1 and 2 interchanged, unverified for $k=-7,-9,-11,\ldots$	
$A_3(t,y); k=-1$	given by $A_3(t,y)$, $-3 \le k < -1$ case with a factor $2\ln t$ replacing $1/(k+1)=1/p_1$ in A_3 and in T^0_1	
$A_3(t,x); k=-1$	given by $A_3(t,y)$, $k=-1$ case with indices 1 and 2 interchanged	
	Type I	
$A_3(t,x); -1 \le k < 0$	$B\tau^{1-k}\sin\omega\xi$	$T^0_0=\frac{-4t^{-2(p_1+p_2)}}{s^{2p_3}}p_3{}^2B^2\sin^2(\omega s^{p_1}x)=-T^1_1=-T^2_2=T^3_3$ $T^0_1=T^0_0 t\omega\frac{s^{p_1}}{p_3}\cot(\omega s^{p_1}x)$, $T^1_0=-t^{-2p_1}T^0_1$
$A_3(t,y); -3 \le k \le -1$	given by $A_3(t,x)$ case with indices 1 and 2 interchanged, r replacing s and y replacing x	
$A_2(t,z); -\infty < k \le -3$	given by $A_3(t,y)$ case with indices 2 and 3 interchanged, q replacing r and z replacing y	
	Additional $k=0$ modes	
$A_1(t,z)$	$B\sin\omega z\cos\omega\tau$	given by $A_3(t,x)$ type II, $k=-1$ mode with indices 1 and 3 interchanged and z replacing x
$A_2(t,z)$	given by previous mode with indices 1 and 2 interchanged	
	Traveling wave modes	
$A_2(t,x), A_3(t,x)$	$B\sin(\tau-\xi)$	$T^0_0=-2\left(\frac{t}{s}\right)^{-2/3}B^2\omega^2\cos^2(\omega s^{p_1}x)=-T^1_1$, $T^0_1=t^{-1/3}T^0_0=-T^1_0$
$A_1(t,z), A_2(t,z)$	$B\sin(\tau-z)$	$T^0_0=-2t^{-2}B^2\omega^2\cos^2[\omega(\ln t-z)]=-T^3_3=-T^0_3=T^3_0$

Appendix D of Lifshitz and Khalatnikov,[2] and the stress-energy components T^μ_ν are source terms in those linear equations. Also in Appendix D, those perturbation equations are solved for the $T^{\mu(M)}_\nu$ of matter in an ultrarelativistic state, and it is shown that the resulting $h_{\mu\nu}$ is relatively small compared to the background spacetime.

Here we shall establish the relative smallness of $h_{\mu\nu}$ corresponding to some of the plane waves A_μ, by showing component by component that the stress energy T^μ_ν generated by A_μ grows no faster as $t\to 0$ than the $T^{\mu(M)}_\nu$ for matter. Hence we shall compute T^μ_ν in the limit as $t\to 0$ for the various modes A_μ.

Let the initial condition be given by Eq. (17). For $A_2(t,x)$ use the coordinates of Eq. (7) and let $A_2 = v^{(k)}$. Then using integration by parts on the solution given by Eq. (9), in the limit as $t \to 0$

$$v^{(k)}(\tau, \xi) \cong B \sin\omega\xi \left(1 - \frac{\omega^2 \tau^2}{2(k+1)}\right). \tag{22}$$

Then Eqs. (19) and (20) determine T^μ_ν. For the other modes T^μ_ν is determined similarly. For $A_1(t,y)$ use Eq. (6) and for $A_1(t,z)$ use Eq. (3). For $A_3(t,x)$, $A_3(t,y)$, $A_2(t,z)$ there are two types of solutions. Type I is given by Eq. (16) and type II is given by Eqs. (10) and (11) for $k \neq -1, -3, -5, \ldots$ and by Eqs. (12) to (15) for $k = -1, -3, -5, \ldots$. There are some additional solutions possible for $A_1(t,z)$ and $A_2(t,z)$ for $k=0$, which are similar to the previous solutions for $A_1(t,z)$ and $A_2(t,z)$ of type II, respectively. Finally, for $k=0$, the EPD equation reduces to the wave equation, so that traveling waves are also possible solutions. For $A_2(t,x)$ and $A_3(t,x)$, $k=0$ for $p_3 = \frac{2}{3}$; also for $A_1(t,z)$ and $A_2(t,z)$, $k=0$ for $p_3 = 1$. The results of the computations to determine A_μ and T^μ_ν are given in Table I.

Now we compare these T^μ_ν with $T^{\mu(M)}_\nu$. The stress-energy tensor for matter in the ultrarelativistic state[2] $T^{\mu(M)}_\nu$ has a dependency on t given by

$$T^0_0 \sim T^3_3 \sim t^{-1-p_3}, \quad T^1_1 \sim t^{-2+2p_3}, \quad T^2_2 \sim t^{-1+p_3-2p_2},$$

$$T^0_1 \sim T^0_2 \sim T^0_3 \sim t^{-1}, \quad T^1_2 \sim T^1_3 \sim g^{11} g_{22} T^2_3 \sim t^{-1+p_3-2p_1}.$$

In the perturbation equations solved in Appendix D of Lifshitz and Khalatnikov,[2] only the largest term $T^{3(M)}_3$ of the three $T^{\mu(M)}_\mu$, $\mu = 1,2,3$ was kept in the equations. Hence in our comparison of $T^{\mu(M)}_\nu$ with the T^μ_ν generated by a plane wave, we shall compare the tensors component by component except for T^μ_μ, $\mu=1,2,3$; the exceptions will be compared to $T^{3(M)}_3$.

For the solution mode $A_2(t,x)$, comparison of $T^{\mu(M)}_\nu$ with the results in Table I shows that the components of T^μ_ν for $A_2(t,x)$ are either proportional to smaller powers of $1/t$ than $T^{\mu(M)}_\nu$ or are zero. Hence we conclude that the wave mode $A_2(t,x)$ produces a negligible perturbation $h_{\mu\nu}$ on the Kasner background spacetime $g^{(B)}_{\mu\nu}$.

Also notice in the Einstein field equations (18) the principal terms in $R^\mu_\mu \sim t^{-2}$, $R^0_1 \sim t^{-1}$ for $g^{(B)}_{\mu\nu}$ whereas $T^\mu_\mu \sim t^{-2+2p_3}$, $T^0_1 \sim t^{-1+2p_3}$, so that T^μ_ν is negligible compared to the principal terms in R^μ_ν.

Similarly for wave modes $A_1(t,y)$, $A_1(t,z)$, $A_3(t,x)$ type I, $A_3(t,y)$ type I, $A_2(t,z)$ type I, and $A_3(t,x)$ type II, $-1<k<0$, comparison of T^μ_ν with $T^{\mu(M)}_\nu$ shows that these waves produce a negligible perturbation on the Kasner background.

However, for $A_3(t,y)$ type II, $-3 \leq k \leq -1$, $T^\mu_\mu \sim t^{-2(p_2+p_3)}$, $T^{0(M)}_0 \sim T^{3(M)}_3 \sim t^{-1-p_3}$, and $1+p_3 < 2p_2 + 2p_3$ since $p_1 < p_2$. Hence T^μ_μ grows faster than $T^{\mu(M)}_\mu$ as $t \to 0$. In fact in Einstein's equations (18), the terms in $R^\mu_\mu \sim t^{-2}$, which are of lower power in $1/t$ than T^μ_μ; also $R^0_\mu \sim t^{-1}$ whereas $T^0_2 \sim t^{1-2(p_2+p_3)}$. Hence the Kasner spacetimes are *unstable* under a perturbation produced by the wave mode $A_3(t,y)$ type II.

By similar comparisons, the modes $A_2(t,z)$ type II, $A_3(t,y)$ type II, $k=-1$, $A_3(t,x)$ type II, $k=-1$, additional $k=0$ modes for $A_1(t,z)$ and $A_2(t,z)$ and the traveling wave modes are found to produce unstable perturbations.

IV. DISCUSSION

Electromagnetic plane-wave perturbations in a Kasner spacetime have been shown to be solutions to the classical Euler-Poisson-Darboux equation, the theory for which has been extensively developed by Weinstein *et al.*[8,9] For $-\infty < k < +\infty$, there are plane waves which produce a negligible perturbation on the Kasner background spacetime near the singularity. However, for $-\infty < k \leq -3$ there are wave modes $A_2(t,z)$ and $A_3(t,y)$ and also there are traveling wave modes which produce unstable perturbations to the Kasner background near the singularity. It would be interesting to determine the evolution of a Kasner spacetime undergoing such a perturbation as the singularity is approached.

In the Khalatnikov-Lifshitz generic case[1] where mixmaster perturbations are taken into account, perturbation terms due to mixing do not occur in the $R^0_0 = 8\Pi(T^0_0 - \frac{1}{2}T)$ equation, whereas for $A_2(t,z)$, $A_3(t,z)$, there is a T^0_0 component of the same perturbing power as the other diagonal components. Hence it would be interesting to find out whether there are similar electromagnetic perturbations in the generic case. If so, they might cause the generic spacetime to evolve differently as the singularity is approached so as to allow thorough mixing. Hence such perturbations might contribute toward a theoretical explanation of the observed isotropy of the microwave background radiation.[7]

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