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Quantization of a scalar field in the Kerr spacetime*

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The interaction of a scalar field with the Kerr gravitational field is studied. The properties of the Klein-Gordon equation in the Kerr metric are reviewed, and a two-component formalism is developed. This formalism expresses the one-particle quantum theory for a massive scalar particle in the Kerr metric. A semiclassical analysis of the spontaneous emission of particles by a Kerr black hole is given. The quantization of a scalar field in the Kerr metric is developed, and a treatment of the spontaneous particle creation is given. The particular quantization given here leads to emission only into the classical superradiant modes and hence no emission by a Schwarzschild black hole. In the case in which $\omega M \ll 1$, where M is the mass of the black hole and ω the frequency of a given mode, an explicit expression may be given for the rate at which particles are emitted into each mode. In the case in which the particle's mass is zero and $a \ll M$, a = angular momentum per unit mass of the black hole, the total rate of loss of energy of the black hole is shown to be proportional to a^6/M^8. A discussion of the problem of the vacuum energy is given. It is shown that the energy of the vacuum state of a scalar field in a Kerr spacetime $a \ll M$ is the same as that for a Schwarzschild ($a = 0$) spacetime.

I. INTRODUCTION

Quantum field theory in a curved spacetime is known to lead to the prediction that nonstationary gravitational fields are capable of producing particle-antiparticle pairs.[1-3] Since the gravitational field couples universally to all other fields, it is expected that in an expanding universe, for example, production of all types of particles is possible.[4] A quantum field theory formulated in a given background spacetime describes the interaction of the quantized field with the classical gravitational field associated with the curvature of spacetime. In the case of static or stationary spacetime it would appear that there would be no particle production. One would expect that positive- and negative-frequency components of a field would not mix, and hence that a system prepared in the vacuum state at one time would always stay in the vacuum state. As we will see later, this is not always the case.

Since the Kerr metric represents the gravitational field of a rotating black hole, it is an excited state of a system which we might expect to be able to release energy spontaneously. At the classical level, it is known that the rotational energy of a rotating black hole can be extracted. One means for doing so is the superradiant scattering process discovered by Zel'dovich[5] and by Misner[6] in which a wave (e.g., scalar, electromagnetic, or gravitational) in certain modes is scattered off the black hole and is found to carry away energy from the black hole. In particular, if the black hole has mass M and angular momentum $L=Ma$, and if the wave has frequency ω and a component of angular momentum of m along the rotation axis of the black hole, then energy is extracted by the scattering process if

$$\omega < m\Omega_h . \tag{1}$$

Here[7]

$$\Omega_h = a/2Mr_+ \tag{2}$$

is the frequency of dragging of inertial frames at the horizon at $r=r_+$, where

$$r_+ = M + (M^2 - a^2)^{1/2}. \tag{3}$$

The amplification factor has been calculated numerically by Press and Teukolsky.[8,9] Some analytic calculations have been done by Starobinsky[10] for scalar waves and by Starobinsky and Churilov[11] for electromagnetic and gravitational waves. For an extreme Kerr ($a=M$) case, it is found that the amplification factor for scalar waves is about 0.1% for the most favorable modes. This factor is about 4.5% for electromagnetic waves and 138% for gravitational waves. Rotational energy can be extracted from the black hole until it is a Schwarzschild black hole whose mass is equal to or greater than the irreducible mass, M_{ir}, of the original Kerr black hole. The relationship between mass, irreducible mass, and angular momentum is[12]

$$M^2 = M_{ir}{}^2 + \frac{L^2}{4M_{ir}{}^2} .$$

From a quantum-mechanical point of view, superradiant scattering is stimulated emission of quanta, and there must also be a corresponding spontaneous-emission process. This process is the spontaneous production of pairs of particles of energy $\lesssim m\Omega_h$. The rate for spontaneous emission can be determined from the amplification factors for superradiant scattering by the same arguments that are used to relate the spontaneous and stimulated emission coefficients for atoms interacting with the radiation field. For the emission of massless scalar particles from an extreme Kerr black hole, the rate of loss of energy is of the order of $10^{-3}/M^2$.[13] This corresponds to a rate of about 10^{-7} W for a black hole of $M = 10^{-5}$ cm $= 10^{21}$ g, which will emit predominantly visible light. It is clear that this process is significant only for microscopic black holes. Very small black holes ($M \lesssim 10^{-13}$ cm) which were formed at the beginning of the universe would have lost their angular momentum by now as a result of particle production.

Unruh[14] has given a quantum field-theoretic treatment of the particle production process for massless scalar particles and for neutrinos. In the latter case, it is found that even though there is no supperradiant scattering of neutrino waves, there is still quantum-mechanical pair production by a rotating black hole. In this paper a somewhat different treatment of the quantization of a scalar field in the Kerr spacetime will be given. In the case when the mass of the field vanishes, the results given here agree with those of Unruh.

Hawking[15,16] has recently given a treatment of a quantized scalar field in the gravitational field of an object undergoing gravitational collapse to form a black hole. He finds that the total number of particles produced into each mode during the collapse process is infinite, corresponding to a constant rate of production which continues for an infinite amount of time. This result is interpreted to mean that the collapsing object actually loses all of its mass by particle production and goes out of existence. The rate of energy loss is proportional to $1/M^2$, so that it is of the same order of magnitude as the energy loss by an $a = M$ Kerr black hole due to the process considered in this paper. Hawking's work has been extended by Gibbons[17] to calculate the rate of charge loss by a charged black hole. It is not clear at this time what relationship this process has to analyses such as that presented here which begin with the static or stationary geometry of an already formed black hole. It will be seen in Sec. III that the quantization presented here predicts no particle production by a Schwarzschild black hole. Since an observer at infinity will require an infinite amount of time to see the gravitational collapse process reach completion, it is perhaps conceivable that a collapsing object could behave quite differently from a black hole that has been in existence since the beginning of the universe. More work is needed on this subject before a final assessment can be given.

In Sec. II, the Klein-Gordon equation in the Kerr metric is studied and some of the properties of its solutions are reviewed. In Sec. III, a two-component formalism for the Klein-Gordon equation is developed which expresses the quantum theory of a single scalar particle in the Kerr gravitational field. A semiclassical analysis of the spontaneous particle emission is given in Sec. IV. In Sec. V a quantum field-theoretic treatment of the particle emission will be given. The problem of the vacuum energy will be discussed in Sec. VI and a discussion of the results is given in Sec. VII.

II. KLEIN-GORDON EQUATION

The Klein-Gordon equation for a scalar particle of mass μ is

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\,g^{\alpha\beta}\frac{\partial\psi}{\partial x^\beta}\right) - \mu^2\psi = 0. \tag{4}$$

We will take the Kerr metric to be in terms of the Boyer-Lindquist coordinates:

$$ds^2 = \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left((r^2+a^2)\sin^2\theta + \frac{2Mr}{\rho^2}a^2\sin^4\theta\right)d\phi^2 - \frac{4Mr}{\rho^2}a\sin^2\theta\,d\phi\,dt - \left(1 - \frac{2Mr}{\rho^2}\right)dt^2, \tag{5}$$

where $\Delta = r^2 - 2Mr + a^2$ and $\rho^2 = r^2 + a^2\cos^2\theta$ and $\sqrt{-g} = \rho^2\sin\theta$. Equivalently,

$$\left(\frac{\partial}{\partial s}\right)^2 = \rho^{-2}\left[\Delta\left(\frac{\partial}{\partial r}\right)^2 + \left(\frac{\partial}{\partial\theta}\right)^2 + (\sin^{-2}\theta - a^2\Delta^{-1})\left(\frac{\partial}{\partial\phi}\right)^2 - \frac{4Mar}{\Delta}\frac{\partial^2}{\partial\phi\,\partial t} - [\Delta^{-1}(r^2+a^2)^2 - a^2\sin^2\theta]\left(\frac{\partial}{\partial t}\right)^2\right]. \tag{6}$$

If we let $\psi = R(r)S(\theta)e^{im\phi}e^{-i\omega t}$, we find that Eq. (4) separates[18,19] and yields ordinary differential equations for $R(r)$ and $S(\theta)$:

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) + [\omega^2(r^2+a^2)^2 - 4Ma\,\omega mr - \mu^2r^2\Delta + m^2a^2 - (\omega^2a^2 + \lambda_{lm})\Delta]R = 0 \tag{7}$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left(\lambda_{lm} + c^2\cos^2\theta - \frac{m^2}{\sin^2\Theta}\right)S = 0, \tag{8}$$

where $c^2 = a^2(\omega^2 - \mu^2)$. Equation (8) is the equation for the oblate spheroidal harmonics[20] $S_{lm}(ic, \cos\theta)$ with eigenvalue λ_{lm}, where l and m are integers and $|m| \le l$. In the case where $a = 0$, the S_{lm} reduce to associated Legendre functions $P_l^m(\cos\theta)$, and λ_{lm} becomes $l(l+1)$.

The radial equation for R has regular singular points at $r = r_- \equiv M - (M^2 - a^2)^{1/2}$ and at $r = r_+$ and an irregular singular point at $r = \infty$. Relatively little is known about the properties of these functions, but some useful information can be obtained by the following transformations: let

$$R = \frac{U}{(r^2 + a^2)^{1/2}}, \tag{9}$$

and define a new radial coordinate r^* by

$$\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}. \tag{10}$$

Then Eq. (7) becomes

$$\frac{d^2U}{dr^{*2}} - V(r)U = 0, \tag{11}$$

with

$$V(r) = -\omega^2 - (r^2 + a^2)^{-2}[m^2a^2 - 4Lm\omega r - \mu^2 r^2\Delta - \Delta(\lambda_{lm} + \omega^2 a^2)] + \Delta(r^2 + a^2)^{-3}[\Delta + 2r(r - M)] - 3r^2\Delta^2(r^2 + a^2)^{-4}. \tag{12}$$

As $r \to \infty$ $(r^* \to \infty)$, $V \to \mu^2 - \omega^2$, so that

$$R \sim \frac{e^{\pm ikr}}{r}, \quad k = (\omega^2 - \mu^2)^{1/2}. \tag{13}$$

As $r \to r_+$ $(r^* \to -\infty)$, $V \to -(\omega - m\Omega_h)^2$, so

$$R \sim \frac{e^{\pm i(\omega - m\Omega_h)r^*}}{r} \tag{14}$$

is the form of R near the horizon.

The linearly independent solutions of Eq. (11) in the case when $\omega > \mu$ may be taken to be $U_\pm$, which have the asymptotic forms

$$U_+ \sim \begin{cases} e^{-ikr^*} + A_+ e^{ikr^*}, & r^* \to \infty \\ B_+ e^{-i\tilde{\omega}r^*}, & r^* \to -\infty \end{cases} \tag{15}$$

and

$$U_- \sim \begin{cases} B_- e^{ikr^*}, & r^* \to \infty \\ e^{i\tilde{\omega}r^*} + A_- e^{-i\tilde{\omega}r^*}, & r^* \to -\infty \end{cases} \tag{16}$$

where $\tilde{\omega} = \omega - m\Omega_h$.

The Wronskian relations for Eq. (11) state that

$$W = U_1 \frac{dU_2}{dr^*} - U_2 \frac{dU_1}{dr^*} = \text{constant} \tag{17}$$

for any two solutions U_1 and U_2. This leads to the relations

$$1 - A_+(k)A_+^*(k^*) = \frac{\tilde{\omega}}{k} B_+(k)B_+^*(k^*), \tag{18a}$$

$$1 - A_-(k)A_-^*(k^*) = \frac{k}{\tilde{\omega}} B_-(k)B_-^*(k^*), \tag{18b}$$

$$kB_-(k) = \tilde{\omega}B_+(k), \tag{18c}$$

$$A_+^*(k^*)B_-(k) = -\frac{\tilde{\omega}}{k} A_-(k)B_+^*(k^*). \tag{18d}$$

In this form, the relations are valid even if ω is complex provided that $\omega(k^*) = \omega^*(k)$.

If $\omega < \mu$, then the only allowable solution of Eq. (11) is the one which decays exponentially at large r. That is, if $\kappa = (\mu^2 - \omega^2)^{1/2}$, then

$$U \sim \begin{cases} e^{-\kappa r^*}, & r^* \to \infty \\ Ae^{i\tilde{\omega}r^*} + Be^{-i\tilde{\omega}r^*}, & r^* \to -\infty. \end{cases} \tag{19}$$

This solution leads to the Wronskian relation $|A|^2 = |B|^2$ if κ is real.

The solution U_+ describes a wave which impinges on the black hole from infinity; on the horizon it is purely ingoing in the frame of a physical observer.[21] The solution U_- is a wave which emerges from the black hole; it does not describe a physical classical wave, but is needed to form a complete set of solutions of the Klein-Gordon equation.

The solution U would presumably describe in a bound state near a Kerr black hole. The fact that $|A| = |B|$ says that if such a state is to exist, it must consist of equal amounts of ingoing and outgoing components on the horizon. One interpretation of this result is there cannot be modes of definite frequency $\omega < \mu$ in the Kerr metric. It is reasonable to expect that if one imposes the boundary condition that the solution be purely ingoing at the horizon, the only possible solutions would be resonant states of indefinite frequency. These states describe particles which eventually fall through the horizon of the black hole. A numerical investigation of such states has been done by Deruelle and Ruffini.[22]

If a wave packet which is sharply peaked in frequency is scattered off of the black hole, Eq. (15) tells us that a fraction $|A_+|^2$ of it will be reflected back to infinity and a fraction $1 - |A_+|^2$ will be absorbed by the black hole. However, if $\tilde{\omega} < 0$, then $|A_+|^2 > 1$ as may be seen from Eq. (18a). Thus, the modes for which $\tilde{\omega} < 0$ $(\omega < m\Omega_h)$ are the superradiant modes in which the wave is amplified. We will let

$$A_{klm} \equiv |A_+|^2 - 1 \tag{20}$$

denote the amplification coefficient.

The question arises as to whether there are modes of complex frequency in the Kerr metric. No such modes are known, but there is no proof that none exist. Such modes might be similar to those discovered by Schiff, Snyder, and Weinberg[23] in the case of a charged scalar field interacting with a square-well potential. We will assume in this paper that no such modes exist. The existence of such modes would not necessarily affect the spontaneous emission of energy to infinity if $\mathrm{Re}\,\omega < \mu$. Detweiler and Ipser[24] have made a search which did not reveal any complex frequency modes of $\mathrm{Re}\,\omega > 0$ in the case of a massless scalar field.

The orthogonality relation satisfied by two solutions $S_{l_1|m|}(ic_1, \cos\theta)$ and $S_{l_2|m|}(ic_2, \cos\theta)$ of Eq. (8) is

$$\int_0^\pi [(\lambda_{l_1 m} - \lambda_{l_2 m}) + a^2(\omega_1{}^2 - \omega_2{}^2)\cos^2\theta]\, S_{l_1|m|}(ic_1, \cos\theta) S_{l_2|m|}(ic_2, \cos\theta)\sin\theta\, d\theta = 0. \tag{21}$$

Likewise, the radial functions satisfy the relation

$$\int_{r_+}^\infty \left\{\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\right](\omega_1{}^2 - \omega_2{}^2) - \frac{4Lr}{\Delta}(m_1\omega_1 - m_2\omega_2) + \lambda_{l_2 m_2} - \lambda_{l_2 m_2} + \frac{a^2}{\Delta}(m_1{}^2 - m_2{}^2)\right\} R_{\omega_1 l_1 m_1} R_{\omega_2 l_2 m_2}\, dr = 0. \tag{22}$$

Thus, if $f_{\omega_1 l_1 m_1}$ and $f_{\omega_2 l_2 m_2}$ are two solutions of Eq. (4) of real frequency, they satisfy the orthogonality relation

$$\int f_{\omega_1 l_1 m_1} f^*_{\omega_2 l_2 m_2}[g^{03}(m_1 + m_2) - g^{00}(\omega_1 + \omega_2)]\sqrt{-g}\, d^3x \propto \delta_{\omega_1\omega_2}\delta_{l_1 l_2}\delta_{m_1 m_2}, \tag{23}$$

where $\delta_{\omega_1\omega_2}$ is a Kronecker δ if the frequencies are discrete and a Dirac δ if they are continuous.

This suggests the following definition for the inner product of any two solutions of Eq. (4):

$$\langle f_1, f_2\rangle = \tfrac{1}{2} i \int (f_1 f_2^{*,0} - f_1{}^{;0} f_2^*)\sqrt{-g}\, d^3x. \tag{24}$$

The solutions of definite ω, l, and m are orthogonal to one another under this inner product. The inner product is independent of time for any two solutions, f_1 and f_2, of Eq. (4). It will often be convenient to deal with a complete set of wave-packet states $\{F_\lambda\}$ which are orthonormal with respect to the inner product. Since the value of the inner product is preserved as the wave packets propagate through spacetime, the wave packets may be chosen so that they are orthonormal when they are at infinity or on the horizon.

One such set of wave-packet states may be constructed as follows. Let $g_{nj}(x, t)$ be defined by

$$g_{nj} = \frac{1}{(2\pi\epsilon)^{1/2}} \int_{j\epsilon}^{(j+1)\epsilon} e^{-2\pi i n\epsilon^{-1}k} e^{i(kx-\omega t)}\, dk. \tag{25}$$

where n and j are any integers. In the case $\mu = 0$, this set of wave packets was used by Hawking.[16] These functions are orthonormal:

$$\int_{-\infty}^\infty g_{nj} g^*_{n'j'}\, dx = \delta_{nn'}\delta_{jj'}. \tag{26}$$

They also form a complete set of positive-frequency solutions of the free Klein-Gordon equation in one dimension. For any k', $-\infty < k' < \infty$, let

$$a_{nj}(k', t) = \left(\frac{2\pi}{\epsilon}\right)^{1/2} e^{2\pi i k'\epsilon^{-1} n} e^{-i\omega' t} \times \int_{j\epsilon}^{(j+1)\epsilon} e^{i\omega t} e^{2\pi i n\epsilon^{-1} k}\, dk, \quad j\epsilon \le k' < (j+1)\epsilon$$
$$= 0, \quad \text{otherwise}. \tag{27}$$

Then using the identity[25]

$$\sum_{n=-\infty}^\infty \frac{e^{in\theta}}{n-a} = -\pi\frac{e^{ia(\theta-\pi)}}{\sin a\pi}, \quad 0 < \theta < 2\pi \tag{28}$$

we may see that

$$e^{i(k'x-\omega' t)} = \sum_{n,j=-\infty}^\infty a_{nj} g_{nj}. \tag{29}$$

Since the functions $e^{i(k'x-\omega' t)}$ form a complete set of positive-frequency solutions, the g_{nj} must also be complete. Similarly, the g^*_{nj} form a complete set of negative-frequency solutions.

These wave packets have a momentum spread of $\Delta k = \epsilon$. At time $t = 0$, g_{nj} is peaked about $x = 2\pi n\epsilon^{-1}$. If the packet is sharply peaked in momentum, its center moves at a group velocity $V_g = d\omega/dk = k/\omega$ in the $+x$ $(-x)$ direction if $k > 0$ $(k < 0)$.

In the region $r \gg M$, any solution of Eq. (4) for which $\omega \ge \mu$ may be expressed in terms of the functions $g_{nj}(r^*, t)$ and $g^*_{nj}(r^*, t)$ (in combination with the angular functions). The function U_+ corresponds to a wave packet which initially is localized at infinity (but still sharply peaked in frequency) and is ingoing. In the distant future it consists of an ingoing wave packet localized near the horizon and an outgoing wave packet localized at infinity. Correspondingly, U_- initially represents an outgoing packet localized on the horizon and goes over into an outgoing portion at infinity and an ingoing portion on the horizon. Wave packets localized on the horizon may be described in a manner similar to those at infinity. Define a set of wave packets $h_{nj}(r^*, t)$ by replacing k by $\tilde{\omega}$ in Eq. (25). The h_{nj} and h_{nj}^* (when combined with angular functions) form a complete set of solutions of Eq. (4) localized near the horizon. Thus a set of wave-packets solutions of Eq. (4) are specified by the g_{nj} and h_{nj}. This is a complete set of solutions for which $\omega \geq \mu$. At some initial time we specify the solution to be a particular one of the g_{nj} or h_{nj} (or g_{nj}^* or h_{nj}^*) and to be localized either at infinity (g_{nj}) or on the horizon (h_{nj}). Its future development is then determined by Eq. (4).

A wave packet which is localized at infinity has positive norm if it is of positive frequency and negative norm if of negative frequency, as usual. However, a wave packet which is localized on the horizon has positive norm if $\tilde{\omega} > 0$ and negative norm otherwise. Here we are assuming that the packets are sharply peaked in frequency and have, in the limit in which they are monochromatic, time dependence of the form $e^{-i\omega t}$. Positive frequency is taken to mean $\omega > 0$. Thus, in the Kerr metric positive frequency and positive norm are not always equivalent.

III. TWO-COMPONENT FORMALISM

The quantum mechanics of a single particle is most suitably formulated in terms of the two-component formalism.[26] This formulation uses a preferred time coordinate and gives up manifest covariance, nevertheless, it is more suitable for interpretation of the theory than the covariant formulation based on Eq. (4). A two-component equation equivalent to Eq. (4) is

$$H\Psi = i\frac{\partial \Psi}{\partial t}, \tag{30}$$

where the two-component wave function is

$$\Psi = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{31}$$

Here

$$\begin{aligned} u &= \frac{1}{\sqrt{2}}\left[\psi + i\,\frac{(-g^{00})^{1/2}}{\mu g^{00}}\left(g^{00}\frac{\partial\psi}{\partial t} + g^{03}\frac{\partial\psi}{\partial\phi}\right)\right], \\ v &= \frac{1}{\sqrt{2}}\left[\psi - i\,\frac{(-g^{00})^{1/2}}{\mu g^{00}}\left(g^{00}\frac{\partial\psi}{\partial t} + g^{03}\frac{\partial\psi}{\partial\phi}\right)\right]. \end{aligned} \tag{32}$$

The two-component Hamiltonian is

$$H = -\frac{1}{2\mu}\,\frac{\sigma_3 + i\sigma_2}{(-g^{00})^{1/2}}\left[\frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{-g}\,\frac{\partial}{\partial x^j}\right) - \frac{(g^{03})^2}{g^{00}}\,\frac{\partial^2}{\partial\phi^2}\right] + \frac{\mu}{(-g^{00})^{1/2}}\sigma_3 - i\,\frac{g^{03}}{g^{00}}\,I\frac{\partial}{\partial\phi}, \tag{33}$$

where

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

An inner product for the two-component wave functions can be defined by

$$\langle \Psi_1, \Psi_2 \rangle = \tfrac{1}{2}\mu \int \Psi_1^\dagger \sigma_3 \Psi_2 \sqrt{{}^3g}\; d^3x, \tag{34}$$

where $\Psi^\dagger = (u^*, v^*)$ and ${}^3g = \det(g_{ij}) = gg^{00}$. This is equivalent to the inner product defined in Eq. (24). The expectation value of an operator is defined in the usual way:

$$\langle A \rangle = \frac{\langle \Psi, A\Psi \rangle}{\langle \Psi, \Psi \rangle}. \tag{35}$$

The norm of a state Ψ of definite m and $\omega > 0$ can be expressed as

$$\langle \Psi, \Psi \rangle = \int (mg^{03} - \omega g^{00})\,\psi^*\psi\sqrt{-g}\; d^3x. \tag{36}$$

This norm is positive if

$$\omega > m\,\frac{g^{03}}{g^{00}} \tag{37}$$

everywhere, but may not be otherwise. The reason for this is that the single-particle theory breaks down if $\langle \Psi, \Psi \rangle \leq 0$ and must be replaced by a second-quantized theory.

The single-particle theory does apply well in the low-velocity, weak-field limit in which

$$\omega \simeq \mu, \quad g^{03} \simeq -\frac{2L}{r^3}, \text{ and } (-g^{00})^{1/2} \simeq 1 + \frac{M}{r}.$$

From Eq. (32), we find that $u \gg v$ in this limit, so that Eq. (30) reduces to

$$-\frac{1}{2\mu}\nabla^2 u + \mu\left(1 - \frac{M}{r}\right)u + ig^{03}\frac{\partial u}{\partial\phi} = i\frac{\partial u}{\partial t}. \tag{38}$$

If $\chi = ue^{i\mu t}$, then

$$-\frac{1}{2\mu}\nabla^2\chi - \frac{M\mu}{r}\chi + \frac{2L}{r^3}L_z\chi = i\frac{\partial\chi}{\partial t}. \tag{39}$$

This is a Schrödinger equation whose potential is a sum of an attractive Coulomb potential due to the static part of the gravitational field and a term which couples the orbital angular momenta of the rotating body and the particle. This equation describes a particle in the field of a rotating mass whose gravitational field is weak. Its energy spectrum is that of a hydrogenic atom with a spin-orbit perturbation.

We return now to the case of gravitational fields of arbitrary strength. We compute here the dependence of the single-particle energy levels on a by first-order perturbation theory. Let Ψ_0 be a state corresponding to some solution ψ_0 $(=\psi_{\omega_0 l_0 m_0})$ of Eq. (4) for the Schwarzschild ($a=0$) case. Regard this as an unperturbed state to which we apply the perturbation of increasing a slightly. The first-order perturbed energy is the expectation value of H for $a>0$ in the state Ψ_0:

$$\omega = \langle H\rangle_{\Psi_0} = \omega_0 \langle \Psi_0, \Psi_0\rangle^{-1} \int \psi_0 \psi_0^* \left(\omega_0 + \frac{2Lm_0}{r^3}\right)\left(1-\frac{2M}{r}\right)^{-1} \times \sqrt{-g}\, d^3x, \tag{40}$$

where

$$\langle \Psi_0, \Psi_0\rangle = \omega_0 \int \psi_0\psi_0^*\left(1-\frac{2M}{r}\right)^{-1}\sqrt{-g}\,d^3x. \tag{41}$$

It has been assumed that $a \ll M$ so that

$$g^{00} \simeq -\left(1-\frac{2M}{r}\right)^{-1} \text{ and } g^{03} \simeq -\frac{2L}{r^3}\left(1-\frac{2M}{r}\right)^{-1}.$$

To first order in a, the mass M is the same as that for the Schwarzschild case. In the case of an $a \ll M$ black hole, Eq. (40) simplifies to

$$\omega = \omega_0 + m_0 \frac{L}{4M^3} = \omega_0 + m_0 \Omega_h, \tag{42}$$

since the dominant contribution to the integral comes near $r=2M$.

This result holds for states ψ_0 which are nonzero on the horizon; otherwise the second term in Eq. (42) does not appear, i.e., $\omega \simeq \omega_0$. Since ω can be negative even though $\omega_0 > 0$, this means that the addition of some angular momentum to the black hole can cause certain single-particle levels of positive energy to become negative-energy levels. The norm of the solution of the Klein-Gordon equation will change sign when this happens. In the case when $a \ll M$, we may see from Eq. (42) that the solutions of the Klein-Gordon equation which in Sec. II were referred to as being positive-frequency, negative-norm solutions are in fact solutions whose norm would be positive if $a=0$. That is, these are the solutions whose norm will change from positive to negative if the value of a is adiabatically increased from zero to its actual value. It is the existence of such solutions which signals the breakdown of the single-particle theory and gives rise to the spontaneous pair creation.

IV. SEMICLASSICAL TREATMENT OF PARTICLE PRODUCTION

A semiclassical analysis of the particle production by a rotating black hole may be given using the connection between stimulated and spontaneous emission of bosons. In the case that the emission rate is sufficiently low (as in the case of emission of scalar particles by a Kerr black hole), an analysis based on the use of complex frequencies may be given. Enclose a rotating black hole in a reflecting spherical cavity of radius $r_0 \gg M$. The classical waves in this cavity are described by U_+, which is required to vanish at $r=r_0$. If $A_+ = e^{i\delta}|A_+|$ then the eigenvalues of k are

$$k_n = \frac{1}{2r_0}[(2n+1)\pi - \delta + i\ln|A_+|], \quad n=0, 1, \ldots. \tag{43}$$

We assume here that the amplification coefficient for the superradiant modes is small so that $A_{klm} \ll 1$ or $\ln|A_+| \ll 1$. Let $K_n = \mathrm{Re}k_n$ and $I_n = \mathrm{Im}k_n$. Then

$$\omega_n = (k_n^2 + \mu^2)^{1/2} \simeq \alpha_n + i\beta_n, \tag{44}$$

where $\alpha_n = (K_n^2+\mu^2)^{1/2}$ and $\beta_n = K_n I_n (K_n^2+\mu^2)^{-1/2}$.

Thus, the eigenfrequencies of the normal modes of the cavity are complex. As expected, the superradiant modes grow and the nonsuperradiant modes decay. Consider a particular superradiant mode. The energy density, and hence the total energy in the cavity, in that mode will be proportional to $e^{2\beta_n t}$. Thus, if $N_{nlm}(t)$ is the number of quanta present in mode n, l, m at time t, then

$$\frac{dN_{nlm}}{dt} = 2\beta_{nlm}N_{nlm}. \tag{45}$$

This is the contribution from stimulated emission.

However, this means that the spontaneous emission rate into this mode is $2\beta_{nlm}$. This is a universal relation between the stimulated and spontaneous emission rates for any boson. Consider a system of bosons interacting with some external perturbation. Let $|0\rangle$ be the vacuum state of the system and $a_\lambda^\dagger$ and $b_\lambda^\dagger$ be the creation operators for particles and antiparticles, respectively. (In the case of neutral particles, $b_\lambda^\dagger$ is also a creation operator for a particle, but corresponding to a different mode than $a_\lambda^\dagger$.) The production of particles can only occur if the full Hamiltonian for the interacting system can be expressed in a form in which it has the off-diagonal term $ca_\lambda^\dagger b_{\lambda'}^\dagger$. The probability per unit time that the system will make a transition from the state

$$|n_\lambda, 0\rangle = (n_\lambda!)^{-1/2}(a_\lambda^\dagger)^{n_\lambda}|0\rangle$$

to the state

$$|n_\lambda+1, 1\rangle = [(n_\lambda+1)!]^{-1/2}(a_\lambda^\dagger)^{n_\lambda+1}b_{\lambda'}^\dagger|0\rangle$$

is proportional to the squared matrix element

$$|\langle n_\lambda, 0|H|n_\lambda+1, 1\rangle|^2 = |c|^2(n_\lambda+1). \tag{46}$$

When $n_\lambda = 0$, this is the spontaneous emission rate; otherwise it is the sum of the stimulated and spontaneous rates. Hence, the ratio of the stimulated emission rate into a state of n_λ particles to the spontaneous emission rate is n_λ.

Returning to the particular case of the black hole in a cavity, we see that the rate of spontaneous emission into the mode n, l, m must be

$$\frac{d\tilde{N}_{nlm}}{dt} = 2\beta_{nlm}. \tag{47}$$

The rate at which energy is lost by the black hole is

$$-\dot{M} = \sum_{nlm} 2\beta_{nlm}\alpha_n, \tag{48}$$

where the sum is over all superradiant modes.

In the limit that $r_0 \to \infty$, $\sum_n \to (r_0/\pi)\,dK$ and $dK = (K^2+\mu^2)K^{-1}d\omega$. Thus,

$$-\dot{M} = \frac{1}{\pi}\int_\mu^{m\Omega_h} \omega \ln|A_+|\,d\omega \simeq \frac{1}{2\pi}\int_\mu^{m\Omega_h} \omega A_{klm}\,d\omega. \tag{49}$$

This result is an approximate form of the exact result obtained below, Eq. (77). The fact that $A_{nlm} \ll 1$ has been used not only in Eq. (44), but is essential for the entire approach to be meaningful. Only if $|\mathrm{Im}\omega| \ll |\mathrm{Re}\omega|$ is it meaningful to regard the complex-frequency modes as though they were real-frequency modes which have been slightly perturbed by the presence of the black hole.

V. FIELD QUANTIZATION IN THE KERR METRIC

A. Quantization

The Lagrangian density for a neutral scalar field is

$$\mathcal{L} = -\tfrac{1}{2}(g^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} + \mu^2\psi^2). \tag{50}$$

The canonical momentum is (for an alternative definition, see Appendix A)

$$\pi = \frac{\delta\mathcal{L}}{\delta\psi_{,0}} = -\psi^{,0}, \tag{51}$$

and the Hamiltonian density is

$$\begin{aligned}\mathcal{H} &= \pi\psi_{,0} - \mathcal{L}\\ &= \tfrac{1}{2}(g^{ij}\psi_{,i}\psi_{,j} - g^{00}\psi_{,0}\psi_{,0} + \mu^2\psi^2).\end{aligned} \tag{52}$$

The Hamiltonian is

$$H = \int \mathcal{H}\sqrt{-g}\,d^3x, \tag{53}$$

where the integration is over a t = constant spacelike hypersurface. It may also be expressed, after use of Eq. (4) and an integration by parts, as

$$H = \tfrac{1}{2}\int [2g^{03}\psi_{,03}\psi - g^{00}(\psi_{,0}\psi_{,0} - \psi_{,00}\psi)]\sqrt{-g}\,d^3x. \tag{54}$$

If ψ is a wave-packet solution of the Klein-Gordon equation (or a field operator expanded in terms of such solutions), the surface term obtained in the integration vanishes.

The quantization of the scalar field is effected by imposing the commutation relations on the t = constant hypersurfaces:

$$[\psi(x^i, t), \pi(y^i, t)] = i\delta(x^i, y^i), \tag{55}$$

where $\delta(x^i, y^i)$ is a three-dimensional Dirac δ function which satisfies the condition $\int \delta(x^i, y^i)\sqrt{-g}\,d^3y = 1$. Let $\{F_\lambda\}$ be a complete set of positive-norm wave-packet solutions of Eq. (4) which are orthonormal with respect to the inner product

$$\langle F_\lambda, F_{\lambda'}\rangle = \delta_{\lambda\lambda'}. \tag{56}$$

Then $\{F_\lambda^*\}$ will form a complete orthonormal set of negative-norm solutions:

$$\langle F_\lambda^*, F_{\lambda'}^*\rangle = -\delta_{\lambda\lambda'} \tag{57}$$

and

$$\langle F_\lambda, F_{\lambda'}^*\rangle = 0. \tag{58}$$

The field operator may be expanded as

$$\psi = \sum_\lambda \frac{1}{\sqrt{2}}(a_\lambda F_\lambda + a_\lambda^\dagger F_\lambda^*), \tag{59}$$

where

$$a_\lambda = \sqrt{2}\langle \psi, F_\lambda\rangle \text{ and } a_\lambda^\dagger = -\sqrt{2}\langle \psi, F_\lambda^*\rangle. \tag{60}$$

The creation and annihilation operators satisfy the usual commutation relations:

$$[a_\lambda, a_{\lambda'}^\dagger] = \delta_{\lambda\lambda'}. \tag{61}$$

Let us further assume that the wave packets F_λ are sharply peaked in frequency, that is, that $\Delta\omega \ll 1/M$. Such wave packets, while still normalizable functions, may be treated as monochromatic waves as far as their interaction with the Kerr geometry is concerned.

The time dependence of F_λ will then be of the form $e^{-i\Omega_\lambda t}$. As was discussed in Sec. II, positive frequency and positive norm are not always equivalent in the Kerr metric, so Ω_λ is not always positive. In particular, $\Omega_\lambda = \omega_\lambda\tilde{\omega}_\lambda/|\tilde{\omega}_\lambda|$ in the case of a wave packet which is localized on the horizon. Wave packets which are localized when at infinity always have $\Omega_\lambda = \omega_\lambda$.

If one now substitutes the expansion Eq. (59) into Eq. (54), the Hamiltonian becomes

$$H = \tfrac{1}{2}\sum_\lambda \Omega_\lambda(a_\lambda a_\lambda^\dagger + a_\lambda^\dagger a_\lambda). \tag{62}$$

If the vacuum state of the system is defined to be

$|0\rangle$, where $a_\lambda|0\rangle=0$ for all λ, then it is an eigenstate of the Hamiltonian, as are all other states of definite particle number. However, since $\Omega_\lambda<0$ for some modes, $|0\rangle$ is not the state of lowest energy of the system, and in fact no such state exists. By adding additional particles to such a mode, states of ever decreasing energy may be attained. The presence of negative-energy modes suggests that although the vacuum is an eigenstate of the Hamiltonian, it is still unstable with respect to emission of particles into these modes. In a certain sense this is true, as will be seen below. The fact that H is diagonal stems from the use of exact solutions of the Klein-Gordon equation in the Kerr metric in the expansion of ψ. If, on the other hand, we regarded the presence of the black hole as a perturbation on a field which satisfies the free, Minkowski-space Klein-Gordon equation, we could expand ψ in terms of solutions of this latter equation. Then H would be off-diagonal and have terms of the sort mentioned in Sec. IV. This is, of course, the usual approach for treating the interaction of an atom with the electromagnetic field. In the black-hole case, however, it would be very messy if carried through in detail. A more satisfactory approach is the one given below, or the equivalent one given by Unruh.[14]

B. Particle Production

As a prelude to the discussion of particle production by the Kerr metric, it will be convenient to introduce some further notation for labeling our wave packets. Let $F_\lambda=F_{(\gamma\delta)njlm}$, where l and m are usual quantum numbers arising from the eigenvalues of the angular equations, j labels the mean momentum of the wave packet, and n labels its mean position at some time. We require that F_λ be, at some time, localized entirely at infinity or entirely on the horizon, and when it is so localized to be entirely ingoing or entirely outgoing. When this occurs, if it is at infinity $\gamma=+$, and if it on the horizon $\gamma=-$; if it is outgoing $\delta=+$, and if it is ingoing $\delta=-$. The specific form of F_λ may be left unspecified for the present, but could, in particular, be taken so that the r^* and t dependence of F_λ is described by the g_{nj} or h_{nj} of Sec. II when F_λ is at infinity or on the horizon. In a slight departure from the notation used in Sec. II, we here take j to describe only the magnitude of the mean momentum of the packet. The label γ is redundant since n contains the same information, but it will be convenient to also use γ.

Localization here means that the wave packet can be thought of as being "in one piece." Thus, a $F_{(++)njlm}$ packet and a $F_{(--)njlm}$ packet always stay localized, but the $F_{(+-)njlm}$ and $F_{(-+)njlm}$ packets do not. These latter packets will consist of two disjoint positions when $t\to\infty$, one at infinity and one on the horizon.

We may introduce a second set of wave packets, the $G_\lambda=G_{(\gamma\delta)njlm}$, which will be chosen so that as $t\to\infty$, the localized portions of F_λ are G_λ's. In particular, we may let

$$G_{++}=F_{++} \tag{63a}$$

and

$$G_{--}=F_{--}\,. \tag{63b}$$

We will normally suppress the labels n, j, l, and m. This and the following transformations are diagonal in these labels.

If $\tilde{\omega}>0$, let

$$F_{+-}=A_+G_{++}+B_+\left(\frac{\tilde{\omega}}{k}\right)^{1/2}G_{--}, \tag{64a}$$

$$F_{-+}=B_-\left(\frac{k}{\tilde{\omega}}\right)^{1/2}G_{++}+A_-G_{--}\,, \tag{64b}$$

and if $\tilde{\omega}<0$, let

$$F_{+-}=A_+G_{++}+B_+\left(\frac{-\tilde{\omega}}{k}\right)^{1/2}G^*_{--}\,, \tag{65a}$$

$$F_{-+}=B^*_-\left(\frac{k}{-\tilde{\omega}}\right)^{1/2}G^*_{++}+A^*_-G_{--}\,. \tag{65b}$$

Using the Wronskian relations, Eqs. (18a)–(18d), these expressions may also be written as

$$G_{++}=A^*_+F_{+-}+B^*_-\left(\frac{k}{\tilde{\omega}}\right)^{1/2}F_{-+}\,, \tag{66a}$$

$$G_{--}=B^*_+\left(\frac{\tilde{\omega}}{k}\right)^{1/2}F_{+-}+A^*_-F_{-+}\,, \tag{66b}$$

if $\tilde{\omega}>0$, and, if $\tilde{\omega}<0$, as

$$G_{++}=A^*_+F_{+-}-B^*_-\left(\frac{k}{-\tilde{\omega}}\right)^{1/2}F^*_{-+}\,, \tag{67a}$$

$$G_{--}=-B_+\left(\frac{-\tilde{\omega}}{k}\right)^{1/2}F^*_{+-}+A_-F_{-+}\,. \tag{67b}$$

The G_λ have been defined so that $\langle G_\lambda, G_{\lambda'}\rangle=\delta_{\lambda\lambda'}$; they form a complete set of orthonormal positive-norm solutions of the Klein-Gordon equation for $\omega>\mu$.

Consider a particular packet F_{+-}. It comes in from infinity and scatters off the potential barrier; as $t\to\infty$ it consists of a portion, A_+G_{++}, which has been scattered back to infinity and a portion

$$B_+\left(\frac{\tilde{\omega}}{k}\right)^{1/2}G_{--}\quad\left[B_+\left(\frac{-\tilde{\omega}}{k}\right)^{1/2}G^*_{--}\right]$$

if $\tilde{\omega}>0\,[\tilde{\omega}<0]$ which is going down the black hole. Similarly, the F_{-+} also split into two portions. When $\tilde{\omega}<0$, we see that these portions need not have positive norm in themselves, even though the entire packet does.

These packets must be complemented with a complete set of solutions of Eq. (4) with $0 \leq \omega \leq \mu$ in order that we have a truly complete set. Henceforth, $\{F_\lambda\}$ or $\{G_\lambda\}$ will denote this complemented set. However, we will not need to refer explicitly to the $\omega \leq \mu$ solutions. The field operator may be expanded in terms of either the $\{F_\lambda\}$ or the $\{G_\lambda\}$:

$$\psi = \sum_\lambda \frac{1}{\sqrt{2}}(a_\lambda F_\lambda + a_\lambda^\dagger F_\lambda^*) = \sum_\lambda \frac{1}{\sqrt{2}}(b_\lambda G_\lambda + b_\lambda^\dagger G_\lambda^*). \tag{68}$$

If we write

$$F_\lambda = \sum_\lambda (\alpha_{\lambda\lambda'} G_{\lambda'} + \beta_{\lambda\lambda'} G_{\lambda'}^*), \tag{69}$$

then since $a_\lambda = \sqrt{2}\langle F_\lambda, \psi\rangle$ and $b_\lambda = \sqrt{2}\langle G_\lambda, \psi\rangle$, we have that

$$a_\lambda = \sum_{\lambda'} (\alpha_{\lambda\lambda'} b_{\lambda'} - \beta_{\lambda\lambda'} b_{\lambda'}^\dagger). \tag{70}$$

The fact that $[a_\lambda, a_{\lambda'}^\dagger] = [b_\lambda, b_{\lambda'}^\dagger] = \delta_{\lambda\lambda'}$ imposes the condition

$$\sum_\lambda (\alpha_{\lambda_1\lambda}\alpha_{\lambda_2\lambda}^* - \beta_{\lambda_1\lambda}^*\beta_{\lambda_2\lambda}) = \delta_{\lambda_1\lambda_2}. \tag{71}$$

We must now decide which creation and annihilation operators represent physical particles. The case of wave packets localized at infinity presents no problems; here we may rely upon the usual particle interpretation of quantum field theory in Minkowski space. Thus, if F_λ is such a packet, $a_\lambda^\dagger|0\rangle$ may be interpreted as a one-particle state in Fock space corresponding to a single particle at infinity. The case of wave packets localized on the horizon is more ambiguous. Here particles cannot be said to be free, although formally the solutions of the Klein-Gordon equations have the form of free-particle solutions. A natural definition which presents itself is to continue to use modes which have time dependence of the form $e^{-i\omega t}$ (this corresponds to the use of the timelike Killing vector of the Kerr geometry to define modes). Packets of positive norm represent particles and those of negative norm represent antiparticles. (In the case of a charged field, the association of a particle with a positive-norm solution as opposed to a positive-frequency solution is necessary in order that all particle states have the same sign of charge and all antiparticle states have the opposite sign.) Thus if F_λ is localized on the horizon, $a_\lambda^\dagger|0\rangle$ is still interpreted as a physical one-particle state.

Consider the history of a particular wave packet, F_{+-} for which $\tilde{\omega} < 0$. At $t \to -\infty$ it is localized at infinity and at $t \to +\infty$, it consists of a positive-norm portion outgoing at infinity and a negative-norm portion ingoing on the horizon. In the latter case, the associated one-particle state in Fock space is to be regarded as a superposition of the state for an outgoing particle and the state for an ingoing (to the horizon) antiparticle. (Particles and antiparticles are of course identical in the case of a neutral scalar field.) This means that at $t \to +\infty$, the physical creation and annihilation operators for the mode λ are $b_\lambda^\dagger$ and b_λ, not $a_\lambda^\dagger$ and a_λ. The meaning of $t \to \infty$ for a given wave packet is a time long after it has scattered off of the potential barrier. For each packet, we choose some time in its history after it has scattered off of the potential barrier at which we make the reassignment of the associated physical creation operator from $a_\lambda^\dagger$ to $b_\lambda^\dagger$. In the case of F_{++} and F_{--} packets, this reassignment is trivial: $b_\lambda^\dagger = a_\lambda^\dagger$. In the case of F_{+-} and F_{-+} packets for which $\tilde{\omega} > 0$, it merely replaces a_λ by a linear combination of a_λ's, and hence transforms the basis of the one-particle sector of Fock space without changing the vacuum state. However, for F_{+-} and F_{-+} packets with $\tilde{\omega} < 0$ it performs a Bogoliubov transformation which introduces a new vacuum state, since b_λ is a linear combination of a_λ and $a_\lambda^\dagger$. From Eqs. (70), (69), and (65a)–(65b) we have in this case that

$$a_{+-} = A_+ b_{++} - B_+\left(\frac{-\tilde{\omega}}{k}\right)^{1/2} b_{--}^\dagger, \tag{72a}$$

$$a_{-+} = A_-^* b_{--} - B_-^*\left(\frac{k}{-\tilde{\omega}}\right)^{1/2} b_{++}^\dagger, \tag{72b}$$

or, equivalently,

$$b_{++} = A_+^* a_{+-} + B_-^*\left(\frac{k}{-\tilde{\omega}}\right)^{1/2} a_{-+}^\dagger, \tag{73a}$$

$$b_{--} = A_- a_{-+} - B_-\left(\frac{k}{-\tilde{\omega}}\right)^{1/2} a_{+-}^\dagger. \tag{73b}$$

We will adopt the Heisenberg picture, so the state of the system is the same for all time. If at $t \to -\infty$ it is $|0\rangle$, where $a_{+-}|0\rangle = a_{-+}|0\rangle = 0$, then at $t \to \infty$, $|0\rangle$ will actually contain particles defined relative to the b_λ's. The expected number of such particles will be

$$N_\lambda = \langle 0|b_\lambda^\dagger b_\lambda|0\rangle. \tag{74}$$

The rate of spontaneous emission by the black hole to infinity is determined by

$$N_{++} = \langle 0|b_{++}^\dagger b_{++}|0\rangle = A_{klm}. \tag{75}$$

N_{++} is the total number of particles that are created into the mode $\lambda = (++)njlm$. We may regard the contribution of a given mode to the total particle production rate as being tallied when the packet G_{++} passes some value of $r = r_0 \gg M$. The total particle production rate will depend on the rate at which the packets G_{++} pass this surface, as well as on N_{++}. Let us choose, for the particular form of the G_{++}, packets whose r and t dependence for $r >> M$ is given by the packets g_{nj} of Sec. II. These packets are spaced a distance $\Delta r = 2\pi\epsilon^{-1}$ apart and travel at a velocity of $V_g = k/\omega$. Thus, the number of packets of fixed j, l, and m which cross the surface per unit time moving outward is $u = \epsilon k/2\pi\omega$. The number of packets per unit k interval is $\rho = \epsilon^{-1}$. Consequently, the total rate at which particles are created into the momentum interval dk with quantum numbers l and m is

$$N_{++} u \rho \, dk = \frac{1}{2\pi} N_{++} \, d\omega. \tag{76}$$

Finally, the total rate of emission of energy by the black hole to infinity is

$$-\dot{M} = \frac{1}{2\pi} \sum_{l,m} \int_{\mu}^{m\Omega_h} \omega A_{klm} \, d\omega. \tag{77}$$

The sum on m is taken only over values for which $m\Omega_h > \mu$.

A charged scalar field may be treated in the same manner as a neutral field. The only essential difference is that the rate of production of particles into each mode is twice that for neutral particles; the presence of the charge does not affect the interaction of the field with the black hole, it only doubles the number of degrees of freedom of the field. Both particles and antiparticles have equal probabilities of being emitted to infinity, so the net charge flux out of the black hole is zero. (This will not be true if the black hole has a net charge at the outset.)

Starobinsky[10] has obtained an analytic expression for the amplification coefficient A_{klm} in the region $\omega M << 1$ in the case $\mu = 0$. This result may be generalized to the case when $\mu \neq 0$. (See Appendix B.) The result is

$$A_{klm} = \frac{\pi (r_+{}^2 + a^2)(r_+ - r_-)^{2l} 2^{2l+3} (l!)^2 \eta \tilde{\omega} k^{2l+1}}{[(2l)!]^2 [(2l+1)!]^2 (1 - e^{2\pi\eta})} \times \prod_{j=1}^{l} (j^2 + \eta^2) \prod_{n=1}^{l} (n^2 + 4Q^2), \tag{78}$$

where $\eta = -2M\mu^2/k$ and $Q = (r_+{}^2 + a^2)\tilde{\omega}/(r_- - r_+)$.

This formula is valid for both superradiant and nonsuperradiant modes of $\omega M << 1$. In the latter case, $-A_{klm}$ is positive and is the fraction of the wave absorbed by the black hole. If $a << M$, then Eq. (78) is valid for all superradiant modes. In this case, the mode $l = m = 1$ gives the dominant contribution to the spontaneous emission of particles by the black hole;

$$A_{k11} = \frac{32\pi M^4 \tilde{\omega} \eta k^3 (1 + \eta^2)}{9(1 - e^{2\pi\eta})}, \tag{79}$$

since $Q^2 << 1$ in this case. For a black hole with $a << M$, the number of neutral scalar particles emitted per unit time per unit frequency interval is approximately $(2\pi)^{-1} A_{k11}$. If $\mu = 0$, then

$$A_{k11} = -\tfrac{16}{9} M^4 \tilde{\omega} \omega^3. \tag{80}$$

If we insert this expression into Eq. (77), the result is

$$-\dot{M} = \frac{a^6}{\pi 2^{10} 3^3 5 M^8} \simeq 10^{-6} \frac{a^6}{M^8}. \tag{81}$$

This is the rate of energy loss due to emission of massless scalar particles by a black hole with $a << M$.

Starobinsky[10] has also given expressions for A_{klm} in the $\mu = 0$ case in the region near $\omega = m\Omega_h$. A particularly interesting aspect of this result is that A_{klm} may oscillate rapidly in this region.

We may also treat stimulated emission of particles in this framework. Suppose that the state of the system is not $|0\rangle$, but rather

$$|N\rangle = \frac{(a_{-+}^{\dagger})^N}{\sqrt{N!}} |0\rangle. \tag{82}$$

There are now N particles initially present in one of the ingoing wave-packet modes. The expected number of particles emitted to infinity in the corresponding outgoing wave-packet mode is

$$N_{++} = \langle N | b_{++}^{\dagger} b_{++} | N \rangle = A_{klm}(N+1). \tag{83}$$

This reflects not only the contribution from spontaneous emission, A_{klm}, but also a contribution from stimulated emission $N A_{klm}$. In the classical limit, $N \to \infty$, the wave packet is amplified by a factor of A_{klm}.

The Bogoliubov transformation Eq. (73a)–(73b) introduces a new Fock space of eigenstates of $b_{\lambda}^{\dagger} b_{\lambda}$. Let $|0)$ be the vacuum state of this Fock space, so that

$$b_{++}|0) = b_{--}|0) = 0. \tag{84}$$

We may express $|0\rangle$ in terms of the vectors in the new space. Let

$$\alpha = A_+, \quad \beta = -B_+ \left(\frac{-\tilde{\omega}}{k} \right)^{1/2}. \tag{85}$$

Then

$$|0\rangle = D \sum_{n=0}^{\infty} \left(-\frac{\beta}{\alpha} \right)^n \frac{1}{n!} (b_{++}^{\dagger} b_{--}^{\dagger})^n |0). \tag{86}$$

It may be verified from Eq. (72a)–(72b) that $a_{++}|0\rangle = a_{--}|0\rangle = 0$. The transformation Eqs. (72a)–(72b) leaves all other a_{λ} invariant, so $a_{\lambda}|0) = 0$ for all other modes. The norm of $|0\rangle$ is

$$\langle 0|0\rangle = |D|^2 \sum_{n=0}^{\infty} \left| \frac{\beta}{\alpha} \right|^{2n} = |D|^2 \sum_{n=0}^{\infty} \left(\frac{A_{klm}}{|A_+|^2} \right)^n. \tag{87}$$

If $A_{klm} < |A_+|^2$ for the mode in question, then $|0\rangle$ has finite norm relative to the new Fock space. This condition is in fact fulfilled for a scalar field in the Kerr metric where the amplification coefficient A_{klm} is much less than one for all the superradiant modes. Thus, the problem of inequivalent representations of the commutation relations which often arises in field theory[27] does not occur here.

Equation (86) shows that $|0\rangle$ is a superposition of states containing various numbers of pairs of particles. Each pair consists of one member which is emitted to infinity and one which goes down the black hole. The particles which go down the black hole have negative energy as seen from infinity; hence, energy conservation is assured, and the emission of energy by the black hole is compensated by its loss of mass.

Equation (86) only describes the effect of a single Bogoliubov transformation. However, each time that a wave packet of a superradiant mode crosses the $r=r_0$ surface we need to perform such a transformation. Let $|0\rangle$ be the physical vacuum state at $t=0$ (so that the $a_\lambda^\dagger$ are the physical creation operators). At $t=\tau$, a certain number of modes will have had their physical creation operators redefined to be the $b_\lambda^\dagger$. Adopt the wave packet G_λ used above as the set of basis functions. Then the outgoing packets which cross $r=r_0$ in time τ range from $n=n_0$ to $n=n_0+k\tau\epsilon/2\pi\omega$ for fixed j. Since the packets are sharply peaked in momentum, we may let $k=j\epsilon$. We may thus express $|0\rangle$ in terms of the states of definite physical particle number at time τ as

$$|0\rangle=\prod\left[D_\lambda\sum_{p=0}^{\infty}\left(-\frac{\beta_\lambda}{\alpha_\lambda}\right)^p\frac{1}{p!}(b^\dagger_{(++)njlm}b^\dagger_{(--)njlm})^p|0\rangle\right], \tag{88}$$

where the product is taken over those values of n, j, l, and m for which $\tilde{\omega}<0$ and $n_0\leq n\leq n_0+k\tau\epsilon/2\pi\omega$. The D_λ may be chosen so that

$$|D_\lambda|^{-2}=\sum_{n=0}^{\infty}\left(\frac{A_{klm}}{|A_+(\lambda)|^2}\right)^n. \tag{89}$$

Then $\langle 0|0\rangle=1$.

VI. VACUUM ENERGY

An important question in the theory of a quantized field in a curved spacetime is that of the vacuum energy. On the basis of Poincaré invariance, it can be argued[28] that the energy of the vacuum state must be zero for a field in Minkowski space, but there does not seem to be any reason why this should be so in other spacetimes. For this reason, normal ordering has not been used in the expression for H. This issue is not directly relevant for the treatment of particle production, but is of importance in determining the limitations of the assumption that the background spacetime is a classical solution of Einstein's equations. If the energy of the vacuum is nonzero, it will modify the original background gravitational field. Formally, the vacuum energy is infinite, so the physical energy in the vacuum will have to be found by some type of renormalization procedure. Several discussions of possible procedures have been given in various contexts,[29-34] but the complete answer is still unknown.

The induction of a nonzero vacuum energy by an external gravitational field is analogous to the Casimir effect[35] in which the presence of two parallel conducting plates induces a nonzero energy into the vacuum state of the electromagnetic field and causes an attractive force between the plates. The modes whose wavelength is much shorter than the separation of the plates are unaffected by the presence of the plates, so the vacuum energy shifts by a finite amount when the separation changes. The energy density between the plates is in fact constant, and is[13]

$$U=-\pi^2/720l^4, \tag{90}$$

where l is the separation of the plates. Here the configuration in which the plates are an infinite distance apart is taken to have zero energy.

There is a means by which one may in principle measure this energy density. If we think of this vacuum energy as being associated with a fluctuating electromagnetic field, then we can write

$$U=\tfrac{1}{2}(\langle E^2\rangle+\langle B^2\rangle). \tag{91}$$

Since the field is fluctuating in all directions, $\langle\vec{E}\rangle=\langle\vec{B}\rangle=0$. Another way of visualizing this is to consider that the effect of the plates is to exclude the lower-wavelength modes from the region between the plates. Consequently, the value of $\langle E^2\rangle$ must shift when the plates are introduced, and if it is set equal to zero in the absence of the plates it must become negative in their presence. Any physical system which is sensitive to the magnitude, but not direction of an electric (or magnetic) field will then detect the electromagnetic vacuum energy. An example of such a system is a hydrogen atom; there will be a second-order Stark shift of the order of $a_0{}^3\langle E^2\rangle$ if $\langle E^2\rangle\neq 0$, where a_0 is the Bohr radius. Thus an atom which is placed between two conducting plate 100 Å apart will show a second-order Stark shift of the order of 10^{-8} eV. This shift is probably too small to actually observe, especially since there would be other effects due to interactions with the walls. Since $\langle B^2\rangle\neq 0$, there will also be a quadratic Zeeman effect. Typical energies here are of the order of $\alpha^2 a_0{}^3\langle B^2\rangle$ ($\alpha=\frac{1}{137}$, the fine-structure constant) or about 10^{-12} eV.

If the effect of an external gravitational field on the vacuum is similar to the Casimir effect, it might be expected that only wavelengths which are of the order of or larger than the distance l over which the curvature of spacetime changes will contribute to the vacuum energy. A local observer can transform the gravitational field away within a region of length l, so that much shorter wavelengths will behave as though space were flat and will not contribute. This means that the vacuum energy is expected to be of the order of $1/l$ in a region of volume l^3, so that the energy density should be of the order of $1/l^4$. This vacuum energy can be thought of as arising from the uncertainty in the definition of creation and annihilation operators by a local observer.[16] Those modes whose wavelength is much less than l do not feel the effects of the curvature and can unambiguously be defined. However, those modes whose wavelength is greater than or equal to l are not unambiguously defined. This leads to an uncertainity in the definition of the energy density of any state which is the zero-point energy density $\approx 1/l^4$.

In the case of a black hole, $l \approx M$, so the vacuum energy will be a significant fraction of the total energy when $M \approx 10^{-32}$ cm. This argument indicates that an analysis which assumes that the background gravitational field is described by a classical solution of Einstein's equations and neglects the effects of the vacuum energy will be valid if $M \gg 10^{-32}$ cm.

It can easily be seen that the vacuum energy of a quantized scalar field in a slowly rotating Kerr solution must, to first order in a, be the same as that of a Schwarzschild solution of the same mass (to first order in a, the mass of a Kerr solution is the same as its irreducible mass). Equation (42) shows that for every state ($m > 0$) which increases in energy as a increases from zero, there is another state ($m < 0$) which decreases by an equal amount. Hence the sum of $\frac{1}{2}\omega$ over all modes does not change to first order as a function of a. This result is necessary if the vacuum energy is to be an analytic function of the angular momentum of the Kerr field at $a = 0$ since the energy can depend only on the magnitude and not the direction of the angular momentum.

The energy density induced into the vacuum state of the electromagnetic field by a background gravitational field should in principle be measurable by the second-order Stark effect as is the Casimir energy density. In the vicinity of a black hole of mass M, a hydrogen atom will experience a shift of the order of $a_0{}^3/M^4$ due to the vacuum energy. If $M = 100$ Å, then this shift is of the order of 10^{-6} eV.

VII. DISCUSSION AND CONCLUSIONS

We have seen that a quantum field theory in the Kerr spacetime leads to the prediction that a rotating black hole will spontaneously emit particles and eventually lose its angular momentum. That this should be the case is apparent on semiclassical grounds, as was discussed in Sec. IV. However, the field-theoretic treatment still contains ambiguities with regard to the choice of creation and annihilation operators. In this treatment (and in the treatment given by Unruh[14]), positive-frequency modes have been defined with respect to the timelike Killing vector. In the case of wave packets which are localized at infinity this is just the usual Minkowski-space definition, so there is no ambiguity in this case. However, in the case of wave packets which are localized on the horizon, the situation is less clear. Formally, the solutions of Eq. (11) have the appearance of free-particle solutions here, but this does not necessarily mean that the physical definition of particle number should be the same as at infinity. The definition used here predicts no particle production by a Schwarzschild black hole. Thus, a Kerr black hole will eventually become a Schwarzschild black hole whose mass is equal to or greater than the irreducible mass of the original black hole. Boulware[35] has analyzed the quantization of a scalar field in the Schwarzschild spacetime and came to the conclusion that there should, in fact, be no particle production in this case.

There is still a difficulty in understanding the relation of the work on quantization of fields in the vacuum Schwarzschild and Kerr solutions and the work of Hawking[15,16] on particle production in gravitational collapse. Hawking predicts that a black hole formed by collapse continues to emit particles long after the actual collapse process. There seem to be two possible interpretations of this result. One is that this emission has nothing to do with the collapse and should show up in a correct quantum field theory in the vacuum Schwarzschild spacetime. The other is that the collapse is essential for the production of the particles; the emission appears to an observer at infinity to continue for a long time only because it takes this long for the particles formed in the collapse to reach him. At the present it is an open question as to which interpretation is correct. It does seem to be the case that one can devise quantizations which do yield particle production in the vacuum Schwarzschild geometry if one introduces definitions of positive-frequency modes on the past horizon other than that based on the timelike Killing vector.[36] What remains to be seen is whether a compelling physical criterion can be found in favor of one definition.

Another question that arises is whether the presence of the horizon is essential in order that one get spontaneous particle creation in a Kerr metric. If one had a rotating body with a Kerr metric (or approximately Kerr metric) exterior field, could it also spontaneously lose its rotational energy? If there is an ergoregion (where the Killing vector which is timelike at infinity becomes spacelike), then there seems to be no reason why there would not be pair creation. The presence of an ergoregion with or without a horizon can lead to classical superradiant scattering, and hence to spontaneous emission. If there is no horizon, the matter in the body must be able to interact with the scalar field and absorb the ingoing wave (which carries negative energy) in order that there be superradiant scattering. In the absence of an ergoregion, the situation is less clear. In a theory in which the scalar field interacts only with the gravitational field of the rotating body and not with the matter constituting the body, there will be no superradiant scattering and no spontaneous emission without a horizon. However, if the field interacts with the matter of the body as well, this may not be true. If the effect of these interactions is to exclude the scalar field from the interior of the body, the classical solutions of the Klein-Gordon equation will have to satisfy appropriate boundary conditions at the surface of the body. As was argued by Zel'dovich,[5] such boundary conditions may lead to a superradiant scattering, which will in turn lead to spontaneous emission of quanta. The existence of an ergoregion does not seem to be essential in this case.

The analysis presented in the preceding sections assumes the stationary nature of the Kerr geometry, and thus ignores the time dependence of the gravitational field caused by the reaction of the quantized field back on the metric. This should be a suitable approximation as long as the time scale on which the pair production process occurs is much longer than the time scale on which the metric changes significantly. Another way of expressing this is to say that the average energy carried away by a particle must be much smaller than the total mass of the black hole, which will be the case if $M \gg 1$. When $M\approx 1$ (10^{-5} g in conventional units), this approach breaks down; however, it is not surprising that a semiclassical theory in which the gravitational field is classical does not hold at this level since quantum effects associated with the gravitational field itself are expected to come in at distances of the order of 10^{-32} cm. Furthermore, the discussion of the vacuum energy given in Sec. VI indicates that reaction of the quantized field on the background gravitational field via vacuum energy will become large in this domain. When this happens, the Kerr solution will have to be replaced by some other self-consistent solution of Einstein's equation taking account of the vacuum energy.

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APPENDIX A

Rather than defining the canonical momentum π to be conjugate to $\psi_{,0}$, as was done in Eq. (51), one might also consider defining it to be conjugate to $-\psi^{,0}$:

$$\pi = -\frac{\delta \mathcal{L}}{\delta \psi^{,0}} = \psi_{,0}\,. \tag{A1}$$

In Minkowski space, the two definitions are identical. There seems to be no *a priori* reason to regard either one as being the preferred generalization to a Riemannian spacetime. In fact, Eq. (A1) may also be used to construct a quantum field theory.

Let $\{F_\lambda\}$ be a complete set of positive-norm solution of the Klein-Gordon equation which have time dependence of the form $e^{-i\omega_\lambda t}$. They are assumed to be orthonormal:

$$\langle F_\lambda, F_{\lambda'}\rangle = \delta_{\lambda\lambda'}\,. \tag{A2}$$

Define the $\tilde{\delta}$ function by

$$\langle F_\lambda(x), \tilde{\delta}(x,y)\rangle = \omega_\lambda F_\lambda(y). \tag{A3}$$

An explicit expression for $\tilde{\delta}$ is

$$\tilde{\delta}(x,y) = \sum_\lambda \omega_\lambda F_\lambda(x) F_\lambda^*(y). \tag{A4}$$

Now the quantization of the scalar field may be carried out by imposition of the condition

$$[\psi(x), \pi(y)] = i\tilde{\delta}(x,y) \tag{A5}$$

if $x^0 = y^0$. If ψ is expressed as

$$\psi = \sum_\lambda \frac{1}{\sqrt{2}}(a_\lambda F_\lambda + a_\lambda^\dagger F_\lambda^*), \tag{A6}$$

then, as in Sec. V, $a_\lambda = \sqrt{2}\langle F_\lambda, \psi\rangle$ and satisfy the commutation relations

$$[a_\lambda, a_{\lambda'}^\dagger] = \delta_{\lambda\lambda'}\,. \tag{A7}$$

This may be verified by subsituting Eq. (A6) into Eq. (A5) and using Eqs. (A7) and (A4).

APPENDIX B

We will follow the method of Starobinsky[10] to calculate the amplification coefficient in the case that $\omega M \ll 1$. Assume also that $a \neq M$ and let

$$x = \frac{r - r_+}{r_+ - r_-}. \tag{B1}$$

In the region in which $x \ll l/\omega(r_+ - r_-)$, Eq. (7) becomes approximately

$$x(x+1)\frac{d}{dx}\left(x(x+1)\frac{dR}{dx}\right) + [Q^2 - \lambda_{lm} x(x+1)]R = 0, \tag{B2}$$

where $Q = (r_+{}^2 + a^2)\tilde{\omega}/(r_- - r_+)$. The spheroidal harmonic eigenvalue λ_{lm} is, to second order in ka,

$$\lambda_{lm} = l(l+1) + K, \tag{B3}$$

where

$$K = \tfrac{1}{2}\left[1 - \frac{(2m-1)(2m+1)}{(2l-1)(2l+3)}\right](ka)^2. \tag{B4}$$

For $l \neq 0$, K is a very small correction to λ_{lm} and will be dropped at a later stage in the calculation, but it is desirable to retain it for the present.

A solution of Eq. (B2) is

$$R = (-1)^l \left(\frac{x}{x+1}\right)^{iQ} F(-l-K,\, l+1+K;\, 1-2iQ;\, K+1), \tag{B5}$$

where $F(a, b; c; z)$ is the hypergeometric function. This solution satisfies the boundary condition of being purely ingoing on the horizon $(x \to 0)$. We now wish to examine the behavior of R in the region $\max(l, Q) \ll x \ll l/[\omega(r_+ - r_-)]$. Use the identity

$$F(a, b; c; z) = (1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} F\left(a, c-b; a-b+1; \frac{1}{1-z}\right) + (1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} F\left(b, c-a; b-a+1; \frac{1}{1-z}\right), \tag{B6}$$

and the fact that $F(a, b; c; z) \to 1$ as $z \to 0$. Equation (B6) is valid only if $\arg|(1-z)| < \pi$, so we need to add a small imaginary part $i\delta$ to x in Eq. (B5). The result is that if $x \gg \max(l, Q)$,

$$R \simeq c_1 x^{l+K} + c_2 x^{-l-1-K}, \tag{B7}$$

where we let $\delta \to 0$ at the end of the calculation. The coefficients are given by

$$c_1 = (-1)^K \frac{\Gamma(1-2iQ)\Gamma(2l+1+2K)}{\Gamma(l+1+2K)\Gamma(l+1-2iQ+K)}, \tag{B8}$$

$$c_2 = (-1)^{1-K}\frac{\Gamma(1-2iQ)\Gamma(-2l-1-2K)}{\Gamma(-l-K)\Gamma(-l-2iQ-K)}. \tag{B9}$$

Use of the relation $\Gamma(1+z) = z\Gamma(z)$ and the fact that $\lim_{z\to 0}\Gamma(2z)/\Gamma(z) = \frac{1}{2}$ enables us to write

$$c_1 = (2l)!\left(l!\prod_{n=1}^{l}(n - 2iQ)\right)^{-1} \tag{B10}$$

and

$$c_2 = -l!\,[2(2l+1)!]^{-1}\prod_{n=0}^{l}(n + 2iQ), \tag{B11}$$

where $K \ll 1$ has now been dropped.

However, if $r \gg M$ $(x \gg 1)$ the radial equation, Eq. (7), becomes

$$\frac{d^2U}{d\rho^2} + \left[1 - \frac{\eta}{\rho} - \frac{l(l+1)}{\rho^2}\right]U = 0, \tag{B12}$$

where $U = rR$, $\rho = kr$, and $\eta = -2M\mu^2/k$. The solutions of this equation are the Coulomb wave functions,[37] $F_l(\eta, \rho)$ and $G_l(\eta, \rho)$. In the limit $\rho \to 0$,

$$F_l(\eta, \rho) \to C_l(\eta)\rho^{l+1} \tag{B13}$$

and

$$G_l(\eta, \rho) \to D_l(\eta)\rho^{-l}, \tag{B14}$$

where

$$C_l(\eta) = 2^l e^{-\pi\eta/2}|\Gamma(l+1+i\eta)|/(2l+1)! \tag{B15}$$

and

$$D_l(\eta) = [(2l+1)C_l(\eta)]^{-1}. \tag{B16}$$

Consider the solution

$$U = b_1 F_l(\eta, \rho) + b_2 G_l(\eta, \rho) \tag{B17}$$

of Eq. (B12). In the region $\max(l, Q) \ll x \ll l/[\omega(r_+ - r_-)]$, it may be matched to the solution Eq. (B7) of Eq. (B2). This leads to the relations

$$b_1 = c_1[k^{l+1}(r_+ - r_-)^l C_l(\eta)]^{-1} \tag{B18}$$

and

$$b_2 = c_2 k^l (r_+ - r_-)^{l+1}/D_l(\eta). \tag{B19}$$

The asymptotic forms of the Coulomb wave functions for $\rho \to \infty$ are

$$F_l(\eta, \rho) \to \sin(\rho + \delta_l), \tag{B20}$$

$$G_l(\eta, \rho) \to \cos(\rho + \delta_l), \tag{B21}$$

where δ_l is a phase angle. A term $\propto \ln\rho$ has been dropped in the arguments on the right-hand side of Eqs. (B20) and (B21). Since U has the asymptotic form

$$U \propto e^{-i\rho} + A_+ e^{i\rho} \tag{B22}$$

as $\rho \to \infty$, we find that

$$|A_+|^2 = \left| \frac{b_2 - ib_1}{b_2 + ib_1} \right|^2 . \tag{B23}$$

If we now use Eqs. (B10), (B11), (B18), and (B19), we find that, to lowest order in kM, the amplification coefficient A_{klm} is

$$A_{klm} = |A_+|^2 - 1$$
$$= [(2l)!]^{-2} \left(4Q(r_+ - r_-)^{2l+1} k^{2l+1} \times (l!)^2 C_l^{\,2}(\eta) \sum_{n=1}^{l} (n^2 + 4Q^2) \right). \tag{B24}$$

We may show that

$$C_l^{\,2}(\eta) = \frac{\pi 2^{2l} e^{-\pi\eta} \eta \prod_{j=1}^{l} (j^2 + \eta^2)}{[(2l+1)!]^2 \sinh \pi\eta} . \tag{B25}$$

Thus,

$$A_{klm} = \frac{\pi (r_+^{\,2} + a^2)(r_+ - r_-)^{2l} 2^{2l+3} (l!)^2 \eta \tilde{\omega} k^{2l+1}}{[(2l)!]^2 [(2l+1)!]^2 (1 - e^{2\pi\eta})} \times \prod_{j=1}^{l} (j^2 + \eta^2) \prod_{n=1}^{l} (n^2 + 4Q^2). \tag{B26}$$

For superradiant modes, $A_{klm} > 0$ and is the amplification coefficient; for nonsuperradiant modes $A_{klm} < 0$ and is the absorption coefficient. In the limit that $\mu \to 0$, Eq. (B26) becomes

$$A_{klm} = - \frac{(r_+^{\,2} + a^2)(r_+ - r_-)^{2l} 2^{2l+2} (l!)^4 \tilde{\omega} \omega^{2l+1}}{[(2l)!]^2 [(2l+1)!]^2} \times \prod_{n=1}^{l} (n^2 + 4Q^2) , \tag{B27}$$

which is the result given by Starobinsky.[10]

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