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Scaling, large-P_T distribution, and fireball motion*

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Fireball properties have been investigated using results of previous analyses of inclusive p-p data in terms of a Bose-type distribution modified by a scaling parameter. A discussion is presented on the relation between the scaling parameter and the longitudinal velocity of the fireball. It is found that the transverse motion may account for large-P_T distributions at 90° c.m. angle and that this motion is not detectable for $P_T < 1.5$ GeV/c. The fireball mass is found to increase as $(\gamma_{c.m.}-1)^{1/2}$, $\gamma_{c.m.}$ being the Lorentz transformation factor from lab to c.m. system.

I. PRELIMINARY

In an attempt to account for the inclusive single-particle distributions in terms of the transverse momentum P_T and the longitudinal momentum P_L in the c.m. system, we have proposed the following Bose-type distribution:[1]

$$\frac{d^2\sigma}{dP_T^{\,2}dP_L} \propto \frac{1}{e^{\epsilon(\lambda)/T}-1}, \tag{1}$$

where T is the temperature, the Boltzmann constant is set at $k_B = 1$, and

$$\epsilon(\lambda) = (P_T^{\,2} + \lambda^2 P_L^{\,2} + m^2)^{1/2}, \tag{2}$$

m being the mass of the secondary meson and λ a dimensionless parameter related to the scaling law.[2,3] Note that for $\lambda = 1$, $\epsilon(1) = E$ being the total energy of the meson, relation (1) reduces to the Bose-Einstein distribution, and that the scaling property requires $\lambda \propto 1/P_{max}$, P_{max} being the maximum of the c.m. momentum of the secondary meson.

Analyses of inclusive p-p data indicate that the products λP_{max} remain practically constant for incident momentum greater than 20 GeV/c, and at CERN Intersecting Storage Rings (ISR) energies.[4] It has been noticed that the Feynman-Yang scaling can be expressed by the following empirical relation:

$$\lambda\gamma_{c.m.} \to 2, \tag{3}$$

where $\gamma_{c.m.}$ denotes the Lorentz-transformation factor for the colliding p-p system. It should be mentioned that the parameter λ is crucial for good fits to P_T distributions at fixed P_L. However, it has been noted that the distribution thus modified is still not adequate to account for large P_T distributions of recent ISR experiments, which we propose to investigate in the present note.

Recently, Yu has proposed another interpretation of our parameter λ, Namely, $1/\lambda$ is the Lorentz-contraction factor of the interaction volume.[5] The isobar mass derived by Yu is given by $M^* = \lambda E$, E being the total energy of the colliding proton, namely $E = m_p\gamma_{c.m.}$. We note that this relation combined with the observed scaling property (3) leads to $M^* \to 2m_p$. As mentioned in a previous paper (Ref. 4, II), Yu's approach is doubtful, because he has treated the temperature as an invariant in the Lorentz transformation.

In this paper we present a simple and correct derivation of the relation between the parameter λ in (1) and the Lorentz factor for the fireball motion along the longitudinal direction. In doing so, it has been noticed that our distribution (1) is formally equivalent to the distribution used by Fermi[6] and Landau[7] to account for the conservation of angular momentum (see note added in proof) and that the Bose distribution thus reformulated can also be extended to account for the large-P_T distribution. Thus, we are able to describe the salient features of the single-particle distributions by means of a simple Bose-type distribution. Finally, a discussion will be presented on some basic properties of the distribution under consideration.

II. FIREBALL MOTION IN THE LONGITUDINAL DIRECTION

We proceed to investigate the mass motion of secondary mesons emitted by p-p collisions. Because of symmetry, we shall consider only those secondaries moving in either one of the colliding directions. If there is a mass motion of the emitted secondaries, we have to assume the existence of a *fireball* of mass M^* which moves with a velocity $\vec{v}$ in units of the velocity of light c, with respect to the c.m. system of the colliding protons. In this case, a meson of energy E and momentum $\vec{P}$ in the c.m. system will have in the fireball rest system an energy E^* given by the following Lorentz transformation:

$$E^* = \gamma_F(E - \vec{v}\cdot\vec{P}), \tag{4}$$

where $\gamma_F = (1-v^2)^{-1/2}$.

Let T^* be the temperature of the fireball in its own rest frame; then its temperature T measured in the c.m. system of the colliding protons is given by[8] (see Ref. 9)

$$T = T^*/\gamma_F. \tag{5}$$

Consequently, the ratio

$$\frac{E-\vec{v}\cdot\vec{P}}{T} = \frac{E^*}{T^*} \tag{6}$$

is an invariant, and is positive definite, since $|\vec{v}|<1$. Therefore, in the case of a mass motion due to a fireball, the Bose-Einstein distribution assumes the following covariant form:

$$\frac{d^2\sigma}{dP_T^2 dP_L} \propto \frac{1}{e^{(E-\vec{v}\cdot\vec{P})/T}-1}. \tag{7}$$

As a first application, we assume that the mass motion takes place only along the longitudinal direction, the magnitude of $\vec{v}$ being b;[10] then

$$E-\vec{v}\cdot\vec{P} = E - bP_L. \tag{8}$$

We may relate b to the parameter λ of our distribution (1) by equating the above expression to $\epsilon(\lambda)$ of (1). This leads to the following relationship:

$$1-(1-\lambda^2)\beta^2 = (1-b\beta)^2, \tag{9}$$

where $\beta = P_L/E$ is the longitudinal velocity of the secondary under consideration. For simplicity, we assume $\beta = 1$; then we get $b = 1 \pm \lambda$. Since $0<\lambda<1$, and b must be less than 1, we have to choose the minus sign and obtain

$$b = 1-\lambda, \tag{10}$$

from which we deduce the Lorentz factor for the fireball as follows:

$$\gamma_F = \frac{1}{[\lambda(2-\lambda)]^{1/2}}. \tag{11}$$

We may find an approximate expression for the fireball mass by replacing λ in the above expression by (3); this gives

$$\frac{M^*}{m_p} = \frac{\gamma_{\text{c.m.}}}{\gamma_F} \simeq 2(\gamma_{\text{c.m.}}-1)^{1/2}, \tag{12}$$

which indicates that, by virtue of the empirical scaling law (3), the mass of each fireball formed by p-p collision increases as the $\frac{1}{2}$ power of the available energy in the c.m. system. As is well known, this is Stefan's law for black-body radiation. Note that $M^*/m_p > 1$; this requires $\gamma_{\text{c.m.}} > 1.25$. A plot of M^*/m_p vs $\gamma_{\text{c.m.}}$ is shown in Fig. 1.

Finally, we note that if $\lambda = 1$, we find from (11) that $\gamma_F = 1$. In this case there is no mass motion; in other words, there is no fireball formation. Now, as mentioned before, $\lambda = 1$ leads to $\epsilon(\lambda) = E$; then distribution (1) becomes identical with the Bose-Einstein distribution. This indicates that in its original form, the Bose-Einstein distribution is not adequate to describe the longitudinal-momentum distribution, although it is adequate for the P_T distribution. We recall that it is this motivation which has led us to modify the Bose distribution by introducing the scaling parameter λ (see Ref. 1, Sec. 8).

Let us now estimate the fireball mass (in units of m_p) using the values of λ obtained from our previous analyses of p-p data, Ref. 4, I and II. For this purpose we compute γ_F by means of (11), then compute M^*/m_p by means of (12). The results thus obtained[11] are presented in Fig. 1. It is to be noted that all the points follow actually very close to the dashed curve representing Stefan's law (12).

III. FIREBALL MOTION IN THE TRANSVERSE DIRECTION

Next, we propose to investigate the mass motion in the transverse direction. In this regard, we note that recent ISR data on P_T distributions at 90° in the c.m. system and in the plane of the colliding proton beam are particularly suitable for our purpose: Since $P_L = 0$, we have to deal only with P_T. Let a denote the transverse component of the fireball velocity; then

$$\vec{v}\cdot\vec{P} = aP_T \tag{13}$$

and

$$\left(\frac{d^2\sigma}{dP_T^2 dP_L}\right)_{90°} \propto \frac{1}{e^{(E-aP_T)/\Theta}-1}. \tag{14}$$

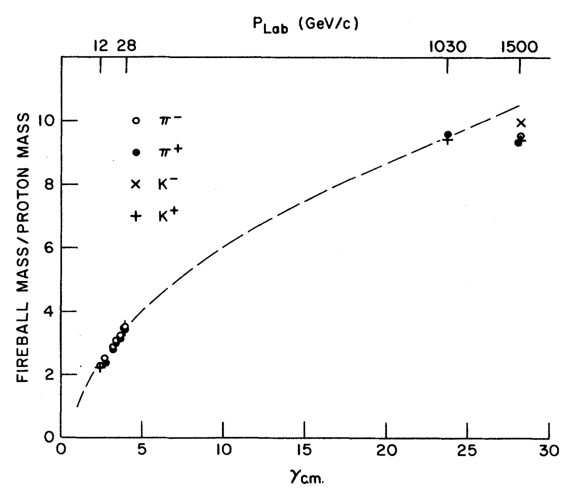

FIG. 1. Estimates of fireball mass by means of the scaling parameter λ [see Eq. (11)]. The dashed curve represents Stefan's law, Eq. (12).

Here, we have denoted the temperature by Θ to distinguish it from T in (1). We take this precaution to remind us that the two parameterizations of the temperature, by (1) on the one hand and by (14) on the other, are different. We shall show that they are related as follows: Let $\langle P_T \rangle_a$ denote the average of P_T evaluated according to (14); then, using the relativistic approximation $P_T/E = 1$ and noting that $\langle P_T \rangle_{a=0}$ is identical with $\langle P_T \rangle_{\lambda=0}$, which is the average of P_T according to (1) with $\lambda = 0$, a change of variable leads to

$$(1-a)\langle P_T \rangle_a = \langle P_T \rangle_{\lambda=0}, \tag{15}$$

which gives an implicit relation between the two estimates of temperature T and Θ for an experimental P_T distribution at 90° under consideration.

The data we have analyzed with (14) are as follows: the Saclay-Strasbourg Collaboration,[11] the British-Scandinavian-CERN Collaboration,[12] and the CERN-Columbia-Rockefeller Collaboration.[13] The parameters a (in units of c) and Θ (in GeV) have been estimated by least-squares fits to the P_T distributions provided by those experiments. The results thus obtained are presented in Table I.

A comparison of the values of parameter a listed in Table I indicates that apart from the Saclay-Strasbourg experiment, all other values are definitely different from zero. From the specifics of the experiments we note that what makes the Saclay-Strasbourg experiment different is that their measurements cover a P_T range less than 0.78 GeV/c, whereas other experiments aim at *large* P_T, greater than 1.38 amd 2.59 GeV/c for the British-Scandinavian-CERN Collaboration and for the CERN-Columbia-Rockefeller Collaboration, respectively. That $a \simeq 0$ for the Saclay-Strasbourg Collaboration is to be expected, since these data have previously been analyzed with the modified Bose distribution (1), and good fits have been obtained with $T = 0.117 \pm 0.006$ GeV (Ref. 4, II). This view is further supported by previous results of analyses of various P_T distributions presented in Ref. 1 and 4. We therefore have the feeling that the transverse fireball motion is practically imperceptible for P_T distributions not exceeding, say, 1.5 GeV/c, which is about four times the average value of P_T over the entire distribution.

As regards other values of a in Table I, we notice that they are certainly different from zero and that their variations from one set of data to another are rather small. This is in contrast with the longitudinal mass motion, for which we have found a definite energy dependence, namely $b \simeq 1 - 2/\gamma_{\text{c.m.}}$ according to (10) and (3).

We also notice that the values of Θ are larger than those obtained by means of (1) for $P_T < 1$ GeV/c (see Ref. 4, II). This, in conjunction with the fact that $a \neq 0$, has the effect to reduce the drop of the cross section for large P_T as compared with our original distribution (1). Owing to this, we are now able to get reasonable fits to large-P_T distributions.

As an illustration, we present in Fig. 2 the result of a simultaneous fit to the two sets of data at $\sqrt{s} = 53$ GeV: the π^- data of the British-Scandinavian-CERN Collaboration and the π^0 data of the CERN-Columbia-Rockefeller Collaboration. The parameters are $\Theta = 0.176 \pm 0.006$ GeV and

TABLE I. Estimates of parameters Θ and a. ISR data of P_T distribution $x = 0$ (Refs. 12–14).

Experiment	$\sqrt{s}$ (GeV)	Range of P_T (GeV/c)	Secondary particle	Θ (GeV)	a (in units of c)	χ^2/point
Saclay-Strasbourg	52.7	0.22–0.78	π^+, π^-	0.112 ± 0.005	0.002 ± 0.005	0.8
British-Scandinavian-CERN	44	1.38–2.94	π^+	0.152 ± 0.005	0.398 ± 0.034	1.6
			π^-	0.153 ± 0.005	0.310 ± 0.031	1.5
	53	1.38–4.14	π^+	0.148 ± 0.005	0.324 ± 0.034	1.7
			π^-	0.150 ± 0.005	0.320 ± 0.038	2.5
CERN-Columbia-Rockefeller	23.5	2.80–4.64	π^0	0.152 ± 0.004	0.378 ± 0.028	2.3
	30.6	2.96–6.36	π^0	0.184 ± 0.005	0.540 ± 0.030	1.5
	44.8	2.64–5.86	π^0	0.171 ± 0.003	0.415 ± 0.020	1.4
	52.7	2.59–9.01	π^0	0.185 ± 0.003	0.489 ± 0.014	1.4
	62.4	2.89–4.56	π^0	0.179 ± 0.005	0.444 ± 0.029	1.2

$a = 0.450 \pm 0.010$. Within fitting errors, these estimates are consistent with those of individual fits listed in Table I.

IV. DISCUSSION

To sum up the results presented above, we may write a general expression which will account for the fireball motion in both longitudinal and transverse directions. For this purpose, we have to replace $\epsilon(\lambda)$ in (1) by

$$\epsilon(\lambda) - aP_T\theta(P_T - C), \tag{16}$$

where θ is a step function with its cut at a value of P_T equal to $C \simeq 1.5$ GeV/c and the temperature T in (1) by

$$T + (\Theta - T)\theta(P_T - C). \tag{17}$$

Needless to say that instead of λ one may use another parameter b, the longitudinal velocity of the fireball; then (16) becomes covariant as follows:

$$E - bP_L - aP_T\theta(P_T - C). \tag{18}$$

It should be mentioned that the two components a and b of the fireball velocity here considered are uncorrelated. Thus, strictly speaking, we are not in a position to obtain a complete picture of the fireball motion. However, we note that the transverse motion, which, according to our investigation, is not perceptible for $P_T < 1.5$ GeV/c, constitutes actually a very small part of the cross section. Therefore, we may regard the fireball mass derived from the longitudinal motion, expressions (11) and (12), as a good approximation and proceed to investigate the dependence of the multiplicity on the mass of the fireball.

For this purpose we plot the average multiplicity of negative secondaries $\langle n_- \rangle$ against the fireball mass in units of m_p as shown in Fig. 3.[14] Clearly, $\langle n_- \rangle = 0$ for $M^*/m_p = 1$, which is the threshold. Now, if we fit the points with a straight line passing through the threshold point as shown by the dashed line in the figure, we find for the slope 0.561 ± 0.006. Thus, we have

$$\langle n_- \rangle = 0.561[2(\gamma_{c.m.} - 1)^{1/2} - 1]. \tag{19}$$

This linearity relation between the average multiplicity and the fireball mass is to be expected according to Fermi's statistical model, although we have arrived at this result by a different approach which is based on the Bose distribution on the one hand and the scaling property on the other. It should be mentioned that if $E/T \ll 1$, then the Bose distribution reduces to the invariant phase space. This leads to a $\log s$ dependence for the average multiplicity.

Finally, it is worth noting that if such a simple distribution as (1) turns out to be adequate to describe the main characteristic features of the single-particle distributions observed in hadron collisions, it is because it describes the statistical-mechanical properties of the mesons through the Bose-Einstein distribution as well as the scaling property by means of the parameter λ, which also accounts for the anisotropy of the angular distri-

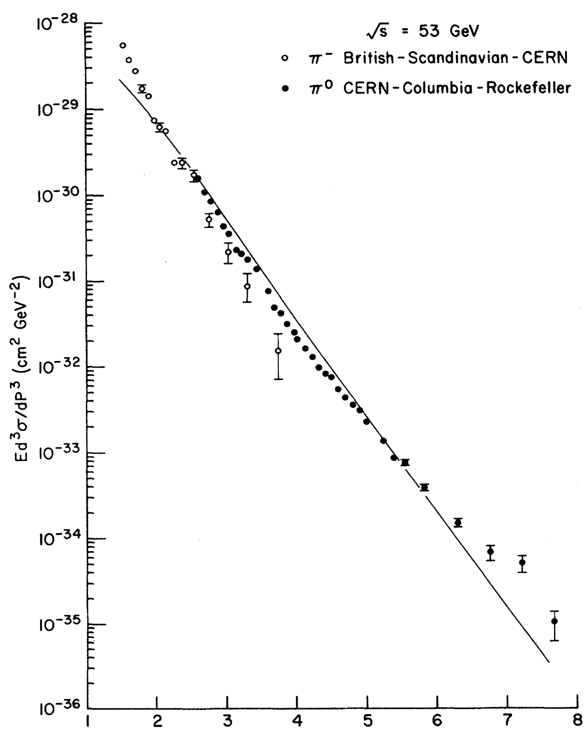

FIG. 2. Simultaneous fits to π^0 and π^+ data at $\sqrt{s} = 53$ GeV. Data from Refs. 12 and 13 are fitted with a Bose-type distribution (14).

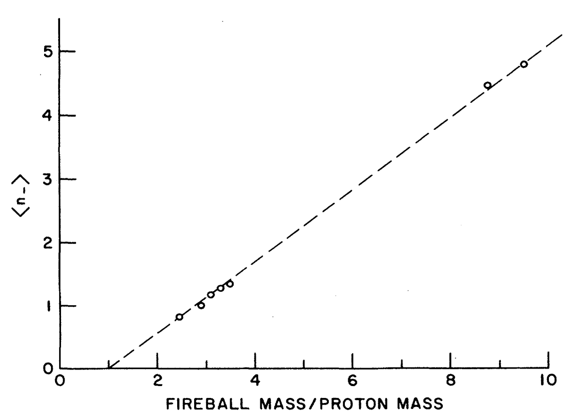

FIG. 3. Average multiplicity of negative particles vs fireball mass. The dashed line is a linear fit passing through the threshold point: $\langle n_- \rangle$ = zero for $M^*/m_p = 1$.

bution as mentioned above (see Sec. II). It should also be mentioned that the Gaussian-form P_T^2 distribution which is customarily used to analyze the P_T distribution can be regarded as an approximate form of (1) and that its range of application is certainly much less as compared with the distribution (1).

V. CONCLUSIONS

An attempt has been made to use recently available p-p data of Fermilab experiments with the 30-in. chamber. The results we have obtained are as follows.

For the Rochester-Michigan experiment, we have used the data complied by Whitmore.[14] The value of $\langle P_L \rangle$ is obtained from the x distribution in Fig. 58 of Ref. 14, and the value of $\langle P_T \rangle$ is read off from Fig. 61 of Ref. 14. Then we deduce the parameter λ of (1) using the following formula [see (33) of Ref. 1]

$$\lambda = \frac{2}{\pi}\frac{\langle P_T \rangle}{\langle P_L \rangle}, \qquad (20)$$

and the temperature T from $\langle P_T \rangle$. In this regard, we recall that the presence of λ in (1) does not affect the relationship between $\langle P_T \rangle$ and T. The results are presented in Table II.

As for the Fermilab-ANL experiment, we have at our disposal the averages of P_T and P_L for π^-.[15] We have also considered the published data on K^0.[16] In this case the value of $\langle P_L \rangle$ is deduced from the slope of the plot $d\sigma/dx$ against x, whereas the temperature T is estimated from $\langle P_T^2 \rangle$ which is estimated from the slope of the plot $d\sigma/dP_T^2 \text{vs} P_T^2$. The relationship between $\langle P_T^2 \rangle$ and T according to (1) is as follows:

$$\langle P_T^2 \rangle = 2mT\frac{K_3(m/T)}{K_2(m/T)}, \qquad (21)$$

where m is the mass of the secondary particle under consideration and K_n denotes the modified Bessel function of the second kind and of order n. We then deduce $\langle P_T \rangle$ from the temperature T. The values thus obtained are listed in Table II.

Let us compare the parameters λ and T for these Fermilab experiments. First, consider the temperature T. We find that all three values are consistent with what we should expect, namely $T \simeq m_\pi = 0.140$ GeV.

Turn now to the parameter λ. We recall that according to scaling we should expect $\lambda\gamma_{\text{c.m.}}$ to be constant and equal to ~2 according to our previous results of analyses of data above and below energies (Ref. 4, I and II). For the Rochester-Michigan experiment, we find that $\lambda\gamma_{\text{c.m.}}$ is about two standard deviations below the expected value. As for the Fermilab-ANL experiments, we obtain for K^0 a good agreement. However, we notice important disparities between the two values of $\lambda\gamma_{\text{c.m.}}$ for π^- and K^0. This is contrary to what we should expect from results of previous analysis, namely $\lambda_\pi = \lambda_K$ at a given energy. It would be interesting to see that future data could elucidate this point.[17]

Note added in proof. Denote $\mu = \cos\theta$, θ being the c.m. angle of a secondary meson. The angular distribution derived from (1) is

$$\frac{d\sigma}{d\mu} \propto \frac{\lambda}{[1-(1-\lambda^2)\mu^2]^{3/2}}.$$

For a detailed discussion on this subject, we refer to T. F. Hoang and R. Singer, ANL Report No. ANL-HEP 7455, 1974 (unpublished).

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TABLE II. Estimates of parameters for Fermilab 30-in. chamber experiments.

Experiment	Secondary particle	$\langle P_T \rangle / \langle P_L \rangle$ (GeV/c)	T (GeV)	λ	$\lambda\gamma_{\text{c.m.}}$	M^*/m_p
Rochester-Michigan 102 GeV/c	π^-	$\frac{0.340 \pm 0.015}{1.071 \pm 0.180}$	0.149 ± 0.010	0.22 ± 0.04	1.57 ± 0.32	4.55 ± 0.40
Fermilab-ANL 205 GeV/c	π^-	$\frac{0.337 \pm 0.004}{0.664 \pm 0.004}$	0.147 ± 0.002	0.32 ± 0.01	3.28 ± 0.01	7.57 ± 0.02
	K^0	$\frac{0.383 \pm 0.005}{1.280 \pm 0.064}$	0.124 ± 0.003	0.19 ± 0.01	1.95 ± 0.10	6.05 ± 0.14

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[9]Note another derivation of (6). Consider the four-vector $\beta^i = u^i/T^*$, where $u^i = \gamma_F(1, \vec{v})$, and the scalar product of β^i with the four-momentum $P^i = (E, \vec{P})$ of a meson. We have in the fireball system $\beta_i P^i = E^*/T^*$, whereas $\beta_i P^i = \gamma_F(E - \vec{v} \cdot \vec{P})/T^*$ in the c.m. system. These two equations are equivalent to (5) and (6).
[10]Clearly, the results thus obtained hold true also for an alternative assumption that $P_T = 0$.
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[14]The values of $\langle n_- \rangle$ are estimated from Fig. 14 of the average charged-particle multiplicity $\langle n_{ch} \rangle$ compiled by J. Whitmore, Phys. Rep. 10C, 273 (1974). We have used $\langle n_- \rangle = \frac{1}{2}(\langle n_{ch} \rangle - 2)$.
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[17]We note that in the context of the fireball interpretation, this equality is self-evident. If follows from (10) and the fact that the center-of-mass motion is independent of the nature of the secondary particles.