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Quasispherical gravitational collapse

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A previously derived class of solutions representing gravitationally collapsing dust clouds is investigated in detail. The class includes all spherically symmetric collapses but is more general and is termed quasispherical. Most features of spherical collapse appear to go over in their essentials to the more general class. Under reasonable initial conditions the singularity is hidden behind an event horizon, but with slightly irregular initial density distributions portions of the singularity become naked. The latter feature is not restricted to the nonspherical case and will in general appear in situations such as collapsing spherical shells of matter.

I. INTRODUCTION

The most generally accepted picture of gravitational collapse (what Thorne[1] has termed the "establishment" viewpoint), that the unimpeded collapse of a compact body such as a star will result in a space-time singularity which is hidden from the external world by an event horizon, is based primarily on discussions of spherical collapse and a general faith that the picture obtained from these highly symmetrical situations is maintained in its essentials for more realistic situations. This faith has been strengthened over the years by many general results concerning gravitational collapse,[1] but the case of the "radicals" (that departures from spherical symmetry will destroy the horizon, leaving a "naked" singularity) cannot be regarded as truly disproved until at least one exact example of nonspherical collapse has been exhibited. In this paper such an example will be given, which does appear to confirm the viewpoint of the "establishment."

However, for fear of misleading the reader with extravagant claims, let me give out a few notes of warning. Firstly, the class of solutions here discussed, though not in general spherically symmetric, do in fact possess an invariantly defined family of spherical 2-surfaces which gives them a character which I call quasispherical. Thus they are not a very general class of solutions, and it may be that only the very strongest claims of the radicals must be modified; that is, it *might* still be true that a general collapse results in naked singularities, only a "set of measure zero" among the class of possible initial conditions resulting in event horizons. Secondly, naked singularities may appear if certain parameters are suitably chosen; this, however, is a feature already noted for spherical collapse by Yodzis *et al.*,[2] and appears to be completely mirrored in the quasispherical case. A criterion will be given here for this to happen which renders it improbable for normal stars (though possibly not for close binary stars, and certainly not for collapsing shells of matter). Thirdly, the matter content in the solutions given here extends to infinity, as it has not been found possible in general to match the metrics up (except in the case of spherical symmetry) with a vacuum metric. However, the total mass *is* finite, for the density may be made to die off as fast as one likes at large distances; in this the solutions may be regarded as being even *more* realistic than if such a matching had been achieved, since no star has a precisely defined boundary (there is always some residual gas in the surrounding space).

II. THE NATURE OF THE METRIC

In a recent paper[3] some new dust solutions of Einstein's equations

$$G_{\mu\nu} = T_{\mu\nu} = \rho u_\mu u_\nu, \quad u_\mu u^\mu = 1$$

were found (units are adopted here such that $c = 8\pi G = 1$). In these solutions the metric is assumed to have the form

$$ds^2 = dt^2 - X^2 dr^2 - Y^2(dx^2 + dy^2),$$

where t is the proper time along the (geodesic) world lines of the fluid and r, x, y are comoving spatial coordinates, constant along each world line [so that $u^\mu = (1,0,0,0)$]. For computational convenience it is convenient to adopt a pair of complex conjugate coordinates

$$\zeta = x + iy, \quad \bar{\zeta} = x - iy$$

so that the metric takes on the form

$$ds^2 = dt^2 - X^2 dr^2 - Y^2 d\zeta\, d\bar{\zeta}, \tag{1}$$

where X and Y are, at the outset, quite general functions of t, r, ζ, and $\bar{\zeta}$. It has been found possible to integrate Einstein's equations explicitly under these assumptions,[3] but for the purposes of this paper we restrict ourselves[4] to the case

$$Y' \equiv \frac{\partial Y}{\partial r} \neq 0.$$

The solution, after some trivial changes in notation from those in Ref. 3, may be shown to have the form

$$Y = \frac{\phi(r, t)}{P(r, \zeta, \bar{\zeta})}, \quad X = \frac{PY'}{W(r)} \tag{2}$$

where W is an arbitrary function of r subject only to the restriction

$$W(r) > 0,$$

and $P(r, \zeta, \bar{\zeta})$ must have the form

$$P = a(r)\zeta\bar{\zeta} + B(r)\zeta + \bar{B}(r)\bar{\zeta} + c(r), \tag{3}$$

where a and c are arbitrary real functions and B is a complex function of r satisfying

$$ac - B\bar{B} = \tfrac{1}{4}\epsilon, \quad \epsilon = \pm 1 \text{ or } 0. \tag{4}$$

We will consider here only the case $\epsilon = +1$, in which case the 2-surfaces $S_{r,t}$ defined by $t =$ const, $r =$ const are spheres of radius $\phi(r, t)$. The demonstration of this fact is provided in an appendix to Ref. 3 and rests on the possibility of showing that in each such 2-surface a bilinear (i.e., fractional linear) transformation of coordinates $\zeta, \bar{\zeta}$ may be found such that the 2-metric reduces to

$$\phi^2(r, t) d\zeta d\bar{\zeta} / \tfrac{1}{4}(\zeta\bar{\zeta} + 1)^2,$$

which on subsequent introduction of polar coordinates

$$\zeta = e^{i\varphi}\cot\tfrac{1}{2}\theta, \quad \bar{\zeta} = e^{-i\varphi}\cot\tfrac{1}{2}\theta$$

reduces to the usual form

$$\phi^2(r, t)(d\theta^2 + \sin^2\theta d\varphi^2).$$

Thus the term "quasispherical" has been adopted to characterize these solutions, though true spherical symmetry (occurrence of threefold group of symmetries isomorphic to the rotation group) only arises in general if a, c, and B can be set equal to constants; in such a case there is no loss of generality in adopting the values $a = c = \frac{1}{2}$, $B = 0$.

Finally, in order for Einstein's equations to be satisfied it is necessary that $\phi(r, t)$ satisfies a "Friedmann" equation

$$\dot{\phi}^2 = W^2 - 1 + \frac{S(r)}{\phi}, \tag{5}$$

where an overdot denotes $\partial/\partial t$ and $S(r)$ is an arbitrary function of r. It is now a straightforward computation to calculate the density

$$\rho = \frac{PS' - 3SP'}{P^3 WXY^2} = \frac{PS' - 3SP'}{\phi^2(P\phi' - \phi P')}. \tag{6}$$

For spherical symmetry these solutions reduce to the well-known Bondi-Tolman metrics.[5] In the general quasispherical case the density is not in general constant over the 2-spheres $S_{r,t}$, but some feeling for the matter distribution may be obtained by calculating the surface density $\sigma = \rho X$ (defined such that the mass contained in the shell bounded by spheres r and $r + dr$ is $dM = dr \iint \sigma Y^2 d\zeta d\bar{\zeta}$). From Eq. (6) we find, on using local coordinates such that $a = c = \frac{1}{2}$, $B = 0$ in $S_{r,t}$,

$$\sigma = \frac{PS' - 3SP'}{PW\phi^2} = \frac{f_1(r)\zeta\bar{\zeta} + F(r)\zeta + \bar{F}(r)\bar{\zeta} + f_2(r)}{W\phi^2(1 + \zeta\bar{\zeta})}$$

with f_1, f_2 real functions and F a complex function of r. On adopting polar coordinates as above, this reduces to the functional form

$$\sigma = g(r, t)[1 + f(r)\cos\theta + g(r)\cos\phi\sin\theta + h(r)\sin\phi\sin\theta],$$

which represents a dipole distribution. However, the axes of the dipoles on different spheres are oriented arbitrarily (up to smoothness) relative to each other, in the sense that no coordinate transformation exists in general to bring the surface densities on different spheres simultaneously to this canonical form at any instant $t =$ const.

From (1) and (6) we can compute the mass within the sphere $r =$ const (at a given time t),

$$\begin{aligned} M(r, t) &= \int_0^r dr \int\int d\zeta d\bar{\zeta} XY^2\rho \\ &= \int_0^r dr \int\int d\zeta d\bar{\zeta}\left(\frac{S'}{WP^2} - 3S\frac{P'}{WP^3}\right) \\ &= \int_0^r dr \left[\frac{S'}{W}\int\int\frac{d\zeta d\bar{\zeta}}{P^2} + \frac{3S}{2W}\frac{d}{dr}\left(\int\int\frac{d\zeta d\bar{\zeta}}{P^2}\right)\right]. \end{aligned}$$

Now $d\zeta d\bar{\zeta}/P^2$ is always the metric of a unit sphere, whence

$$\int\int\frac{d\zeta d\bar{\zeta}}{P^2} = 4\pi,$$

and

$$M(r, t) = M(r) = 4\pi\int_0^r \frac{S'}{W} dr, \tag{7}$$

a formula which is identical with that obtained for spherical symmetry,[6] and one which shows, as to be expected, that the mass within a comoving volume remains constant in time. Furthermore, we have that

$$\rho > 0 \rightarrow M'(r) > 0 \rightarrow \frac{S'(r)}{W(r)} > 0. \tag{8}$$

The total mass may clearly be chosen to be finite, and to die off as fast as one wishes by choosing $S'(r)/W(r) \rightarrow 0$ as $r \rightarrow \infty$ sufficiently rapidly. However, one cannot in general set $\rho=0$ for $r>r_0$ except in the case of spherical symmetry (see Ref. 3). Thus, if it is desired to match a finite portion $r<r_0$ of a general quasispherical dust cloud with a vacuum exterior metric, the exterior metric will necessarily have a form more general than (1). Instead of attempting such a matching we shall restrict ourselves here to dust clouds of finite total mass whose density rapidly approaches zero at infinity.

The cases $\epsilon=0$ or -1 could similarly be analyzed, leading to quasiplanar and quasipseudospherical solutions respectively; the 2-surfaces $r, t=$ const having in these cases the characters of planes and pseudospheres of radius ϕ. However, these cases will have unbounded masses (or at best peculiar topologies arising from rather unnatural compactifications of the 2-surfaces via suitable coordinate identifications) and are therefore discarded as being of little physical significance.

III. REGULARITY CONDITIONS

We now impose a number of conditions on the functions ϕ, W, and S in order to satisfy certain regularity requirements. Firstly we assume that at $t=0$ the function $\phi_0(r)=\phi(r,0)$ is a monotonically increasing function of r, so that a coordinate transformation $r'=f(r)$ may be effected such that

$$\phi_0(r)=r. \tag{9}$$

For the metric (1) to be everywhere C^1 it is clear that the functions $a(r)$, $c(r)$, and $B(r)$ must be continuous and differentiable as $r\to 0$ (with vanishing first derivatives at $r=0$), while for (1) to be locally Euclidean at $r=0$ it is clearly necessary to set

$$W(0)=1. \tag{10}$$

From (5) it now follows that in order that $\dot{\phi}(r,0)$ be bounded as $r\to 0$ we must have $S(r)/r$ bounded, i.e., $S(r)\to 0$ as $r\to 0$. On the other hand, boundedness of the density as $r\to 0$ gives from (6) that

$$\rho\to\frac{S'}{r^2} \text{ is bounded as } r\to 0.$$

Hence

$$S'=O(r^2),\quad S=O(r^3) \text{ as } r\to 0. \tag{11}$$

Finally, if the initial density $\rho_0(r)\equiv\rho(r,0)$ is to be everywhere positive and nonsingular we obtain some further restrictions which are most easily derived after establishing the following useful lemmas.

Lemma 1: A quadratic form

$$Q=\alpha\zeta\bar{\zeta}+\beta\zeta+\bar{\beta}\bar{\zeta}+\gamma,$$

where α,γ are real and β a complex number, has no zeros in the complex ζ plane if and only if its discriminant is positive, i.e.,

$$\Delta(Q)=\alpha\gamma-\beta\bar{\beta}>0.$$

Furthermore, Q is everywhere >0 if and only if $\Delta(Q)>0$ and $\alpha>0$.

Proof: Let

$$\zeta=Ze^{i\theta},\quad \beta=Be^{i\varphi}\quad (Z, B \text{ real});$$

then

$$Q=\alpha Z^2+2B\cos(\theta+\varphi)Z+\gamma=0$$

has a real solution $Z=Z_0$ if and only if there exists θ such that

$$B^2\cos^2(\theta+\varphi)\geq\alpha\gamma,$$

i.e., if and only if

$$\beta\bar{\beta}\geq\alpha\gamma.$$

This establishes the first part of the lemma, and the second part follows trivially.

Corollary 1: If $\epsilon=+1$, then $P>0$ for all values of $\zeta, \bar{\zeta}, r$ if and only if $a(r)>0$ for all r.

Proof: From Eqs. (3) and (4) we have

$$\Delta(P)=ac-B\bar{B}=\tfrac{1}{4}>0$$

and the result follows immediately from Lemma 1.

Corollary 2: If $\epsilon=+1$ and $a(r)>0$ for all r, then P' has a zero on each surface $r=$ const.

Proof: $\Delta(P')=a'c'-B'\bar{B}'$. Now from (4),

$$c=a^{-1}(\tfrac{1}{4}+B\bar{B}),$$

$$c'=a^{-1}(B'\bar{B}+B\bar{B}')-a^{-2}a'(\tfrac{1}{4}+B\bar{B}),$$

whence

$$\Delta(P')=\frac{a'}{a}(B'\bar{B}+B\bar{B}')-\frac{a'^2}{a^2}(\tfrac{1}{4}+B\bar{B})-B'\bar{B}'$$

$$=-\left(\frac{a'}{2a}\right)^2-\left|\frac{a'B}{a}-B'\right|^2\leq 0.\quad \text{Q.E.D.}$$

An easily derived and useful formula on discriminants is, for any function $\theta(r)$, and for $\epsilon=+1$,

$$\Delta(P\theta'-b\,\theta P')=\tfrac{1}{4}\theta'^2+b^2\theta^2(a'c'-B'\bar{B}'). \tag{12}$$

Applied to the density relation (6), these results give the following.

Lemma 2: ρ_0 is nonsingular for all $r>0$, $\zeta, \bar{\zeta}$ if and only if

$$a'c'-B'\bar{B}'>-1/4r^2. \tag{13}$$

If this condition holds and $a(r)>0$ for all r, then $\rho_0>0$ for all $r, \zeta, \bar{\zeta}$ if and only if

$$a'c'-B'\bar{B}'>-S'^2/36S^2. \tag{14}$$

Proof: From (6) ρ_0 is nonsingular if and only if $P-rP'\neq 0$ for all $r, \zeta, \bar{\zeta}$. Using Lemma 1 and Eq.

(12) with $\theta = r$ gives the equivalent condition

$$\Delta(P - rP') = \tfrac{1}{4} + r^2(a'c' - B'\bar{B}') > 0,$$

which proves (13). If (13) holds and $a(r) > 0$ then clearly $P - rP' > 0$ for all r, ζ, $\bar{\zeta}$, since this is certainly true as $r \to 0$ by Corollary 1. Thus, $\rho_0 \neq 0$ if and only if [using Lemma 1 and (12) with $\theta = S, b = 3$]

$$\Delta(PS' - 3SP') = 9S^2(a'c' - B'\bar{B}') + \tfrac{1}{4}S'^2 > 0.$$

Furthermore, it this condition holds then $PS' - 3SP' > 0$ everywhere, since it is certainly so as $r \to 0$ because $S' = O(r^2) > 0$ [Eqs. (8) and (11)] and $S(r) = O(r^3)$. This proves (14).

The result of Lemma 2 together with Corollary 2 may be summed up by saying that if the initial density $\rho_0(r)$ is everywhere regular then

$$0 \geq a'c' - B'\bar{B}' > -\tfrac{1}{4}\min\left(\frac{1}{r^2}, \left(\frac{S'}{3S}\right)^2\right). \qquad (15)$$

IV. COLLAPSE TIMES

Let us assume that initially at $t = 0$ the density is everywhere positive and nonsingular and that all portions of the dust cloud are momentarily collapsing,

$$\dot{\phi}(r, 0) \leq 0.$$

We will say that $t = 0$ is a *moment of time symmetry* if $\dot{\phi}(r, 0) = 0$.

From Eq. (6) we see that the density becomes infinite when

$$\phi = 0 \text{ or } P\phi' - \phi P' = 0.$$

These two conditions categorize what we will call *singularities of the first and second kinds*, respectively. From Eq. (5) it may be readily shown that eventually $\phi = 0$ along each fluid stream line r, ζ, $\bar{\zeta}$ = constants, and we define the *primary collapse time* $t_0 = t_0(r)$ to be the (proper) time taken to reach this point, i.e., t_0 is defined by

$$\phi(r, t_0) = 0. \qquad (16)$$

However, it may happen that before this point is reached a singularity of the second kind occurs. This will happen if the *secondary collapse time* $t_1 = t_1(r, \zeta, \bar{\zeta})$ defined by

$$P(r, \zeta, \bar{\zeta})\phi'(r, t_1) - P'(r, \zeta, \bar{\zeta})\phi(r, t_1) = 0 \qquad (17)$$

satisfies

$$t_1 < t_0.$$

In general we refer to the *collapse time*

$$t_C = \min(t_0, t_1) = t_C(r, \zeta, \bar{\zeta})$$

as the time taken for a particle of fluid to reach a density singularity. We say that the comoving sphere r = const *possesses a secondary collapse point* if $t_C = t_1$ for this value of r and some value of ζ, $\bar{\zeta}$. We then have the following result.

Lemma 3: The comoving sphere r = const possesses a secondary collapse point if and only if for some $t < t_0(r)$

$$\min\left(\frac{1}{r^2}, \left(\frac{S'}{3S}\right)^2\right) > 4(B'\bar{B}' - a'c') \geq \left(\frac{\phi'(r, t)}{\phi(r, t)}\right)^2.$$

Proof: From (17), Lemma 1, and Eq. (12) we have that $t_1 < t_0$ for some r, ζ, $\bar{\zeta}$ if and only if

$$0 \geq \Delta(P\phi' - \phi P') = \tfrac{1}{4}\phi'^2 + \phi^2(a'c' - B'\bar{B}')$$

for some $t < t_0(r)$. Combining this with Lemma 2 we obtain the desired result.

For simplicity let us restrict attention now to two cases:

(a) The zero-energy case, $W(r) = 1$. This is the critical case where all particles just manage to come from infinity (with zero energy) in the infinite past, and Eq. (5) integrates to give

$$\phi = (r^{3/2} - \tfrac{3}{2}S^{1/2}t)^{2/3},$$
$$t_0 = 2r^{3/2}/3S^{1/2}.$$

(b) The time-symmetric case, $\dot{\phi}(r, 0) = 0$. From Eq. (5) we have that $W^2 = 1 - S/r$, and the solution of (5) is a cycloidal function, most conveniently expressed in terms of a parameter η,

$$\phi = r\cos^2\eta,$$
$$t = \frac{r^{3/2}}{S^{1/2}}(\eta + \sin\eta\cos\eta),$$
$$t_0 = \pi r^{3/2}/2S^{1/2} \quad (\eta = \pi/2).$$

In both these cases the criterion of Lemma 3 reduces to the following simpler condition.

Theorem 1: In the case of a zero-energy or time-symmetric collapse, the sphere r = const possesses a secondary collapse point if and only if

$$\frac{S'}{3S} > \frac{1}{r}. \qquad (18)$$

Proof:

(a) Zero-energy case. We easily deduce that

$$\frac{\phi'}{\phi} = \frac{1}{r}\left[1 + \left(1 - \frac{S'r}{3S}\right)\frac{t/t_0}{1 - t/t_0}\right].$$

Hence if

$$\frac{S'}{3S} \leq \frac{1}{r},$$

then

$$\frac{\phi'}{\phi} \geq \frac{1}{r} \geq \min\left(\frac{1}{r}, \frac{S'}{3S}\right)$$

for all $0 < t < t_0$, so that $r = \text{const}$ can possess no secondary collapse points (by Lemma 3).

To show that (18) is sufficient, we need only note that, by Eq. (15),

$$0 \leq \beta^2 = 4r^2(B'\overline{B}' - a'c') < 1.$$

Thus

$$\left|\frac{\phi'}{\phi}\right| \leq [4(B'\overline{B}' - a'c')]^{1/2} = \frac{\beta}{r}$$

for all t in the range

$$0 < t_- \leq t \leq t_+ < t_0(r),$$

where

$$t_\pm = \frac{1 \pm \beta}{(S'r/3S) \pm \beta}\, t_0,$$

from which the theorem follows by Lemma 3.

(b) Time-symmetric case. In this case

$$\frac{\phi'}{\phi} = \frac{1}{r}\left[1 + \tfrac{3}{2}\left(\tan^2\eta + \frac{\eta \sin\eta}{\cos^3\eta}\right)\left(1 - \frac{S'r}{3S}\right)\right],$$

and the argument follows along lines similar to those of the previous case.

It should be pointed out that this theorem holds in particular for the case of spherical symmetry. In that case we can set $a' = c' = B' = 0$, whence $\beta = 0$, $t = t_1(r)$, and if $S'/3S > 1/r$ the whole sphere $r = \text{const}$ collapses simultaneously to the secondary singularity. In the general quasispherical case the secondary singularity is not simultaneous over $r = \text{const}$, but occurs over the range $t_- \leq t \leq t_+$.

The condition (18) has a physical interpretation in terms of the initial density distribution at $t = 0$. Let us define, at time t, the *mean density over the sphere $r = const$* to be

$$\overline{\rho}(r, t) \equiv \lim_{\Delta r \to 0} \frac{M(r + \Delta r) - M(r)}{V(r + \Delta r, t) - V(r, t)} = \frac{M'(r)}{V'(r, t)},$$

where $V(r, t)$ is the volume within the sphere $r = \text{const}$,

$$V(r, t) = \int_0^r dr \int\int d\zeta\, d\overline{\zeta}\, XY^2.$$

A computation similar to that used to deduce Eq. (7) gives

$$V(r, t) = 4\pi \int_0^r dr\, \frac{\phi'\phi^2}{W},$$

whence

$$\overline{\rho}(r, t) = \frac{S'}{\phi'\phi^2},$$

and setting $t = 0$ we find

$$\overline{\rho}_0(r) = S'/r^2.$$

For spherical symmetry we clearly have

$$\overline{\rho}_0(r) = \rho_0(r).$$

Now if $\overline{\rho}_0(r)$ is a decreasing function of r then

$$S(r) = \int_0^r S'(u)du$$
$$> \frac{S'(r)}{r^2} \int_0^r u^2 du = \frac{S'(r)}{3r}.$$

By Theorem 1 this is a sufficient condition for every particle of the fluid to collapse to a singularity of the first kind.

To find a more general criterion, we must define $\langle\overline{\rho}(r, t)\rangle$, the W-weighted volume average of $\overline{\rho}(r, t)$, by

$$\langle\overline{\rho}(r, t)\rangle = \int_0^r \overline{\rho}(r', t) W(r') dV(r', t) \Big/ \int_0^r W(r') dV(r', t)$$
$$= 3S(r)/\phi^3(r, t).$$

Thus at $t = 0$

$$\langle\overline{\rho}_0(r)\rangle = 3S/r^3.$$

The criterion (18) is now equivalent to $\langle\overline{\rho}_0(r)\rangle > 0$, i.e., $\langle\overline{\rho}_0(r)\rangle$ is an increasing function of r. Alternatively it is equivalent to the condition

$$\overline{\rho}_0(r) > \langle\overline{\rho}_0(r)\rangle.$$

Thus $r = \text{const}$ possesses a secondary collapse point if and only if the mean density rises steeply enough somewhere between 0 and r for it to be greater than its W average over the entire region inside $r = \text{const}$. This might be termed a condition of "strong increase" near $r = \text{const}$ (N.B.: $\overline{\rho}_0$ might actually be *decreasing* at r itself, but it must have increased fairly strongly somewhere inside $r = \text{const}$). It holds manifestly throughout a spherical shell of uniform density (vacuum inside the shell).

V. OCCURRENCE OF TRAPPED SURFACES

In order to decide whether the collapses discussed above lead to black holes (hidden singularities) it is necessary to compute the event horizon of observers staying at infinity. As this involves the computation of null geodesics and it proves impracticable to obtain these for the general metric (1), we will concentrate attention instead on the closely related concept of a trapped surface[7] (i.e., a compact spacelike 2-surface both of whose normal future-pointing null geodesic families are converging). Indeed it may be shown[8] that if the 2-surface $S_{r,t}$ ($r = \text{const}$, $t = \text{const}$) is a trapped surface then it and its entire future development lie behind the event horizon provided that the density falls off rapidly enough at infinity for the space to be weakly asymptotically simple.

In order to obtain a criterion for $S_{r,t}$ to be a trapped surface, consider a bundle of affinely parameterred null geodesics, with tangent vector field k^μ normal to $S_{r,t}$. These conditions may be written as

$$k_\mu k^\mu = 0, \quad k^\mu{}_{;\nu} k^\nu = 0 \tag{19}$$

and

$$k^2 = k^3 = 0, \quad (k^0)^2 - X^2(k^1)^2 = 0 \text{ on } S_{r,t}.$$

Here we have numbered coordinates by $x^0 = t$, $x^1 = r$, $x^2 = \zeta$, $x^3 = \bar\zeta$. The choice of affine parameter may clearly be such that (for future-pointing geodesics)

$$k^0 = X, \quad k^1 = \epsilon = \pm 1 \text{ on } S_{r,t}. \tag{20}$$

The convergence or divergence of the null geodesics is determined by the sign of the invariant $k^\mu{}_{;\mu}$ evaluated on $S_{r,t}$ (negative for convergence, positive for divergence). Now

$$\begin{aligned} k^\mu{}_{;\mu} &= k^\mu{}_{,\mu} + \Gamma^\mu_{\nu\mu} k^\nu \\ &= k^0{}_{,0} + k^1{}_{,1} + X\left(\frac{\dot X}{X} + \frac{2\dot Y}{Y}\right) + \epsilon\left(\frac{X'}{X} + \frac{2Y'}{Y}\right) \text{ on } S_{r,t} \end{aligned} \tag{21}$$

since $k^2{}_{,2} = k^3{}_{,3} = 0$ on $S_{r,t}$. On the other hand, forming $\partial/\partial x^0$ of the first equation of (19) and setting $\mu = 1$ in the second equation gives on $S_{r,t}$

$$k^0{}_{,0} - \epsilon X k^1{}_{,0} - \dot X = 0,$$

$$k^1{}_{,0} X + \epsilon(k^1{}_{,1} + 2\dot X) + \frac{X'}{X} = 0.$$

Eliminating $k^1{}_{,0}$ between these two equations and substituting in (21) gives

$$\begin{aligned} k^\mu{}_{;\mu} &= \frac{2}{Y}(X\dot Y + \epsilon Y') \\ &= \frac{2(\phi' P - \phi P')}{\phi P}\left(\frac{\dot\phi}{W} + \epsilon\right). \end{aligned}$$

If initially we assume that $\rho_0(r)$ is nonsingular then the first factor on the right-hand side is positive (see Lemma 2). Hence if

$$0 \geq \dot\phi \geq -W(r) \text{ at } t = 0,$$

or equivalently, by Eq. (5) and (9),

$$S(r) < r, \tag{22}$$

then initially $k^\mu{}_{;\mu} < 0$ for inward geodesics ($\epsilon = -1$) and $k^\mu{}_{;\mu} > 0$ for outward geodesics ($\epsilon = +1$). The inward geodesics clearly remain convergent throughout the collapse, but the outward ones become convergent as well after a time t_H given by

$$\dot\phi(r, t_H) = -W(r),$$

provided this occurs before either ϕ or $\phi' P - \phi P'$ changes sign, i.e., before the collapse time t_C. Substituting into Eq. (5) we see that $S_{r,t}$ is a trapped surface for all $0 < t_H < t < t_C$, where t_H is determined by

$$\phi(r, t_H) = S(r). \tag{23}$$

Clearly, then, a singularity of the first kind ($\phi = 0$) always occurs *after* the formation of a trapped surface ($t_0 > t_H$) and is hidden behind the event horizon. However, if a singularity of the second kind develops it may occur before the trapped surface forms. The light rays in this case will be initially diverging in the vicinity of the singularity, but they may begin to reconverge before they manage to escape to infinity. Accordingly we term this circumstance a *locally naked singularity;* the sense is that such a singularity can be seen by neighboring particles of the fluid. For it to be a truly naked singularity (i.e., *globally naked),* the light rays must never begin to reconverge. It is difficult to give a general criterion for the latter occurrence in our situations, but by the argument given in Theorem 1 a criterion for the occurrence of a locally naked singularity is simply

$$t_- < t_H \tag{24}$$

where t_- is defined by

$$\frac{\phi'(r, t_-)}{\phi(r, t_-)} = \frac{\beta}{r}, \quad \beta^2 = 4r^2(B'\bar B' - a'c').$$

In the case of zero-energy collapse we obtain

$$t_H = \frac{2}{3}\left(\frac{r^{3/2}}{S^{1/2}} - S\right),$$

while for time-symmetric collapse t_H is obtained by setting

$$\cos^2\eta_H = S/r.$$

Using the formulas for t_- obtained in the proof of Theorem 1, the criterion (24) reduces to the inequality

$$1 > \beta > \beta_0,$$

where

$$\beta_0 = 1 - \left(\frac{S'r}{3S} - 1\right)\left[\left(\frac{r}{S}\right)^{3/2} - 1\right]$$

for zero-energy collapse and

$$\beta_0 = 1 - \frac{3}{2}\left(\frac{S'r}{3S} - 1\right)\left(\frac{r}{S} - 1\right)^{1/2} \times \left[\left(\frac{r}{S} - 1\right)^{1/2} + \frac{r}{S}\cos^{-1}\left(\frac{S}{r}\right)^{1/2}\right]$$

for time-symmetric collapse. Hence we have the following theorem.

Theorem 2: A zero-energy or time-symmetric quasispherical collapse will develop a locally naked singularity on the comoving sphere $r =$const if and only if

$$\frac{S'}{3S} > \frac{1}{r}$$

and β is sufficiently close to 1.

Physically the criterion on β is that the density is initially sufficiently asymmetric (i.e., the dipole moment of the local surface density on $r =$ const is high enough).

From the above analysis, locally naked singularities will also occur in spherical symmetry ($\beta = 0$) if

$$\frac{S'r}{3S} > [1 - (S/r)^{3/2}]^{-1}$$

for zero-energy collapse, and

$$\frac{S'r}{3S} > 1 + \frac{2}{3(r/S-1)^{1/2}[(r/S-1)^{1/2} + (r/S)\cos^{-1}(S/r)^{1/2}]}$$

for time-symmetric collapse. In both cases this is merely a still further strengthening of the "strong density increase" condition discussed at the end of Sec. IV. As is shown by specific examples in Ref. 2, such locally naked singularities may be true naked singularities in the spherical case, a feature which no doubt transfers to the quasispherical case. Figure 1 shows an example where there is a strong density increase between r_0 and r_1 leading to both local and global naked singularities.

An easily computed example, already mentioned at the end of Sec. IV, is a spherical shell of matter, $\rho =$const for $r_0 < r < r_1$, $\rho = 0$ for $r < r_0$ and $r > r$. In this case the inner parts of the shell always collapse to a locally naked singularity, and if the shell is thin enough the entire shell (right up to its outer boundary) will collapse to a globally naked singularity.

The examples given in this paper (zero-energy and time-symmetric collapse) are examples of highly coherent collapse situations. The more general case is more complicated to work out but leads to essentially the same conclusions, except that now naked singularities may arise even for situations in which the initial density decreases outwards everywhere, provided the outer parts of the fluid are initially imploding relatively to the inner parts. In such a case crossings may occur for the trajectories of the fluid particles, which will in general result in singularities of the second kind. If these crossings occur before the formation of a trapped surface then they will again be locally naked.

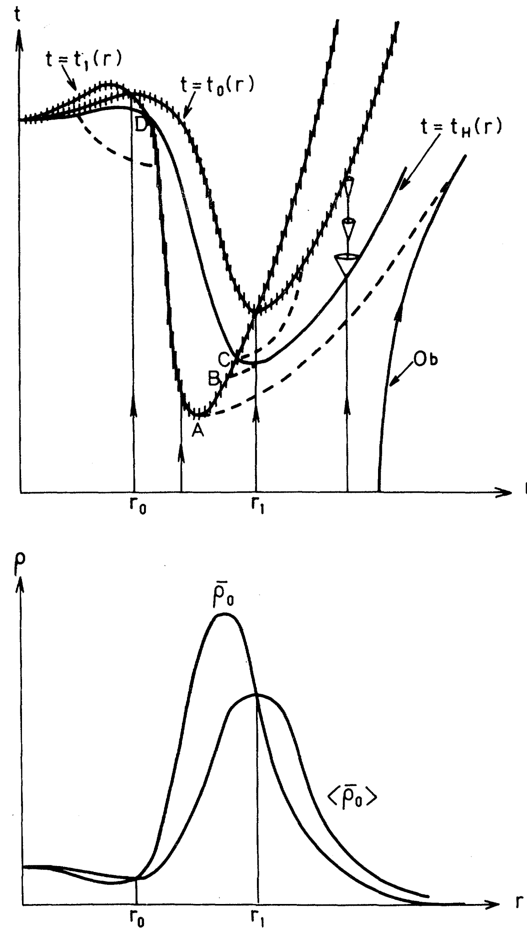

FIG. 1. A zero-energy spherical collapse. Hatched lines represent singularities $t = t_0$, $t = t_1$ of the space-time manifold. Vertical upward-directed lines are paths of fluid particles and dashed lines represent null geodesics, while the line Ob represents the world line of an observer who remains far from the center of the cloud. Between r_0 and r_1 there is a strong density increase at $t = 0$, i.e., $S'r/3S > 1$. The stretch AB of the $t = t_1$ singularity is globally naked (visible to all observers). The stretches BC and AD are locally naked (visible to neighboring particles of the fluid), and all other portions of the singularity surface are hidden.

VI. CONCLUSIONS

Exact examples of nonspherical collapse have been given which show all the essential features of spherical collapse. For reasonable initial conditions, such as density decreasing outwards and coherent enough initial velocities, a black hole will develop. However, portions of the collapse may result in naked singularities if the density distribution is more irregular. The main conclusion is that a strong radial increase in the density in some region of the fluid will herald the onset of a naked singularity; a highly asymmetrical matter distribution tends to make this feature more likely, but is by no means necessary for it to occur. A practical example, where the right kind of condition might exist for the occurrence of a naked singularity, is the collapse of a dumbbell distribution of matter (a kind of simultaneous two-body collapse); although not falling within the class of models discussed here, the density distribution relative to the center of the configuration does appear *prima facie* to satisfy the required conditions.

Whether the occurrence of such a naked singularity is truly a problem for theoretical or observational physicists is left unanswered. However, it should be borne in mind that the naked singularity will in no sense *persist*. An inspection of Fig. 1 shows that any naked parts of the singularity are only briefly visible (over periods no longer than the typical collapse time of the body), soon to be swallowed up in the eventual horizon of the body as a whole. Thus, the results concerning the final state, such as the Israel-Carter theorems, are almost surely correct in an asymptotic sense. However, on the road to the final state we must prepare for cataclysms, albeit short-lived ones.

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