PHYSICAL REVIEW D VOLUME 12, NUMBER 1 1 JULY 1975

Spin tests from angular correlations in sequential decays

M. Daumens, G. Massas,* and P. Minnaert
Laboratoire de Physique Théorique,† Université de Bordeaux I, Gradignan, France
(Received 10 February 1975)

We formulate precisely the principle of spin tests from angular correlations in sequential decays. We show that the spin tests for bosons proposed by Ademollo, Gatto, and Preparata for canonical frames and by Berman, Jacob, and Chung for helicity frames are equivalent.

I. INTRODUCTION

The discovery of particles and resonant states which undergo two-body sequential decay of the type

$$1 \to 2+3, \quad 2 \to 4+5 \tag{1}$$

has incited many physicists, from the late 1950's until the present, to propose more and more refined methods of determining the spin and parity of these states. Generally speaking these methods can be divided into two main classes according to whether they do or do not require the reconstruction of the polarization density matrix of particle 1.

Methods in the first class are based on the following scheme. One first makes a hypothesis on the spin and reconstructs all (or only part of) the polarization density matrix, from the sequential decay angular distribution of process (1). Then the spin test consists in checking that the measured matrix satisfies all the required conditions to be a density matrix (or part of a density matrix), namely, it must be positive[1] or, under some conditions on the production process of particle 1, its rank must be fixed.[2] If these conditions are not satisfied the spin hypothesis must be rejected.[3] Methods in the second class are based on the remark that for each spin value of particle 1 there exist characteristic correlations between the decay angular distribution and the polarization angular distribution of particle 2, i.e., for process (1) there exist spin-dependent correlations among the moments of the sequential decay angular distribution.[4,5]

In this note we formulate precisely the principle of spin tests of this second class, and we show its application for the two main choices of reference frames, namely the canonical and helicity frames. Furthermore we show that the spin tests for bosons of Berman and Jacob[6,7] and Chung[8,9] (BJC) and those of Ademollo, Gatto, and Preparata[10] (AGP) both proceed from this principle and are actually equivalent even though they are expressed in very different forms. We also show how to improve the formulation of the AGP test and simplify its use.

II. CHOICE OF FRAMES AND AMPLITUDES

We assume that particles 3 and 5 in process (1) are spinless bosons with parities ϵ_3 and ϵ_5. We call $j_1^{\epsilon_1}$, $j_2^{\epsilon_2}$, and $j_4^{\epsilon_4}$ the spins and parities of particles 1, 2, and 4, respectively. We denote by θ, ϕ the angles of particle 2 in a rest frame F_1 of particle 1 and by θ', ϕ' the angles of particle 4 in a rest frame F_2 of particle 2. The frame F_2 must be precisely defined relatively to the frame F_1. We call the canonical frame F_2^c that frame deduced from F_1 by the boost L_{21}, and the helicity frame F_2^h that frame deduced from F_1 by the Lorentz transformation $L_{21}\times R(\phi,\theta,0)$. In addition one must choose a set of invariant decay amplitudes to describe the first decay. One may choose the canonical amplitudes A^l or the helicity amplitudes $A(\lambda)$. At this point we want to stress that it is not compulsory to bind the choice of invariant decay amplitudes to the choice of the frame F_2. Indeed, the two sets of invariant amplitudes are related by a linear orthogonal transformation

$$A(\lambda)=\sum_l (\hat{l}/\hat{j}_1)\langle l0j_2\lambda|j_1\lambda\rangle A^l\,, \tag{2a}$$

$$A^l=\sum_\lambda (\hat{l}/\hat{j}_1)\langle l0j_2\lambda|j_1\lambda\rangle A(\lambda)\,, \tag{2b}$$

where we have used the notation $\hat{x}=(2x+1)^{1/2}$, so that either set of invariant amplitudes can be used whatever the choice for F_2. However, the transition matrices for the first step take their simpler form when the choices are associated:

$$T_c(\theta,\phi)^{m_2}{}_{m_1}=\sum_l \langle lmj_2m_2|j_1m_1\rangle Y^l_m(\theta,\phi)A^l\,, \tag{3a}$$

$$T_h(\theta,\phi)^{m_2}{}_{m_1}=\sum_{m_2} (4\pi)^{-1/2}\hat{j}_1 D^{j_1}(\phi,\theta,0)^{m_1}{}^*_{m_2}A(m_2)\,. \tag{3b}$$

III. SEQUENTIAL DECAY ANGULAR DISTRIBUTIONS

The polarization state of the initial particle 1 can be described by the multipole parameters $t^{L_1}_{M_1}$,[11] and it follows that the normalized sequential decay angular distributions for process (1), in the canonical and helicity frames, have the structure

$$I(\Omega,\Omega'_c)=(4\pi)^{-1}\sum_{LL_1L_2} C(L_2)K(L_1L_2L)\sum_{M_1}\langle LML_2M_2|L_1M_1\rangle t^{L_1}_{M_1}{}^* Y^L_M(\Omega)Y^{L_2}_{M_2}(\Omega'_c)\,, \tag{4a}$$

$$I(\Omega,\Omega'_h)=(4\pi)^{-3/2}\sum_{L_1L_2M_2} C(L_2)H(L_1L_2M_2)\sum_{M_1}\hat{L}_1 t^{L_1}_{M_1}{}^* D^{L_1}(\Omega)^{M_1}{}_{M_2}^* Y^{L_2}_{M_2}(\Omega'_h)\,, \tag{4b}$$

where we have used the notation $\Omega=(\theta,\phi)$, $\Omega'=(\theta',\phi')$, and $D^{L_1}(\Omega)=D^{L_1}(\phi,\theta,0)$. $C(L_2)$ represents the real decay coefficients of the second step,[12] and $K(L_1L_2L)$ and $H(L_1L_2M_2)$ are the canonical and helicity decay coefficients for the first step. Their expression in terms of the corresponding invariant decay amplitudes, normalized to $\sum_l |A^l|^2=\sum_\lambda |A(\lambda)|^2=1$, can be written

$$K(L_1L_2L)=\hat{L}_1\hat{j}_1\hat{j}_2\sum_{l,l'}(-1)^{l'}\hat{l}\hat{l}'\begin{Bmatrix} j_2 & j_2 & L_2\\ j_1 & j_1 & L_1\\ l & l' & L\end{Bmatrix}\langle l0l'0|L0\rangle A^l A^{l'*}\,, \tag{5a}$$

$$H(L_1L_2M_2)=\hat{L}_1\sum_{m_2,n_2}\langle j_2n_2L_2M_2|j_2m_2\rangle\langle j_1n_2L_1M_2|j_1m_2\rangle A(m_2)A(n_2)^*\,. \tag{5b}$$

Using Eq. (2) the canonical coefficients can be expressed in terms of helicity amplitudes and the helicity coefficients in terms of the canonical amplitudes. Indeed, at fixed values of L_1 and L_2 the two sets of coefficients $K(L)$ and $H(M_2)$ are related by the linear orthogonal transformation

$$K(L_1L_2L)=\sum_{M_2}(\hat{L}/\hat{L}_1)\langle L0L_2M_2|L_1M_2\rangle H(L_1L_2M_2) \tag{6a}$$

$$H(L_1L_2M_2)=\sum_{L}(\hat{L}/\hat{L}_1)\langle L0L_2M_2|L_1M_2\rangle K(L_1L_2L)\,. \tag{6b}$$

The moments of the sequential decay angular distributions (4) are defined by[13]

$$\mathcal{K}(L_1M_1L_2L)=4\pi\sum_{M,M_2}\langle LML_2M_2|L_1M_1\rangle \times\langle Y^L_M(\Omega)Y^{L_2}_{M_2}(\Omega'_c)\rangle\,, \tag{7a}$$

$$\mathcal{H}(L_1M_1L_2M_2)=(4\pi)^{1/2}\hat{L}_1\langle D^{L_1}(\Omega)^{M_1}{}_{M_2}^* Y^{L_2}_{M_2}(\Omega'_h)\rangle\,, \tag{7b}$$

so that they are proportional to the multipole parameter $t^{L_1}_{M_1}$,

$$\mathcal{K}(L_1M_1L_2L)=C(L_2)K(L_1L_2L)^* t^{L_1}_{M_1}\,, \tag{8a}$$

$$\mathcal{H}(L_1M_1L_2M_2)=C(L_2)H(L_1L_2M_2)^* t^{L_1}_{M_1}\,. \tag{8b}$$

These relations are the basic equations for reconstructing the density matrix of particle 1 from the measured moments of the sequential decay angular distributions; they allow spin tests of the first class. As we shall see now they are also the basic equations to derive spin tests of the second class.

IV. PRINCIPLE OF THE SPIN TESTS

To derive spin tests for particle 1, one assumes that the spin and parities j_2, j_4, ϵ_2, ϵ_3, ϵ_4, ϵ_5 and the decay coefficients $C(L_2)$ are well known. The principle of the tests is based on the remark that for fixed L_1 and M_1 there exist several moments

$$\mathcal{K}(L_2,L)\quad (0\le L_2\le 2j_2,\ \ |L_1-L_2|\le L\le L_1+L_2)$$

or

$$\mathcal{H}(L_2,M_2)\quad (0\le L_2\le 2j_2,\ -L_2\le M_2\le L_2)\,,$$

providing one with a multiple determination of the same multipole parameter $t^{L_1}_{M_1}$. Then, from Eq. (8) the ratios of two different nonvanishing $\mathcal{K}$ or $\mathcal{H}$ moments, at fixed L_1, M_1, are independent of the polarization of particle 1, but they may depend on the spin j_1 through ratios of two K's or two H's. If these ratios can be given analytic forms of j_1, one obtains equations for the spin j_1 with coefficients depending on the experimental moments.

Note that it is easier to get analytic expressions for the ratios if the K's and H's are expressed in terms of helicity amplitudes rather than canonical invariant amplitudes since then their dependence on j_1 is contained in only *one* Clebsch-Gordan coefficient. This is what we shall do in the following discussion.

(i) Let us assume first that the decay in the first step involves only one independent helicity amplitude. Then the dynamical dependence disappears from the expressions of K or H, and at fixed L_1 any ratio of two K's or two H's which depends on j_1 yields an equation for the spin j_1 in terms of the $\mathfrak{K}$ or $\mathcal{H}$ moments.

(ii) Assume now that the decay in the first step involves two independent helicity amplitudes denoted by ζ_1 and ζ_2, and normalized to $|\zeta_1|^2+|\zeta_2|^2=1$. Then the dynamical dependence of the K's and H's can be expressed linearly in terms of the helicity decay parameters α', β', γ',

$$\alpha'=2\,\mathrm{Re}\zeta_1^*\zeta_2,\quad \beta'=2\,\mathrm{Im}\zeta_1^*\zeta_2,\quad \gamma'=|\zeta_1|^2-|\zeta_2|^2, \tag{9a}$$

$$\alpha'^2+\beta'^2+\gamma'^2=1\,. \tag{9b}$$

One has

$$K=a\alpha'+b\beta'+c\gamma'+d\,, \tag{10a}$$

$$H=a'\alpha'+b'\beta'+c'\gamma'+d'\,, \tag{10b}$$

with coefficients a, b, c, d, a', b', c', d' depending in general on j_1. The ratio of two nonvanishing $\mathfrak{K}$'s or $\mathcal{H}$'s, at fixed L_1,M_1, gives a linear equation in α', β', γ'. Dynamics-independent spin tests can then be derived in two different ways: *Either* one can write a system of three independent equations in α', β', γ', and condition (9b) on the solution yields an equation for j_1 in terms of the $\mathfrak{K}$ or $\mathcal{H}$ experimental moments (spin tests of this type have been proposed for fermions by Byers and Fenster[14] and by Ademollo and Gatto[15]), *or* one can write systems of two independent equations in one decay parameter or three independent equations in two decay parameters. The consistency conditions of such systems yield equations for the spin j_1 in terms of the $\mathfrak{K}$ or $\mathcal{H}$ moments.

Spin tests of the latter type have been proposed by AGP[10] for bosons. But, since AGP worked with canonical amplitudes and $9j$ symbols, they could not write down the consistency conditions as analytic equations in j_1. Their procedure for spin determination was then by trial and error: They used numerical values for the $9j$ symbols and they tested the compatibility of the system in the canonical parameters α and γ for successive values of j_1. A considerable improvement of the method is obtained by using helicity-invariant amplitudes, since then one can get analytic expressions for the spin j_1 in terms of the experimental $\mathfrak{K}$ moments. See an example in Sec. V below.

A similar analysis was made implicitly by BJC,[6,7,8,9] although they do not use the parameters α', β', γ'. Their method consists in writing linear combinations of $\mathcal{H}$ moments, with given L_1,M_1, such that their ratios are independent of the quantities $|\zeta_1|^2$, $|\zeta_2|^2$, or $\zeta_1^*\zeta_2$. In our language this corresponds to the fact that with helicity frames the system of linear equations is easily reduced to subsystems of two equations in one decay parameter. Thus the consistency conditions can be readily written down and give simple equations for the spin j_1 in terms of the experimental $\mathcal{H}$ moments. See Sec. V.

To make sure that the AGP and BJC formulations of spin tests are actually equivalent, one may verify, in each case, that the expression of j_1 in terms of the $\mathfrak{K}$ moments can be derived from that in terms of the $\mathcal{H}$ moments by using the linear orthogonal transformation (6) which at fixed L_1, M_1, and L_2 relates the moments $\mathfrak{K}(L_1M_1L_2L)$ and $\mathcal{H}(L_1M_1L_2M_2)$.[16] Furthermore, when the test is expressed as the ratio of mean values of explicit expressions in angles θ, ϕ, θ', ϕ', one may verify that the test in terms of the canonical angles θ_c', ϕ_c' can be derived from the test in terms of the helicity angles θ_h', ϕ_h' by means of the transformation which relates these angles, namely,

$$\cos\theta_h'=\cos\theta\cos\theta_c'+\sin\theta\sin\theta_c'\cos(\phi-\phi_c')\,, \tag{11a}$$

$$\sin\theta_h'\cos\phi_h'=-\sin\theta\cos\theta_c'+\cos\theta\sin\theta_c'\cos(\phi-\phi_c')\,, \tag{11b}$$

$$\sin\theta_h'\sin\phi_h'=-\sin\theta_c'\sin(\phi-\phi_c')\,. \tag{11c}$$

V. EXAMPLE OF AGP AND BJC FORMULATIONS OF SPIN TESTS

Let us consider, as an example[17] of the two possible formulations of spin tests, the sequential decay discussed by AGP and BJC:

$$j_1^{\epsilon_1}\to 1^-0^-,\quad 1^-\to 0^-0^-\,. \tag{12}$$

We define $\eta=\epsilon_1(-1)^{j_1}$. If $\eta=-1$ the first decay involves two amplitudes, whereas if $\eta=+1$ it involves only one. A simultaneous study of both cases, following the procedure of Sec. IV, yields a spin parity test for each value of L_1 even and M_1. Fixing $L_1=2$, $M_1=0$ in both formulations one can write the test in the form

$$\frac{\eta j_1(j_1+1)}{j_1(j_1+1)-3}=\frac{\langle W_N\rangle}{\langle W_D\rangle}\,. \tag{13}$$

For the canonical frame the quantities $\langle W_N\rangle$ and $\langle W_D\rangle$ are linear combinations of the $\mathfrak{K}$ moments

$$\langle W_N\rangle=2\sqrt{7}\,\mathfrak{K}(2020)+2\sqrt{10}\,\mathfrak{K}(2022)+\sqrt{2}\,\mathfrak{K}(2024)\,, \tag{14a}$$

$$\langle W_D\rangle=-2\sqrt{7}\,\mathfrak{K}(2002)+\sqrt{7}\,\mathfrak{K}(2020)-\sqrt{10}\,\mathfrak{K}(2022)+3\sqrt{2}\,\mathfrak{K}(2024)\,, \tag{14b}$$

while for the helicity frame they are linear combinations of the $\mathcal{H}$ moments, and the ratio (13) gives Chung's formula

$$\frac{\eta j_1(j_1+1)}{j_1(j_1+1)-3}=\frac{-2\sqrt{5}\,\mathcal{H}(2022)}{2\mathcal{H}(2000)-\sqrt{5}\,\mathcal{H}(2020)}\,, \qquad (15)$$

which has been used in several experimental papers.[18] In addition the functions W_N and W_D can be written explicitly in terms of the decay angles. For the canonical frame one finds

$$\langle W_N\rangle=\langle -\sin^2\theta[\sin^2\theta(3\cos^2\theta'_c-1)-\sin 2\theta\sin 2\theta'_c\cos(\phi-\phi'_c)+(1+\cos^2\theta)\sin^2\theta'_c\cos 2(\phi-\phi'_c)]\rangle\,, \qquad (16a)$$

$$\langle W_D\rangle=\langle(3\cos^2\theta-1)[\tfrac{1}{5}+\cos^2\theta+\cos^2\theta'_c-3\cos^2\theta\cos^2\theta'_c-\sin 2\theta\sin 2\theta'_c\cos(\phi-\phi'_c)-\sin^2\theta\sin^2\theta'_c\cos 2(\phi-\phi'_c)]\rangle\,, \qquad (16b)$$

and for the helicity frame the BJC test can be written in the simple form

$$\frac{\eta j_1(j_1+1)}{j_1(j_1+1)-3}=\frac{-5\langle\sin^2\theta\sin^2\theta'_h\cos 2\phi'_h\rangle}{\langle(3\cos^2\theta-1)(3-5\cos^2\theta'_h)\rangle}\,. \qquad (17)$$

As we remarked previously, one verifies the equivalence of both formulations by using either the linear orthogonal transformation (6) or the transformation (11).

Equation (17) has been given by Koch.[19] Using Eqs. (2.12) and (2.14) of the SLAC report by Berman and Jacob[6] one obtains for the numerator of Eq. (17) the expression

$$\langle W_N\rangle=-4\langle\sin^2\theta\cos 2\phi'_h\rangle \qquad (18)$$

given in two recent experimental papers.[20] The equivalence of both expressions results from the identity

$$\langle\cos 2\phi'_h\rangle=\tfrac{5}{4}\,\langle\sin^2\theta'_h\cos 2\phi'_h\rangle\,, \qquad (19)$$

strictly valid for the decay of a pure spin-one particle into spinless particles. However, since the function $\cos 2\phi'_h$ does not belong to an orthonormal set of functions, the function $\frac{5}{4}\sin^2\theta'_h\cos 2\phi'_h$ is a better estimator from a statistical point of view and with respect to the sensibility to the background.[21]

ACKNOWLEDGMENT

The authors thank Professor S. U. Chung, Professor J. T. Donohue, and Professor A. Rougé for enlightening discussions.

*Present address: CERN, Geneva, Switzerland.

†Equipe de Recherche associée au C.N.R.S. Postal address: Laboratoire de Physique Théorique Université de Bordeaux I, Chemin du Solarium, 33170 Gradignan, France.

[1]T. D. Lee and C. N. Yang, Phys. Rev. 109, 1755 (1958); L. Durand III, L. F. Landovitz, and J. Leitner, *ibid.* 112, 273 (1958); M. Ademollo, R. Gatto, and G. Preparata, *ibid.* 140, B192 (1965); P. Minnaert, *ibid.* 151, 1306 (1966); M. G. Doncel, L. Michel, and P. Minnaert, Nucl. Phys. B38, 477 (1972); M. Daumens, G. Massas, and P. Minnaert, *ibid.* B77, 360 (1974).

[2]R. K. Adair, Phys. Rev. 100, 1540 (1955); Ph. Eberhard and M. L. Good, *ibid.* 120, 1442 (1960); M. Peshkin, *ibid.* 123, 637 (1961); M. Ademollo and R. Gatto, *ibid.* 133, B531 (1964); Ademollo, Gatto, and Preparata, Ref. 1; Minnaert, Ref. 1.

[3]Positivity conditions are rather poor for spin tests since they only give inequalities which can be satisfied for several spin assignments. On the contrary, rank conditions (when they exist) give equalities, so they are strong conditions which should distinguish the correct spin without ambiguities.

[4]This remark was first made in connection with fermion decays by R. Gatto and H. P. Stapp [Phys. Rev. 121, 1553 (1961)] and by R. H. Capps [*ibid.* 122, 929 (1961)]. See also M. Ademollo and R. Gatto, Nuovo Cimento 30, 429 (1963).

[5]For another type of spin test which does not require a reconstruction of the initial density matrix see E. de Rafael, Phys. Lett. 11, 260 (1964).

[6]S. M. Berman and M. Jacob, SLAC Report No. 43, 1965 (unpublished); S. M. Berman, in *High-Energy Physics and Elementary Particles,* edited by N. Byers, S. W. MacDowell, and C. N. Yang (International Atomic Energy Agency, Vienna, Austria, 1965).

[7]See also similar spin tests for fermions in S. M. Berman and M. Jacob, Phys. Rev. 139, B1023 (1965); Janice Button-Shafer, *ibid.* 139, B607 (1965).

[8]S. U. Chung, Phys. Rev. 138, B1541 (1965).

[9]S. U. Chung, CERN Yellow Report No. 71-8, 1971 (unpublished).

[10]M. Ademollo, R. Gatto, and G. Preparata, Phys. Rev. Lett. 12, 462 (1964); Phys. Rev. 139, 1608 (1965).

[11]The multipole parameters $t^{L_1}_{M_1}$ are linear combinations of density matrix elements,

$$t^{L_1}_{M_1}=\sum_{m_1,n_1}\langle j_1 n_1 L_1 M_1|j_1 m_1\rangle\,\rho^{n_1}{}_{m_1}\,.$$

[12]They differ from the corresponding coefficients used by Daumens *et al.*, Ref. 1, by a factor $(4\pi)^{1/2}$, so that for the boson decay $1^-\to 0^-0^-$ of Sec. V the nonvanishing coefficients are $C(0)=1$ and $C(2)=-\sqrt{2}$.

[13]These moments are related to the moments $H(L_2M_2L_1M_1)$ of Chung, Ref. 9, and to the moments $A(LL_2, L_1M_1)$ of Ademollo, Gatto, and Preparata, Ref. 10, by

$$\mathcal{K}(L_1M_1L_2M_2) = \hat{L}_1\hat{L}_2 H(L_2M_2L_1M_1)^*,$$

$$\mathcal{K}(L_1M_1L_2L) = [C(L_2)/\hat{L}_2]A(LL_2, L_1M_1).$$

[14]N. Byers and S. Fenster, Phys. Rev. Lett. 11, 52 (1963).

[15]M. Ademollo and R. Gatto, Phys. Rev. 133, B531 (1964); see in particular their Eq. (3.24).

[16]The relation between the canonical and helicity moments was given by Chung in footnote 21 of Ref. 8.

[17]More examples and a systematic derivation of spin tests will be given in a forthcoming publication.

[18]P. Antich *et al.*, Nucl. Phys. B20, 201 (1970); N. Armenise *et al.*, Nuovo Cimento Lett. 8, 425 (1973); S. U. Chung *et al.*, Phys. Lett. 47B, 526 (1973).

[19]W. Koch, in *Analysis of Scattering and Decay*, edited by M. Nikolic (Gordon and Breach, New York, 1968). See in particular Eq. IV. 153, p. 318.

[20]G. T. Jones, Nucl. Phys. B52, 383 (1973), see Eq. (2.2); R. Barloutaud *et al.*, Nucl. Phys. B59, 374 (1973), see Sec. 4.3.1.

[21]A. Rougé, private communication.