PHYSICAL REVIEW D VOLUME 12, NUMBER 9 1 NOVEMBER 1975

SU(4) σ model

J. Schechter*
Physics Department, Syracuse University, Syracuse, New York 13210

M. Singer†
Physics Department, University of Wisconsin, Madison, Wisconsin 53706
(Received 9 July 1975)

The mass spectrum of pseudoscalar and scalar mesons is discussed in a general linear SU(4) σ model. We study the way in which the spectrum is influenced by the symmetry of the vacuum and by terms which are the analogs of quark mass terms and quark interaction terms. It is found that a paracharmonium-like state can be naturally accommodated if the charmed "quark mass" is much larger than the others. Other possibilities are also investigated.

I. INTRODUCTION

The recent discovery[1] of narrow 1^- resonances in the 3- to 4-GeV region increases the plausibility of the suggestion[2] that four rather than three quarks underlie the structure of hadrons. Already many authors[3] have discussed mass formulas and other properties on the basis of SU(4), the most natural symmetry group.

In the old case of three quarks it was found that the symmetry structure of the presumably basic quark Lagrangian[4] was most readily reflected in the mass spectrum of the pseudoscalar mesons. By using current-algebra techniques it was learned[5] that the spin-zero mass spectrum could be explained if only quark mass terms were responsible for the breaking of chiral SU(3)$\times$SU(3) and if the vacuum were almost SU(3) invariant. The large K to π mass ratio implied that the third (strange) quark was very much more massive than the other two. An alternate way to get the above results involved the construction of phenomenological Lagrangians containing only the spin-zero fields of interest and having the appropriate SU(3)$\times$SU(3) transformation properties. These SU(3) σ models[6-9] had the advantage that they very clearly displayed the structure of the theory and simplified several calculations of complicated physical processes.[10] As a bonus it was found[7,8] that the mass and mixing of the ninth pseudoscalar meson—the η' (960)—could be calculated in agreement with experiment. In the present paper we wish to extend the discussion[8] of a generalized, linear SU(3) σ model to the case[11] of SU(4). Our motivation is not primarily to fit the new data with a phenomenological Lagrangian having the minimum number of parameters; indeed, the existence of high-mass spin-zero mesons has not yet been conclusively established. Rather we wish to find out what these (we hope) soon-to-be-discovered spin-zero particles can tell us about the nature of symmetry breaking. For example, is the straightforward generalization to the case where the symmetry breaker is like an exceedingly heavy fourth (charmed) quark mass term and where the vacuum is almost SU(4) invariant the correct one? Or must we add large amounts of new symmetry-breaking terms and/or consider a more peculiar vacuum? It is also possible that the present techniques are not the appropriate ones for the treatment of the new particles. This too should be reflected in the mass spectrum and would be interesting to discover.

A concise description of the present model and its connection with the quark model is given in Sec. II. Section III presents the formulas and curves which are predicted for the masses while Sec. IV contains the application of these formulas to the cases which seem most interesting. Some approximate formulas which help in understanding the most favored case are also given in Sec. IV.

II. GENERALIZED LINEAR SU(4) σ MODEL

The SU(3) version of this model has been described in detail elsewhere,[7,8] from the present point of view. Since that description used a tensor notation for the chiral SU(3) objects, most of the old equations can be carried over directly with the understanding that summations go from 1 to 4 rather than from 1 to 3. Hence we shall be brief here, though we will try to make the discussion reasonably self-contained.

The Lagrangian will be constructed out of a 16-plet of pseudoscalar fields, ϕ_a^b $(a,b=1,2,3,4)$, and a 16-plet of scalar fields, S_a^b. These transform, respectively, like $(4,4^*)\mp(4^*,4)$ under chiral SU(4). We have the standard identifications $\pi^+=\varphi_1^2$, $K^+=\phi_1^3$, $K^0=\varphi_2^3$, etc. The three isoscalar members of the pseudoscalar multiplet are denoted η, η', and η''. Other particles will be referred to by their tensor symbols. Using a matrix notation such that $\phi_a^b \to \phi_{ab}$ we write the Lagrangian density as

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}(\partial_\mu \varphi \partial_\mu \varphi) - \tfrac{1}{2}\mathrm{Tr}(\partial_\mu S \partial_\mu S) - V_0 - V_{SB}. \quad (2.1)$$

In (2.1) V_0 may be taken to be the most general function without derivatives of the following chiral SU(4)×SU(4) invariants:

$$\begin{aligned}
I_1 &= \mathrm{Tr}[(S+i\phi)(S-i\phi)],\\
I_2 &= \mathrm{Tr}\{[(S+i\phi)(S-i\phi)]^2\},\\
I_3 &= \mathrm{Tr}\{[(S+i\phi)(S-i\phi)]^3\},\\
I_4 &= \mathrm{Tr}\{[(S+i\phi)(S-i\phi)]^4\},\\
I_5 &= \det(S+i\phi)+\det(S-i\phi). \quad (2.2)
\end{aligned}$$

Furthermore, V_{SB} is a symmetry breaker of simple form. If $\mathcal{L}$ is to be renormalizable, V_0 should be restricted to

$$aI_1 + b(I_1)^2 + cI_2 + dI_5, \quad (2.3)$$

where a, b, c, and d are some constants. However, our formulas will hold for the general case too. If we prefer V_0 to have the larger invariance group U(4)×U(4) we should delete its dependence on I_5, which is not invariant under "axial quark-number" transformations.

The symmetry breaker will be constructed using the quark model as a guide. In addition to quark-mass-type terms which transform as $[(4,4^*)+(4^*,4)]$ we will include, for the sake of generality, some terms transforming like $[(1,15)+(15,1)]$. The latter can be thought of as corresponding to symmetry-breaking effects resulting from the unequal effective couplings of different quarks to vector gluons in terms with the generic form $\bar{q}\gamma_\mu q G_\mu$ (q is a quark field and G_μ is a gluon field). The simplest terms with these transformation properties are

$$V_{SB} = -2\sum_a A_a S_a^a + \sum_{a,b} B_a (S_b^a S_a^b + \phi_b^a \phi_a^b). \quad (2.4)$$

Here the A_a are analogous to the quark masses and the B_a are presumably analogous to some kind of effective coupling constants.

We will work in the tree (classical) approximation. Symmetry breaking in the vacuum will be measured by the quantities

$$\alpha_a = \langle S_a^a \rangle_0, \quad (2.5)$$

where the symbol $\langle\ \rangle_0$ means that the enclosed object should be evaluated at the classical equilibrium point. This equilibrium point is formally the solution of the matrix equations

$$\left\langle \frac{\partial V_0}{\partial S}\right\rangle_0 + \left\langle \frac{\partial V_{SB}}{\partial S}\right\rangle_0 = 0. \quad (2.6)$$

The α_a are analogous to the vacuum expectation values $\langle 0|\bar{q}_a q_a|0\rangle$ in the quark model. We will also need equations which express the chiral invariance of V_0. These are seen to be in matrix notation[8]

$$\left[\phi, \frac{\partial V_0}{\partial \phi}\right] + \left[S, \frac{\partial V_0}{\partial S}\right] = 0, \quad (2.7)$$

$$\left[\frac{\partial V_0}{\partial S}, \phi\right]_+ - \left[\frac{\partial V_0}{\partial \phi}, S\right]_+ = -2i\frac{\partial V_0}{\partial I_5}[\det(S+i\phi) - \det(S-i\phi)]\cdot 1. \quad (2.8)$$

By differentiating (2.7) and (2.8) with respect to the fields and evaluating the results at the equilibrium point, we get Ward-type identities[8] among the masses and coupling constants of the model. Here we are only interested in the pseudoscalar and scalar masses, whose squares are given by the appropriate components of $\langle \partial^2 V/\partial\phi^2\rangle_0$ and $\langle \partial^2 V/\partial S^2\rangle_0$ with

$$V = V_0 + V_{SB}. \quad (2.9)$$

From (2.5)–(2.9) we get the following mass formulas[8]:

$$\left\langle \frac{\partial^2 V}{\partial S_b^a \partial S_f^e}\right\rangle_0 = \frac{1}{\alpha_a - \alpha_b}\delta_a^f \delta_e^b \left(\left\langle \frac{\partial V_{SB}}{\partial S_b^b}\right\rangle_0 - \left\langle \frac{\partial V_{SB}}{\partial S_a^a}\right\rangle_0\right) + \left\langle \frac{\partial^2 V_{SB}}{\partial S_b^a \partial S_f^e}\right\rangle_0, \quad (2.10)$$

$$\left\langle \frac{\partial^2 V}{\partial \phi_b^a \partial \phi_f^e}\right\rangle_0 = -\frac{1}{\alpha_a + \alpha_b}\delta_a^f \delta_e^b \left(\left\langle \frac{\partial V_{SB}}{\partial S_a^a}\right\rangle_0 + \left\langle \frac{\partial V_{SB}}{\partial S_b^b}\right\rangle_0\right) - \frac{4U}{\alpha_a + \alpha_b}\delta_a^b \delta_e^f \frac{\alpha_1\alpha_2\alpha_3\alpha_4}{\alpha_e} + \left\langle \frac{\partial^2 V_{SB}}{\partial \varphi_b^a \partial \varphi_f^e}\right\rangle_0. \quad (2.11)$$

In (2.11) we introduced the abbreviation

$$U = \left\langle \frac{\partial V_0}{\partial I_5}\right\rangle_0. \quad (2.12)$$

Now the needed tools are at hand. Our problem may be restated as follows: What does the spin-zero mass spectrum tell us about (i) the four "quark masses" A_a, (ii) the four "effective coupling constants" B_a, (iii) the vacuum symmetry parameters α_a [note $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ means the vacuum is SU(4) invariant], and (iv) the quantity U which breaks U(4)×U(4) down to SU(4)×SU(4)? Actually, we will restrict ourselves further to the case of isospin invariance (neglecting "electromagnetic" effects). Thus we set

$$\alpha_1 = \alpha_2 \equiv \alpha, \quad A_1 = A_2, \quad B_1 = B_2. \tag{2.13}$$

[The isospin-noninvariant case for SU(3) is discussed in Ref. 9.] It is also convenient to define

$$w \equiv \frac{\alpha_3}{\alpha}, \quad w' \equiv \frac{\alpha_4}{\alpha}. \tag{2.14}$$

Note that the leptonic decay constants of the pseudoscalars are given[7,8] in terms of the α_a by

$$\begin{aligned} F_\pi &= 2\alpha, \\ F_K &= \alpha(1+w), \\ F(\phi_1^4) &= \alpha(1+w'), \\ F(\varphi_3^4) &= \alpha(w+w'). \end{aligned} \tag{2.15}$$

III. SPIN-ZERO MASS SPECTRUM

Equations (2.10) and (2.11) give the squared masses of a certain number of spin-zero particles (the "Goldstone bosons") in terms of the U(4)×U(4) symmetry-breaking parameters. The remaining particles, those in the "direction" of the conserved group generators, do not have their masses specified. Of course, if we were to adopt a specific V_0, like the remormalizable choice of (2.3), we could calculate all the masses. At present, however, we shall consider the most general V_0; the resulting model can be considered a phenomenological description of the quark mode. Substituting (2.4) into (2.10) and (2.11) gives the squared masses of particles which carry either charge, strangeness, or charm,

$$m^2(\varphi_a^b) = \left\langle \frac{\partial^2 V}{\partial\phi_a^b \partial\phi_b^a} \right\rangle_0 = \frac{1}{\alpha_a + \alpha_b}\left[2(A_a + A_b) - (\alpha_a - \alpha_b)(B_a - B_b)\right] \quad (a \neq b), \tag{3.1}$$

$$m^2(S_a^b) = \left\langle \frac{\partial^2 V}{\partial S_a^b \partial S_b^a} \right\rangle_0 = \frac{1}{\alpha_a - \alpha_b}\left[2(A_a - A_b) - (\alpha_a + \alpha_b)(B_a - B_b)\right] \quad (a \neq b), \tag{3.2}$$

and the following mass-squared matrix (in a nondiagonal basis) of the four neutral pseudoscalars ($\pi^0, \eta, \eta', \eta''$):

$$M_{ab} \equiv \left\langle \frac{\partial^2 V}{\partial\phi_a^a \partial\phi_b^b} \right\rangle_0 = 2\begin{bmatrix} \frac{A_1}{\alpha_1} - U\frac{\alpha_2\alpha_3\alpha_4}{\alpha_1} & -U\alpha_3\alpha_4 & -U\alpha_2\alpha_4 & -U\alpha_2\alpha_3 \\ -U\alpha_3\alpha_4 & \frac{A_2}{\alpha_2} - U\frac{\alpha_1\alpha_3\alpha_4}{\alpha_2} & -U\alpha_1\alpha_4 & -U\alpha_1\alpha_3 \\ -U\alpha_2\alpha_4 & -U\alpha_1\alpha_4 & \frac{A_3}{\alpha_3} - U\frac{\alpha_1\alpha_2\alpha_4}{\alpha_3} & -U\alpha_1\alpha_2 \\ -U\alpha_2\alpha_3 & -U\alpha_1\alpha_3 & -U\alpha_1\alpha_2 & \frac{A_4}{\alpha_4} - U\frac{\alpha_1\alpha_2\alpha_3}{\alpha_4} \end{bmatrix}. \tag{3.3}$$

In the isospin limit (3.1), (3.2), and (3.3) give us the masses of all the pseudoscalars and all but six of the scalars. These six correspond to the four generators of charge, strangeness, charm, and "quark number" and to the isospin operators $I_\pm$.

Note that the [(1, 15) + (15, 1)] symmetry breaker makes no contribution to M_{ab}. The pion mass squared is seen from (3.1) to be

$$\pi^2 = \frac{2A_1}{\alpha} = \frac{4A_1}{F_\pi}, \tag{3.4}$$

where for compactness we represent the particle mass by its own symbol.

An interesting situation is the case where V_0 is U(4)×U(4) invariant so that $U = 0$. Then we see that (3.3) takes the form

$$\begin{bmatrix} \pi^2 & & & \\ & \pi^2 & & \\ & & 2A_3/\alpha_3 & \\ & & & 2A_4/\alpha_4 \end{bmatrix}$$

so that one of the isoscalars is degenerate with the π. Hence we must have $U \neq 0$ to construct a realistic theory. This creates a problem (the "U(1) problem"[12]) in a model where strong interactions are mediated by color gauge gluons and where symmetry breaking is induced by a unified weak-electromagnetic gauge scheme. Such models predict that the symmetric part of the fundamental quark Lagrangian should have the full U(4)×U(4) invariance. Thus from our present standpoint we should either require the above picture to be modified or regard our nonzero U to be a phenomenological description of other effects which arise in binding the quark to form mesons.

To proceed with determination of parameters in the model, we realize that only six experimental quantities are available:

$$(\pi^0)^2 \equiv 1,\quad (K^0)^2 = 13.60,\quad \eta^2 = 16.54,$$
$$\eta'^2 = 50.35,\quad F_\pi \simeq 1.01\pi^0,\quad F_K/F_\pi \simeq 1.28. \qquad (3.5)$$

On the other hand, we must determine three A_a's, three B_a's, three α_a's, and the quantity U. Thus, to keep things within bounds we shall first suppose that the B_a's are all zero. Furthermore, even though we may immediately find $w = 1.56$ from (2.15) and (3.5) we shall consider it a free parameter which may vary in the range 1 to 2. This is because results depend crucially on this quantity and because the equation for F_K in (2.15) is subject to modification when other (nonzero-spin) particles are introduced into the theory.[7] Thus we end up with seven parameters and five known quantities. We choose to consider w and w' as free parameters so that once these are specified, everything else in the model can be calculated. Now w and w' are not completely arbitrary but

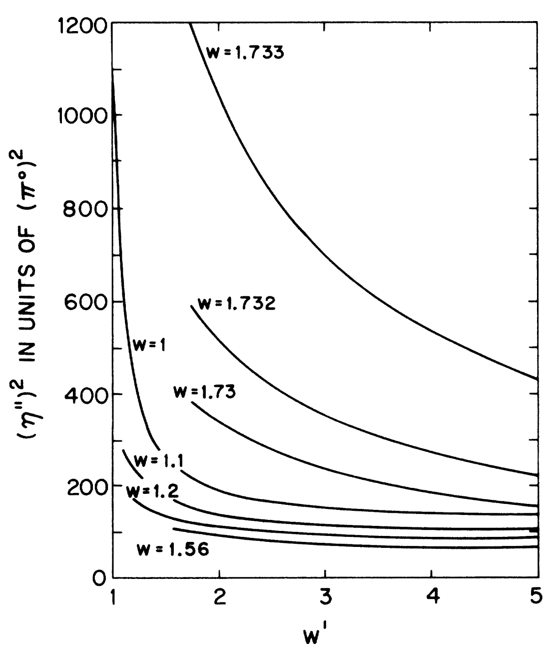

FIG. 1. η''^2 in $(\pi^0)^2$ units plotted against w and w'.

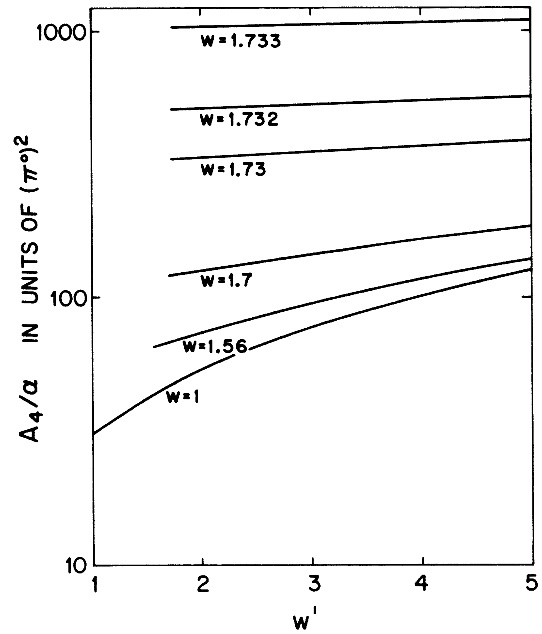

FIG. 2. A_4/α in $(\pi^0)^2$ units plotted against w and w'.

must lead to positive values for all squared masses. Adopting the requirement that the "quark masses" be ordered according to $A_4 > A_3 > A_1 > 0$ we must have

$$w' > w > 1, \qquad (3.6)$$

since, from (3.2), the following squared masses must be positive:

$$m^2(S_1^3) = \frac{2(A_3 - A_1)}{\alpha(w-1)},$$
$$m^2(S_3^4) = \frac{2(A_4 - A_3)}{\alpha(w'-w)}, \qquad (3.7)$$
$$m^2(S_1^4) = \frac{2(A_4 - A_1)}{\alpha(w'-1)}.$$

The pseudoscalars from (3.1) satisfy positivity:

$$K^2 = \frac{2(A_3 + A_1)}{\alpha(w+1)},$$
$$m^2(\phi_3^4) = \frac{2(A_4 + A_3)}{\alpha(w'+w)}, \qquad (3.8)$$
$$m^2(\phi_1^4) = \frac{2(A_4 + A_1)}{\alpha(w'+1)}.$$

The objects in (3.7) and (3.8) have a very simple dependence on the parameters. It is much more complicated to discuss η, η', and η'' since we must diagonalize the matrix M_{ab} of (3.3). This is done in the Appendix. The interesting results are displayed in Figs. 1 and 2 which show η''^2 and A_4/α each given as a function of both w and w'. For fixed w', η''^2 is large around $w = 1$, decreases to a minimum at $w \simeq 1.55$, and then rises sharply near $w = 1.73$. A_4/α, on the other hand, remains low until the region around $w = 1.73$ is reached. Finally, for $w \geq 1.74$, η''^2 is unphysical (complex).

IV. ANALYSIS OF MASS SPECTRUM

In this section we shall consider first the "expected" case in which the η'' is identified as paracharmonium and then consider other possibilities.

A. Very heavy charmed quark

From Fig. 2 we notice that the charmed "quark mass" A_4 will be very large around $w = 1.73$. We can show directly that when A_4 is very large not only is η'' close to φ_4^4 (paracharmonium) but that w must be around its value in the SU(3) σ model and the η - η' mixing must be close to zero. To see this let us chop up M of (3.3) into a 3×3 submatrix $\tilde{M}$ and the remainder as follows:

$$M \equiv \begin{bmatrix} \tilde{M} & C \\ C^T & N \end{bmatrix}. \qquad (4.1)$$

We assume that the (44) matrix element N, which is the only one involving A_4, is much larger in magnitude than all others. Then to lowest order M is partially diagonalized by the matrix a as follows:

$$aM(a^{-1}) \simeq \begin{bmatrix} & & & 0 \\ & \tilde{M} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & N \end{bmatrix}, \qquad (4.2)$$

where

$$a = \begin{bmatrix} 1 & \dfrac{-C}{N} \\ C^T/N & 1 \end{bmatrix}. \qquad (4.3)$$

From (4.2) we identify

$$\eta''^2 \simeq N \simeq \frac{2A_4}{\alpha w'}. \qquad (4.4)$$

Since the elements of C/N are, by assumption, small, we see from (4.3) that η'' is essentially φ_4^4. Furthermore, we note that the $\pi^0\eta\eta'$ submatrix is given by $\tilde{M}$. Now Eq. (3.10) of Ref. 8, which is the $\pi^0\eta\eta'$ matrix of the SU(3) σ model, coincides exactly with $\tilde{M}$ when we replace $6V_4$ in that matrix by $\alpha_4 U$. Thus all the previous results of Refs. 7 and 8 hold; namely, if η' is to have its correct value w must be around 1.73 (not too bad experimentally) and the η-η' mixing angle must be very small. Furthermore,

$$\alpha w' U \simeq 6(-1.85)\pi^0, \qquad (4.5)$$

the number -1.85 being V_4 in the SU(3) σ model. In this way we understand why the numerical analysis gave large η''^2 around $w = 1.73$. Thus the present model can accomodate the charmonium picture in a very natural way. No $[(1,15)+(15,1)]$ symmetry breaker is apparently needed and the charmed particle masses are given in (3.7) and (3.8). Note that if η''^2 is specified the approximate formulas (4.4) and (4.5) are not sufficient to specify w' for a known w. In principle the numerical analysis can do this but we see from Fig. 1 that η''^2 is very sensitive to w around $w = 1.73$ so that experimental uncertainties in the input would prevent us from finding w to sufficient accuracy. However, if charmed pseudoscalars (say) are found, we can find w' from (3.8). At any rate $w' \gtrsim w$, corresponding to a vacuum which still is relatively close to being SU(4) invariant, is not ruled out for large η'' by this analysis. An interesting feature is that we have a natural mechanism giving η' as almost a pure SU(3) singlet while η'' is, like paracharmonium, very far from being an SU(4) singlet.

It is of some interest to give formulas for the $\eta\eta'\eta''$ mixing angles in this case. By the above discussion, all these angles are small. Hence we may write

$$\begin{bmatrix} \pi^0 \\ \eta \\ \eta' \\ \eta'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -x & -y \\ 0 & x & 1 & -z \\ 0 & y & z & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_1^1 \\ \varphi_2^2 \\ \varphi_3^3 \\ \varphi_4^4 \end{bmatrix}, \qquad (4.6)$$

where x, y, and z are, respectively, the $\eta-\eta'$, $\eta-\eta''$, and $\eta'-\eta''$ mixing angles. Using (4.6) to diagonalize M gives

$$x=\frac{1}{\sqrt{2}}\frac{1}{\eta'^2-\eta^2}\frac{1}{(w+2)}[3\pi^2-(2w+1)\eta^2+2(w-1)\eta'^2],$$

$$y=\frac{2}{\sqrt{6}}\frac{1}{\eta''^2-\eta^2}\frac{(1-w)}{w'(w+2)}(\pi^2/2+\eta^2/2-\eta'^2),$$

$$z=\frac{1}{\sqrt{3}}\frac{1}{\eta''^2-\eta'^2}\frac{(2w+1)}{w'(w+2)}(-\pi^2/2-\eta^2/2+\eta'^2). \tag{4.7}$$

Next, we give a sample choice of "reasonable" parameters. Taking $\eta''^2=500(\pi^0)^2$ ($\eta''\simeq 3.0$ GeV) and, for definiteness,[13] $w=w'=1.7314$ we find that the "input" quantities, π, K, η, η', and F_π are fitted when the system is described by the following parameters (see Appendix):

$$\alpha=0.505\pi,\quad \alpha_3=w\alpha,\quad \alpha_4=w'\alpha,$$

$$A_1=0.253\pi^3,\quad A_3=9.11\pi^3,\quad A_4=214\pi^3,$$

$$U=-12.8.$$

One then has the predictions for the charmed spin-zero objects

$$m(\varphi_1^4)=2.4\text{ GeV},\quad m(\varphi_3^4)=2.2\text{ GeV},$$

$$m(S_1^4)=4.6\text{ GeV},\quad m(S_3^4)\to\infty,$$

and the mixing angles of (4.7)

$$x=0.95°,\quad y=0.43°,\quad z=2.18°.$$

The mass of S_3^4 is infinite [see (3.7)] solely because of our choice $w=w'$ so that result should not be taken seriously. It is amusing to see that even though the symmetry of the vacuum is not too far from SU(4) the "quark masses" stand in the ratio

$$A_1:A_3:A_4=1:36:845.$$

Explicit values of the system parameters for various choices of $w=w'$ are listed in Table I.

As previously mentioned, the above fits require $w=1.73$ or by (2.15) $F_K/F_\pi=1.36$ which differs slightly from the experimental value of 1.28. Actually, by including some $(1,15)+(15,1)$ symmetry breaker in addition we may get $F_K/F_\pi=1.28$ while still retaining the paracharmonium picture (small x, y, and z). In the last part of the Appendix it is shown that the addition of terms with $B_1-B_3\neq 0$ will shift the value of w from 1.73. The choice $B_3-B_1\simeq 1.23(\pi^0)^2$ is the suitable one. Equations (4.7) for the (small) mixing angles still hold.

B. Other possibilities

Suppose we consider the case when w and w' are both very close to 1. This corresponds to an almost perfectly SU(4)-invariant vacuum. Then η''^2 will, from Fig. 1, still be large. However, Fig. 2 shows that A_4 will be small so that the masses of the charmed mesons will be of the same order of magnitude as the ordinary ones [see (3.8)]. To avoid this we may allow B_4 to be large (keeping $B_1=B_2=B_3$). Then, according to (3.2) and (3.3) our result for η'' will not be affected but we may make the charmed pseudoscalars as massive as we like. However, (see Table I) η'' will, in this case, be very far from φ_4^4.

As another possibility, suppose that we wish to have the "canonical" SU(4) structure for the η' and η''; i.e., $\eta\simeq(1/2\sqrt{3})(\varphi_1^1+\varphi_2^2+\phi_3^3-3\phi_4^4)$ and $\eta''\simeq\frac{1}{2}(\phi_1^1+\varphi_2^2+\phi_3^3+\varphi_4^4)$. With $B_1=B_2=B_3$, numerical calculations show that $1.4<w<1.733$, $w'<5$, and $\eta''\simeq 1$ GeV. Results are given in Table II and in the Appendix. If we require $w'\simeq w\simeq 1.56$ *and* η'' large, then by setting $B_3-B_1\simeq -4.2(\pi^0)^2$, we get that $\eta''\simeq 3.2$ GeV. Again the charmed mesons are given large masses by a large term B_4.

Finally, suppose that $B_1=B_2=B_3$ and $w\simeq w'\simeq 1.56$.

TABLE I. System parameters for various values of $w=w'$.

$w=w'$	$(\eta'')^2$ (units of $(\pi^0)^2$)	x (degrees)	y (degrees)	z (degrees)	$m^2(\phi_1^4)$ (units of $(\pi^0)^2$)	$m^2(\varphi_3^4)$ (units of $(\pi^0)^2$)	A_4/α (units of $(\pi^0)^2$)	$U\alpha^2$
1.0	1099.0	−20.0	16.5	57.2	32.1	44.7	31.6	−134.4
1.2	172.8	−12.7	10.7	51.1	36.5	45.1	39.6	−15.2
1.4	114.2	−6.7	6.5	40.6	42.4	47.3	50.4	−7.2
1.56	105.9	−2.7	4.0	27.7	51.5	52.8	65.4	−4.8
1.6	108.6	−1.8	3.4	23.6	55.9	55.9	72.1	−4.4
1.7	153.5	0.29	1.7	10.3	90.4	82.0	121.5	−3.6
1.71	173.6	0.49	1.4	8.4	103.8	92.4	140.1	−3.5
1.72	214.2	0.69	1.1	6.1	130.3	113.2	176.7	−3.4
1.730	385.3	0.90	0.57	2.9	239.8	199.4	326.8	−3.3
1.731	453.3	0.93	0.47	2.4	283.0	233.4	386.0	−3.3
1.732	589.6	0.97	0.36	1.8	369.7	301.7	504.4	−3.3
1.733	1191.2	1.2	0.17	0.84	751.4	602.7	1026.3	−3.2

Then we predict that $\eta'' \simeq 1.4$ GeV which invites speculation that the η'' is the E(1420). The charmed mesons can be given large masses by taking B_4 to be large, as above.

Note added in proof. A candidate for η'' has recently been found at $m(\eta'')=2.80$ GeV (reported by H. Harari, Stanford Conference, 1975). Adopting this value, we may compute all the quantities discussed in Sec. IV as functions of w only. Three typical sets of values are given in Table III. Note that we predict $m(\phi_3^4)<m(\phi_1^4)$, as discussed in the Appendix.

APPENDIX

It is first necessary to find the eigenvalues $(\pi^2, \eta^2, \eta'^2, \eta''^2)$ of M_{ab} [Eq. (3.3)] which satisfy

$$\det(M-\lambda\cdot 1)=(\lambda-\pi^2)(\lambda-\eta^2)(\lambda-\eta'^2)(\lambda-\eta''^2). \tag{A1}$$

Since we have assumed isotopic spin invariance, we use (3.4) and divide out $(\lambda-\pi^2)$ from both sides of (A1). This leaves us with a cubic equation for λ. Equating the coefficients of the powers of λ on the right- and left-hand sides of this cubic equation gives

$$\begin{aligned}
&\eta^2+\eta'^2+\eta''^2=r_1+s_1\frac{A_4}{\alpha}+u_1U\alpha^2,\\
&\eta^2\eta'^2\eta''^2=s_2\frac{A_4}{\alpha}+u_2U\alpha^2+v_2A_4U\alpha,\\
&\eta^2\eta'^2+\eta^2\eta''^2+\eta'^2\eta''^2=r_3+s_3\frac{A_4}{\alpha}+u_3U\alpha^2+v_3A_4U\alpha,
\end{aligned} \tag{A2}$$

where

$$\begin{aligned}
&r_1=\frac{w'}{2}\quad s_3=2\left(\frac{A_1}{\alpha}+\frac{A_3}{\alpha w}\right),\quad s_1=\frac{2}{w'},\\
&v_3=-4\left(2w+\frac{1}{w}\right),\quad u_1=\frac{w'}{2}v_3-ws_1,\\
&s_2=-\frac{1}{w}\quad u_2=\frac{2}{w'}\quad r_3=\frac{8A_1A_3}{\alpha^2ww'},\\
&v_2=-8\left(\frac{A_1}{\alpha w}+2\frac{A_3}{\alpha}\right),\quad u_3=\tfrac{1}{2}w'v_2-2\frac{w}{w'}r_1.
\end{aligned} \tag{A3}$$

Since the equations (A.2) are linear in $(\eta'')^2$ but second order in U and A_4, we did not solve for $(\eta'')^2$ directly but rather we solved for A_4/α by eliminating U in two different ways to get the following two equations:

TABLE II. System parameters for "canonical" SU(4) structure: $z\simeq 60°$; x and $y<7°$.

w	w'	x (degrees)	y (degrees)	z (degrees)	$(\eta'')^2$ (units of $(\pi^0)^2$)	$m^2(\phi_1^4)$ (units of $(\pi^0)^2$)	$m^2(\phi_3^4)$ (units of $(\pi^0)^2$)
1.45	5.24	−5.3	4.0	60.0	65.5	44.9	46.6
1.50	6.76	−4.1	3.1	60.0	61.9	46.0	47.1
1.55	9.14	−2.9	2.3	60.0	58.3	47.0	47.7
1.60	13.4	−1.8	1.6	60.1	55.6	48.0	48.3
1.65	22.6	−0.7	0.9	60.0	53.4	49.0	49.0
1.70	60.1	0.3	0.3	60.0	51.5	50.0	50.0
1.73	63.0	1.3	0	60.0	50.5	50.3	50.3

$$\begin{aligned}
&\frac{A_4}{\alpha}=w'[a_1(\eta'')^2+b_1]+\frac{1}{w'}[c_1(\eta'')^2+d_1],\\
&\frac{A_4}{\alpha}=w'\left[\frac{a_2(\eta'')^2+b_2}{c_2(\eta'')^2+d_2}\right].
\end{aligned} \tag{A4}$$

The constants a_1 through d_2 are given by

$$\begin{aligned}
a_1&=\tfrac{1}{2}-(1+2w^2)c_2\Big/\left[8w\left(2\frac{A_1}{\alpha}-2\frac{A_3}{\alpha w}\right)^2\right],\\
b_1&=\tfrac{1}{4}\left(\frac{A_1}{\alpha w}+2\frac{A_3}{\alpha}\right)c_2\Big/\left(2\frac{A_1}{\alpha}-2\frac{A_3}{\alpha w}\right)^2,\\
c_1&=-\frac{w}{8}c_2\Big/\left(2\frac{A_1}{\alpha}-2\frac{A_3}{\alpha w}\right)^2,\\
d_1&=-\frac{w}{8}d_2\Big/\left(2\frac{A_1}{\alpha}-2\frac{A_3}{\alpha w}\right)^2,\\
a_2&=\tfrac{1}{2}d_2+\left(\frac{A_1}{\alpha}+\frac{A_3}{\alpha w}\right)c_2,\\
b_2&=-2\frac{A_1A_3}{\alpha^2 w}c_2,\\
c_2&=\frac{8}{w}\left[\left(\frac{A_1}{\alpha}\right)^2+2\left(\frac{A_3}{\alpha}\right)^2\right]+(\eta\eta')^2\left(4w+\frac{2}{w}\right)\\
&\quad-4(\eta^2+\eta'^2)\left(\frac{A_1}{\alpha w}+2\frac{A_3}{\alpha}\right),\\
d_2&=\frac{8}{w}(\eta^2+\eta'^2)\left[\left(\frac{A_1}{\alpha}\right)^2+2\left(\frac{A_3}{\alpha}\right)^2\right]\\
&\quad-4(\eta\eta')^2\left(\frac{A_1}{\alpha w}+2\frac{A_3}{\alpha}\right)-\frac{16}{w}\left[\left(\frac{A_1}{\alpha}\right)^3+\frac{2}{w}\left(\frac{A_3}{\alpha}\right)^3\right].
\end{aligned}$$

Note that a_1 through d_2 are independent of w'. We can then solve for $(\eta'')^2$ by equating the two expressions for A_4/α in (A4). The result is a simple quadratic equation for $(\eta'')^2$:

$$(\eta'')^4(a_1w'+c_1/w')c_2+(\eta'')^2[(a_1d_2+b_1c_2-a_2)w'+(c_1d_2+d_1c_2)/w']+(d_2b_1-b_2)w'+\frac{d_1d_2}{w'}=0, \tag{A6}$$

TABLE III (added in proof). System parameters for various values of w for $m(\eta'')=2.80$ GeV.

w	w'	$m(\phi_1^4)$ (GeV)	$m(\phi_3^4)$ (GeV)	$m(S_1^4)$ (GeV)	$m(S_3^4)$ (GeV)	x (degrees)	y (degrees)	z (degrees)
1.7307	1.7307	2.21	2.01	4.27	∞	0.92	0.51	2.57
1.732	2.422	2.34	2.16	3.63	5.12	0.95	0.36	1.81
1.733	5.045	2.55	2.43	3.12	3.42	0.97	0.17	0.86

where η'' is now a function of only A_1/α, A_3/α, w, and w'. To complete the picture it is necessary to calculate the η-η'-η'' mixing angles. For small mixing these are given in (4.6); for the general situation we replace the left-hand matrix in (4.6) by

$$Y(y)Z(z)X(x), \tag{A7}$$

where $Y(y)$, $Z(z)$, and $X(x)$ are, respectively, rotations through y, z, and x in the η-η'', η'-η'', and $\eta\eta'$ "planes." The matrix elements of (3.3) can now be expressed in terms of the masses and mixing angles as follows:

$$\begin{aligned}
M_{11}=&\frac{\pi^2}{2}+\frac{\eta^2}{6}(x_-^2\cos^2y+x_+^2\sin^2y\,\sin^2z)+\frac{\eta'^2}{6}x_+^2\cos^2z\\
&+\frac{\eta''^2}{6}(x_-^2\sin^2y+x_+^2\cos^2y\,\sin^2z)+\frac{x_+x_-}{6}(\eta''^2-\eta^2)\sin2y\,\sin z;\\
M_{33}=&\frac{\eta^2}{3}(x_+^2\cos^2y+x_-^2\sin^2y\,\sin^2z)+\frac{\eta'^2}{3}x_-^2\cos^2z\\
&+\frac{\eta''^2}{3}(x_+^2\sin^2y+x_-^2\cos^2y\,\sin^2z]-\frac{x_+x_-}{3}(\eta''^2-\eta^2)\sin2y\,\sin z;\\
M_{44}=&\eta^2\sin^2y\,\cos^2z+\eta'^2\sin^2z+\eta''^2\cos^2y\,\cos^2z;\\
M_{13}=&\frac{-\eta^2x_+x_-}{3\sqrt2}(\cos^2y-\sin^2y\,\sin^2z)+\frac{\eta'^2x_+x_-}{3\sqrt2}\cos^2z\\
&-\frac{\eta''^2x_+x_-}{3\sqrt2}(\sin^2y-\cos^2y\,\sin^2z)+\frac{1}{6\sqrt2}(\eta''^2-\eta^2)(x_-^2-x_+^2)\sin2y\,\sin z;\\
M_{14}=&\frac{\eta^2x_+}{2\sqrt6}\sin^2y\,\sin2z-\frac{\eta'^2x_+}{2\sqrt6}\sin2z+\frac{\eta''^2x_+}{2\sqrt6}\cos^2y\,\sin2z+\frac{x_-}{2\sqrt6}(\eta''^2-\eta^2)\sin2y\,\cos z;\\
M_{34}=&\frac{\eta^2x_-}{2\sqrt3}\sin^2y\,\sin2z-\frac{\eta'^2x_-}{2\sqrt3}\sin2z+\frac{\eta''^2x_-}{2\sqrt3}\cos^2y\,\sin2z-\frac{x_+}{2\sqrt3}(\eta''^2-\eta^2)\sin2y\,\cos z,
\end{aligned} \tag{A8}$$

where

$$\begin{aligned}
&x_+=\sqrt2\cos x+\sin x,\\
&x_-=\cos x-\sqrt2\sin x,\\
&x_+^2+x_-^2=3.
\end{aligned} \tag{A9}$$

We can now solve for x, y, z, and $U\alpha^2$ in terms of known masses and quantities previously calculated, the only unknowns being w, w', and (B_3-B_1). First, by looking at the trace of M, we see that

$$U\alpha^2=-\tfrac12\,\frac{\eta''^2+\eta'^2+\eta^2-\pi^2-2(A_3/\alpha w)-2(A_4/\alpha w')}{2ww'+w'/w+w/w'}. \tag{A10}$$

We then solve for the mixing angles x, y, and z, which are given by

$$\begin{aligned}
\cos^2y=&\frac{\Delta}{\Delta''}\,\frac{(4U^2\alpha^4)(2w^2+1)/\tilde\Delta+\Delta+\tilde\Delta}{(4U^2\alpha^4)(2w^2+1)/\tilde\Delta+2\Delta+\tilde\Delta-\Delta''},\\
\cos^2z=&\frac{\tilde\Delta}{\Delta''\cos^2y-\Delta},\\
x_+=&-\frac{\sqrt3}{4}\,\frac{1}{U\alpha^2}\,\frac{1}{1+2w^2}[-\Delta''\sin2y\,\cos z\\
&+\sqrt2w\sin2z(\Delta''\cos^2y-\Delta)],\\
x_-=&-\frac{\sqrt3}{4}\,\frac{1}{U\alpha^2}\,\frac{1}{1+2w^2}[\sqrt2w\,\Delta''\sin2y\,\cos z\\
&+\sin2z(\Delta''\cos^2y-\Delta)],
\end{aligned} \tag{A11}$$

where

$$\Delta = \eta'^2 - \eta^2,$$
$$\Delta'' = \eta''^2 - \eta^2, \qquad \text{(A12)}$$
$$\tilde{\Delta} = 2\,\frac{A_4}{\alpha w'} - 2U\alpha^2 \frac{w}{w'} - \eta'^2,$$

and from (A9)

$$\cos x = \tfrac{1}{3}(\sqrt{2}\,x_+ + x_-). \qquad \text{(A13)}$$

(i) Let us now consider the case where all B's are equal to zero. By (2.4), (2.11), (2.13), (2.14), and (3.1) we see that

$$2\,\frac{A_1}{\alpha} = (\pi^0)^2;\quad 2\,\frac{A_3}{\alpha} = (K^0)^2 - (\pi^0)^2 + w(K^0)^2. \qquad \text{(A14)}$$

The general results for this model can be seen in Figs. 1 and 2, and in Table I. For $1 < w < 1.7$, A_4/α remains small. An interesting case discussed in Sec. IV occurs when $w \simeq 1.73$. Here the mass of the η'' is quite large. We can understand this by examining the quantity c_2 in (A4) and (A5). When $c_2 = 0$, Eqs. (A6) become

$$\frac{A_4}{\alpha} = \tfrac{1}{2} w'(\eta'')^2 + \frac{d_1}{w'},\quad d_1 \neq 0$$

and (A15)

$$\frac{A_4}{\alpha} = \tfrac{1}{2} w'(\eta'')^2.$$

These two equations are consistent only if the mass of the η'' is infinite. By referring to Ref. 7, Eq. (45), and replacing A_1/α and A_3/α in c_2 by their values in (A14), we see that setting $c_2 = 0$ is equivalent to assuming this sum rule which requires that $w \simeq 1.733$. We also see that A_4/α is infinite in this case. Thus if w is near its old value, we expect the mass of the η'' and A_4/α to be large.

Mixing angles, charmed pseudoscalar masses, and the $U(4)$ breaking parameter U are listed in Table I for various values of $w = w'$. Note that all three mixing angles are small only around $w = 1.7$. It is amusing that in this region Table I gives $m^2(\phi_1^4) > m^2(\phi_3^4)$, which is the opposite of the additive quark model. The reason for this is seen by referring to (3.8) and taking A_4 large.

It is of some interest to ask if this model has solutions where η'' is close to a pure SU(4) singlet, and η close to a member of an SU(3) octet. This situation corresponds to the choice of mixing angles, $x = y = 0$ and $z = 60°$. It can be achieved for w in the range 1.45 to 1.73 and $w' > 5$. Some representative values are shown in Table II. However, all these solutions are characterized by masses of the η'' and the charmed pseudoscalars on the order of 1 GeV. Thus, they are unrealistic.

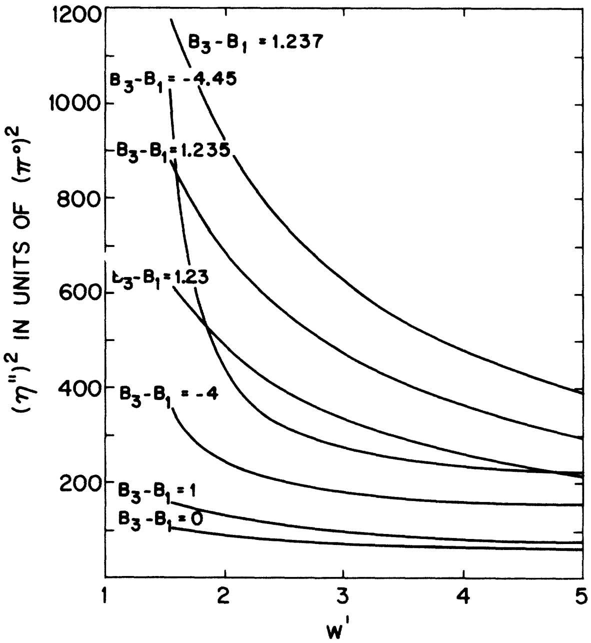

FIG. 3. η''^2 in $(\pi^0)^2$ units plotted against w' and $B_3 - B_1$ for $w = 1.56$.

(ii) Let us now look at the case where the B's are not equal to zero. Equations (A1) through (A13) are still valid, and the only change from the previous discussion results from the identification of A_3/α. By (2.4), (2.11), (2.13), (2.14), and (3.1)

$$2\,\frac{A_3}{\alpha} = (K^0)^2 - (\pi^0)^2 + w(K^0)^2 + (w-1)(B_3 - B_1). \qquad \text{(A16)}$$

This gives $(\eta'')^2$ as a function of w, w', and $(B_3 - B_1)$. We now let $w = 1.56$, its experimental value, and consider $(\eta'')^2$ to be a function of w' and $(B_3 - B_1)$. From numerical calculations we find the following:

(a) for $B_3 - B_1 > 1.24\ (\pi^0)^2$, $(\eta'')^2$ is complex;
(b) for $1.24 > B_3 - B_1 > 1.23$, $(\eta'')^2$ is large;
(c) for $1.23 > B_3 - B_1 > -5$, $(\eta'')^2$ decreases until $B_3 - B_1 = 0$, then starts increasing as $(B_3 - B_1)$ decreases;
(d) for $(B_3 - B_1) < -6$, $(\eta'')^2$ is negative.

In analogy with the previous discussion, these results can be understood by noting that when $(B_3 - B_1) \simeq 1.235$, $c_2 = 0$. Thus we see that this new mass spectrum is similar to the previous one where the value of $(B_3 - B_1)$ now determines the critical value of w (see Fig. 3). It is thus possible to have $w = 1.56 \simeq w'$ with large η''. In this case the numerical calculations show that we can have either paracharmonium for $(B_3 - B_1) \simeq 1.23(\pi^0)^2$ or an SU(4) singlet η'' for $(B_3 - B_1) \simeq -4.2(\pi^0)^2$.

It is interesting to note that this model places strict limits on $(B_3 - B_1)$. It seems that SU(3) cannot be badly broken in the quark-gluon term contribution to the mass matrix.

---

*Work supported in part by the U. S. Energy Research and Development Administration.

†Work supported in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation, and in part by the Energy Research and Development Administration under Contract No. E(11-1)-881, C00-464.

[1]J. J. Aubert *et al.*, Phys. Rev. Lett. 33, 1404 (1974); J.-E. Augustin *et al.*, *ibid.* 33, 1406 (1974).

[2]P. Tarjanne and V. L. Teplitz, Phys. Rev. Lett. 11, 441 (1963); Z. Maki and V. Ohnuki, Prog. Theor. Phys. 32, 144 (1964); Y. Hara, Phys. Rev. 134, B701 (1964); D. Amati, H. Bacry, J. Nuyts, and J. Prentki, Phys. Lett. 11, 190 (1964); B. J. Bjorken and S. L. Glashow, *ibid.* 11, 255 (1964); C. R. Hagen and A. J. Macfarlane, Phys. Rev. 135, 13432 (1964); I. S. Gerstein and M. L. Whippman, *ibid.* 136, 13823 (1964); P. Dittner and S. Eliezer, Phys. Rev. D 8, 1929 (1973); D. A. Dicus and V. S. Mathur, *ibid.* 9, 1003 (1974); S. C. Prasad, *ibid.* 9, 1017 (1974).

[3]M. K. Gaillard, B. W. Lee, and J. L. Rosner, Rev. Mod. Phys. 47, 277 (1975); S. Borchardt, V. S. Mathur, and S. Okubo, Phys. Rev. Lett. 34, 38 (1975); A. DeRujula and S. L. Glashow, *ibid.* 34, 46 (1975); T. Appelquist and H. D. Politzer, *ibid.* 34, 43 (1975); C. G. Callen, R. L. Kingsley, S. B. Treiman, F. Wilczek, and A. Zee, *ibid.* 34, 52 (1975); J. Kogut and L. Susskind, *ibid.* 34, 767 (1975); T. Das, P. P. Divakaran, L. K. Pandit and V. Singh, *ibid.* 34, 770 (1975); E. Eichten, K. Gottfried T. Kinoshita, J. Kogut K. D. Lane and T.-M. Yan, *ibid.* 34, 369 (1975); B. G. Kenny, P. C. Peaslee, and L. J. Tassie, *ibid.* 34, 429 (1975); D. H. Boal, R. H. Graham, J. W. Moffat, and P. J. O'Donnell, *ibid.* 34, 541 (1975); B. J. Harrington, S. Y. Park, and A. Yildiz, *ibid.* 34, 706 (1975); T. P. Cheng and P. B. James, *ibid.* 34, 917 (1975); E. Takusagi and S. Oneda, *ibid.* 34, 1129 (1975); J. Pasupathy and G. Rajasekaran, *ibid.* 34, 1250 (1975); J. S. Kang and H. Schnitzer, Phys. Rev. D 12, 841 (1975); B. Stech, (unpublished); K. Kajantie, C. Montoneu, M. Roos, and N. Tornquist; University of Helsinki Report No. 3-75, 1975 (unpublished).

[4]M. Gell-Mann, Phys. Rev. 125, 1067 (1962).

[5]M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968); S. L. Glashow and S. Weinberg, Phys. Rev. Lett. 20, 244 (1968); V. Mathur and S. Okubo, Phys. Rev. D 1, 2046 (1970); R. Dashen, Phys. Rev. 183, 1245 (1969); P. Auvil and N. Deshpande, *ibid.* 183, 1463 (1969); L. K. Pande, Phys. Rev. Lett. 25, 777 (1970); E. Reya, Rev. Mod. Phys. 46, 545 (1974); R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, Phys. Rev. Lett. 26, 104 (1971).

[6]J. Cronin, Phys. Rev. 161, 1483 (1967); M. Levy, Nuovo Cimento 52A, 23 (1967); S. Gasiorowicz and D. A. Geffen, Rev. Mod. Phys. 41, 531 (1969); W. A. Bardeen and B. W. Lee, Phys. Rev. 177, 2389 (1969); G. Cicogna, F. Strocchi, and R. V. Caffarelli, Phys. Rev. D 1, 1197 (1970), 6, 301 (1972); P. Carruthers and R. W. Haymaker, Phys. Rev. 4, 1808 (1971); R. Olshansky, *ibid.* 4 2440 (1971); W. F. Palmer, *ibid.* 4, 2542 (1972); I. S. Barker, Nuovo Cimento Lett. 3, 267 (1972); L. H. Chan and R. W. Haymaker, Phys. Rev. D 7, 402 (1973); 10, 4143 (1975); S. P. Rosen and A. McDonald, Phys. Rev. 4, 1833 (1971); P. H. Dondi and S. Eliezer, Nucl. Phys. B47, 390 (1972); A. Kapoor, Phys. Rev. D 11, 1841 (1975).

[7]J. Schechter and Y. Ueda, Phys. Rev. D 3, 168 (1971).

[8]J. Schechter and Y. Ueda, Phys. Rev. D 3, 2874 (1971).

[9]W. Hudnall and J. Schechter, Phys. Rev. D 9, 2111 (1974); J. Schechter and Y. Ueda, *ibid.* 4, 733 (1971).

[10]For example, the decay rate (see Ref. 9) of $\eta \to 3\pi$ could be calculated in reasonable agreement with the new experimental data as quoted in S. Weinberg (see Ref. 12). The result of Ref. 9 is $\Gamma(\eta \to \pi^+\pi^-\pi^0) = 255$ eV while the new experimental result is 204 ± 22 eV.

[11]A scale-invariant chiral SU(4)×SU(4) σ model was discussed by B. Hu, Phys. Rev. D 9, 1825 (1974).

[12]R. Mohapatra and J. Pati, Phys. Rev. D 8, 4212 (1973); P. Langacker and H. Pagels, *ibid.* 9, 3413 (1974); I. Bars and M. Halperin, *ibid.* 9, 3430 (1974); J. Kogut and L. Susskind, *ibid.* 10, 3468 (1974); S. Weinberg, *ibid.* 11, 3583 (1975); P. C. McNamee and M. D. Scadron, *ibid.* 10, 2280 (1974).

[13]Actually, w' cannot be exactly equal to w to avoid conflict between the equilibrium condition (2.6) and $A_3 \neq A_4$. This may be seen explicitly by examining the SU(4) generalization of Eq. (1.8) in the first paper of Ref. 9. Thus we should consider w' in this case to be very slightly different from w.