Hadron mass relations in a charmed quark model with broken SU(8) symmetry*

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It is assumed that charmed hadrons exist, and, together with the usual hadrons, can be classified in representations of SU(8). Hadron masses are assumed to arise from quark masses plus two-body interactions between quarks. Mass relations among the hadrons are obtained. It is estimated that charmed vector mesons have masses about 2 GeV and that charmed baryons have masses between 2.2 and 5 GeV.

I. INTRODUCTION

The recent discovery of very narrow mesons at SLAC and Brookhaven[1] has enhanced the possibility of charm[2,3] as a new quantum number in particle physics. In the present paper, we consider a model in which the hadrons are made up of quarks, with a charmed quark in addition to the usual three quarks, and derive expressions for the hadron masses. Our main aim is to obtain relations among the masses of hadrons, including hadrons with charm. It will be recalled that mass relations have proved to be very useful in the past. For example, the Gell-Mann–Okubo mass formula led to the prediction of the mass of the Ω^-; its subsequent discovery led to the acceptance of SU(3) as a broken symmetry of nature. In the same spirit, it is hoped that, with the knowledge of the masses of a few of the charmed hadrons, one can use the mass relations derived here to predict the masses of others.

We assume that the four quarks belong to the fundamental representation of SU(4). We also allow the quarks to have spin, so that we treat the whole problem within the framework of SU(8).[4] In the model, the mass of a hadron belonging to a given representation of SU(8) arises from a contribution which is common to all members of the same representation, plus two contributions which lead to deviations from this central value. The first contribution to the mass splitting arises from the differences in the masses of the individual constituent quarks; the second from differences between the various two-body quark-quark interactions. Calculation of hadron mass relations along these lines have previously been carried out by Lichtenberg[5] within the framework of SU(4), and by Federman, Rubinstein, and Talmi[6] for SU(6). The present work is a generalization of these calculations to include both charm and spin.

We take the usual baryon SU(3) octet and decimet as belonging to the symmetric 120-dimensional representation of SU(8), and the pseudoscalar and vector meson SU(3) nonets as belonging to the mixed $\underline{63} \oplus \underline{1}$ representations of SU(8). The baryon wave functions therefore are completely symmetric under the interchange of the SU(4) and spin indices of their constituent quarks (with oribital angular momentum zero). If one wishes to preserve Fermi statistics for the quarks, one may include color,[7] but otherwise color plays no role in our calculation. Our mass relations for the baryons and mesons are derived below in Secs. II and III, respectively; we also provide some estimates for charmed particle masses. Hadrons with nonzero charm have, of course, still to be found experimentally. However, should they be discovered, we expect our mass relations to provide a check as to whether broken SU(8) is a viable symmetry or not, and if so to give an indication as to where other charmed hadrons are to be found.

II. MASS RELATIONS FOR THE BARYONS

We first consider the baryons, including charmed baryons. The symmetric 120-dimensional representation of SU(8) has the following SU(4) and spin content:

$$\underline{120} \supset {}^4\underline{20}_S + {}^2\underline{20}_M \,, \tag{1}$$

where the left superscript denotes the spin multiplicity and the subscripts S and M denote the symmetry (S = symmetric, M = mixed) of the 20-dimensional representations of SU(4). The SU(3) content of the $\underline{20}_S$ and $\underline{20}_M$ is well known,[8] but for convenience we include it here

$$\underline{20}_S \supset \underline{10}_0 + \underline{6}_1 + \underline{3}_2 + \underline{1}_3 \,, \tag{2}$$

$$\underline{20}_M \supset \underline{8}_0 + \underline{6}_1 + \underline{\bar{3}}_1 + \underline{3}_2 \,, \tag{3}$$

where the subscripts denote the charm of the SU(3) multiplets.

For the individual baryons, we use a notation which has been introduced previously,[5] namely, we use a single symbol for all baryons with the same isospin and strangeness minus charm, and write the charm as a subscript. Thus, for example, Ξ means $I=\frac{1}{2}$, $S=-2$, while Ξ_1 means $I=\frac{1}{2}$,

$S=-1$, $C=1$. The particle content of the $\underline{120}$ SU(8) representation then is the following:

$$\underline{20}_S:\ \underline{10}_0(\Delta, \Sigma, \Xi, \Omega)$$
$$\underline{6}_1(\Sigma_1, \Xi_1, \Omega_1)$$
$$\underline{3}_2(\Xi_2, \Omega_2)$$
$$\underline{1}_3(\Omega_3)\,,$$

$$\underline{20}_M:\ \underline{8}_0(N, \Lambda, \Sigma, \Xi)$$
$$\underline{6}_1(\Sigma_1, \Xi_1, \Omega_1)$$
$$\underline{\bar{3}}_1(\Lambda_1, \Xi_1')$$
$$\underline{3}_2(\Xi_2, \Omega_2)\,.$$

All together, there are 10 baryons in the $\underline{20}_S$, and 11 in the $\underline{20}_M$, yielding a total of 21 baryons in this $\underline{120}$ SU(8) representation.

We denote deviations arising directly from the quark mass differences by m for the u and d quarks (we are neglecting electromagnetic effects), m_s for the s quark, and m_c for the charmed quark. As stated in the Introduction, we also include deviations arising from differences in the two-body interactions; these are denoted by V, with subscripts to indicate the various quark-quark configurations. V_1 and V_0 come from the interactions between u and d quarks in the isospin-1, -0 states, respectively; since the over-all quark wave function is SU(8) symmetric, V_1 and V_0 automatically go with spin-1 and -0 states, respectively, so that no additional notation is necessary to denote the spin dependence. The interaction of u or d with s and c give rise to contributions V_s, V_c, while the interactions of s and c quarks with each other give V_{sc}, V_{ss}, and V_{cc}. Again, V_{ss} and V_{cc} act only in spin-1 states, but V_s, V_c, and V_{sc} act both in spin-1 and spin-0 states of two quarks. We use a superscript a on V_s, V_c, and V_{sc} when two quarks interact in spin-0 states and leave the symbols alone when the quarks interact in spin-1 states.

In order to calculate the contribution of the interactions to the baryon masses, we need to know the baryon SU(8) wave functions. These can be readily calculated. The wave functions are not unique, however, unless it is specified that they are eigenstates of particular subgroups of SU(8). In SU(6), for example, the wave functions of the Λ and Σ^0 baryons are uniquely specified by the requirement that they be eigenstates of the SU(2) subgroup corresponding to isospin. Similarly, we require that the SU(8) wave functions be eigenstates of the ordinary SU(3) subgroup containing isospin and strangeness. In so doing, we have in mind a hierarchy of symmetry breaking, with charm breaking the symmetry more than strangeness, and strangeness more than the z component of isospin.

The baryon masses can be readily calculated, and depend linearly on m, m_s, m_c, and the V's. The expressions for the masses are given in the Appendix. We use the symbol for a baryon to denote its mass, and where necessary to avoid confusion, we use the subscripts S and M to denote that the baryons belong to the $\underline{20}_S$ and $\underline{20}_M$, respectively.

Using the expressions in the Appendix, we have obtained the following four independent relations among the members of the $\underline{20}_S$:

$$\Omega - \Delta = 3(\Xi - \Sigma)\,, \tag{4a}$$

$$\Omega_3 - \Delta = 3(\Xi_2 - \Sigma_1)\,, \tag{4b}$$

$$\Omega_3 - \Omega = 3(\Omega_2 - \Omega_1)\,, \tag{4c}$$

$$\Omega_1 - 2\Xi_1 + \Sigma_1 = \Omega - 2\Xi + \Sigma\,. \tag{4d}$$

We do not get the equal-spacing rule for the SU(3) decimet. However, we do obtain Okubo's second-order decimet mass formula,[9] which is given in Eq. (4a). We have also found the following seven independent formulas relating members of the $\underline{20}_S$ to members of the $\underline{20}_M$

$$\Xi_S - \Sigma_S = \Xi_M - \Sigma_M\,, \tag{5a}$$

$$\Xi_{2S} - \Sigma_{1S} = \Xi_{2M} - \Sigma_{1M}\,, \tag{5b}$$

$$\Omega_{2S} - \Omega_{1S} = \Omega_{2M} - \Omega_{1M}\,, \tag{5c}$$

$$\Xi_S - \Delta = \Xi_M - N + \tfrac{3}{2}(\Lambda - \Sigma_M)\,, \tag{5d}$$

$$\Xi_{2S} - \Delta = \Xi_{2M} - N + \tfrac{3}{2}(\Lambda_1 - \Sigma_{1M})\,, \tag{5e}$$

$$\Omega_S - 2\Xi_S + \Sigma_S = \Omega_{1M} - 2\Xi_{1M} + \Sigma_{1M}\,, \tag{5f}$$

$$\Omega_{1S} - 6\Xi_{1S} + \Sigma_{1S} + 4\Sigma_S = \Omega_{1M} - 6\Xi_{1M}' + \Sigma_{1M} + 4\Sigma_M\,, \tag{5g}$$

where Ξ_{1M} and Ξ_{1M}' belong to the SU(3) $\underline{6}$ and $\underline{\bar{3}}$, respectively, contained in the $\underline{20}_M$. There are no relations involving solely the members of the $\underline{20}_M$; in particular, the Gell-Mann–Okubo octet mass formula does not hold.

The 11 relations in Eqs. (4) and (5) follow directly from the use of SU(8) wave functions for the particles, and our initial assumption of the additivity of the two-body quark-quark interactions. At present, the only relations which can be compared with experiment are those which contain only the masses of the usual octet and deciment, namely Eqs. (4a), (5a), and (5d). These equations are well-known from other sources, having been derived before[6,9,10] within the framework of SU(6). The differences[11] between their left- and right-hand sides are about 8, 22, and 34 MeV, respectively. Considering that the baryons involved in these three relations have masses in the GeV region (about 1 to 1.7 GeV), we can conclude that they are in fact rather well satisfied. It is our hope of course that the other relations which involve the charmed baryons (some of whose masses may be as low as 2.2 GeV; see below) may likewise be satisfied to a reasonable degree, though presumably not as well as for the uncharmed baryons. However, an estimate of charmed baryon masses to say 100 MeV or so would certainly be of great value to experimentalists in locating them.

We now go further and ask what additional assumptions must be made in order to obtain what are in fact the best known relations among the known masses, namely the equal spacing rule for the decimet and the Gell-Mann–Okubo mass formula for the octet:

$$\Omega - \Xi_S = \Xi_S - \Sigma_S = \Sigma_S - \Delta ,$$

$$2(N + \Xi_M) = 3\Lambda + \Sigma_M .$$

From the Appendix, it may be immediately deduced that both of these relations follow with only one additional assumption about the two-body interactions[6]:

$$2V_s = V_{ss} + V_1 . \qquad (6)$$

This extra requirement involves two-body interactions which are symmetric in both SU(3) and spin: it corresponds to equal spacing in the symmetric $\{6\}$ representation of SU(3) with the quarks in their spin-1 configuration. No constraint is necessary on the antisymmetric interactions.

We do not know of any *a priori* reason why quark-quark interactions in spin-1 states should satisfy Eq. (6), other than the fact that Eq. (6) represents a particularly simple form of symmetry breaking. If Eq. (6) is not satisfied, then the Gell-Mann–Okubo octet and decimet formulas will not hold. But it is well known that the actual masses agree with these formulas to a very good approximation, and so we conclude *a posteriori* that in the model Eq. (6) holds. We should like to point out that the situation is no better when the Gell-Mann–Okubo formulas are derived from group theory. In this case, one assumes that the symmetry-breaking term in the mass operator transforms like a component of an octet. The justification, aside from simplicity, is that it works.

In line with our discussion above, we extend this idea to the larger group SU(4) to determine the corresponding relations among the charmed baryons. [Not surprisingly, some of the relations we obtain at this level of approximation are similar to those found[3,12] recently by a different perturbation approach, namely that in which SU(4) is broken by components of its 15-dimensional representation.] The relations analogous to Eq. (6) involving the charmed quark are

$$2V_c = V_{cc} + V_1 ,$$
$$2V_{sc} = V_{ss} + V_{cc} . \qquad (7)$$

Here, of course, we do not have any justification from experiment, and can only argue (1) that we have broken the symmetry of the quark-quark interactions in a particularly simple way, and (2) that Eqs. (7) are analogous to the successful Eq. (6).

Using Eqs. (7), we obtain the following independent relations among the masses of the $\underline{20}_S$:

$$\Xi_1 - \Sigma_1 = \Omega - \Xi = \Xi - \Sigma = \Sigma - \Delta , \qquad (8a)$$

$$\Omega_3 - \Xi_2 = \Xi_2 - \Sigma_1 = \Sigma_1 - \Delta , \qquad (8b)$$

$$\Omega_2 - \Omega_1 = \Omega_1 - \Omega = \Sigma_1 - \Sigma . \qquad (8c)$$

These mass relations should not be considered to be as well justified in our model as those of Eqs. (4) and (5), especially in view of the fact that SU(4) is more badly broken than SU(3).

But SU(3) is itself badly broken, with mass splittings of around 400 MeV within a multiplet. These splittings are very large compared to SU(2) splittings, which are typically less than 10 MeV within an isospin multiplet. We therefore feel that there may be some deeper reason (as yet unknown) why perturbation theory and Eq. (6) work in SU(3), and that this may carry over into SU(4).

From Eqs. (8), we see that the knowledge of the mass of only *one* charmed baryon in the $\underline{20}_S$ (say the Σ_1) would provide the masses of all the other baryons in this representation. [The relations of Eq. (8) are equivalent to those that were obtained previously with a spin-independent interaction.[5] The reason is that only the interactions in the spin-1 state of two quarks are relevant for the $\underline{20}_S$ in our SU(8) model.]

The situation in the $\underline{20}_M$, however, is different from the results of Ref. 5, and we obtain the following relations among the masses of $\underline{20}_M$:

$$2(N + \Xi) = 3\Lambda + \Sigma , \qquad (9a)$$

$$2(N + \Xi_2) = 3\Lambda_1 + \Sigma_1 , \qquad (9b)$$

$$(\Omega_1 - \Xi_1) = (\Xi_1 - \Sigma_1) , \qquad (9c)$$

$$2(\Sigma_1 - \Sigma) = 2\Xi_2 + \Xi_1 - 3\Xi_1' , \qquad (9c)$$

$$(\Omega_2 - \Xi_2) + (\Xi - \Sigma) = 2(\Xi_1 - \Sigma_1) . \qquad (9e)$$

Previously, with spin-independent interactions, 7 relations were obtained[5] among the 11 different baryon masses of the $\underline{20}_M$. Here, with the interactions depending on spin, we obtain only 5 independent relations. If charmed baryons are observed, it will be interesting to see whether the relations of Eq. (9) agree better with experiment than those of Ref. 5.

As remarked above, we need to know the mass of only one charmed baryon in the $\underline{20}_S$ to obtain the masses of all the others in this representation. If one knows in addition the masses of two charmed baryons of the $\underline{20}_M$ (say the Σ_1 and Ω_1), then the masses of all the other members of the $\underline{20}_M$ can be deduced from Eqs. (5) and (9). Thus, in this scheme with approximations (6) and (7), the whole set of charmed baryons in the entire $\underline{120}$ SU(8) representation (there being 13 in all) can be determined once the masses of only three of them are known. It would appear therefore that Eqs. (8) and (9) are a very useful set of relations for estimating the masses of charmed baryons, once the first few have been discovered.

III. MASS RELATIONS FOR THE MESONS

We shall assume that the usual pseudoscalar and vector mesons belong to the mixed $\underline{63}+\underline{1}$ representations of SU(8). The SU(4) and spin content of this 64-plet is

$$\underline{64} \supset {}^3\underline{16} + {}^1\underline{16} , \tag{10}$$

where the $\underline{16}$ is a mixed $\underline{15}+\underline{1}$. The charm content of the $\underline{16}$ is

$$\underline{16} \supset \underline{10}_0 + \underline{\overline{3}}_1 + \underline{3}_{-1} , \tag{11}$$

where the $\underline{\overline{3}}_1$ and $\underline{3}_{-1}$ are antiparticles of each other. The $\underline{10}_0$ of vector mesons contains the nonet of SU(3) including the $\rho, \omega, \varphi, K^*, \overline{K}^*$. We identify the tenth state as the recently discovered ψ (J) meson.[1] As in the work of Ref. 5, we assume that the ρ and ω are comprised only u and d quarks, the φ of strange quarks, and the ψ of charmed quarks.[3] The $\underline{\overline{3}}_1$ mesons consist of an isospin doublet D and singlet F in the notation of Gaillard, Lee, and Rosner.[8] If we assume that the quark-antiquark interactions conserve isospin but nothing more, we get no relations among the meson masses. If, however, the quark-antiquark interactions satisfy in their symmetric configurations the same relations as the quark-quark interactions, namely Eqs. (6) and (7), we obtain the following sum rules

$$2K^* = \rho + \varphi , \tag{12a}$$

$$2D = \rho + \psi , \tag{12b}$$

$$2F = \varphi + \psi . \tag{12c}$$

These are the sum rules that were obtained previously[5] for the vector mesons. The first of these relations agrees well with experiment, since $2K^* = 1.78$ GeV and $(\rho + \varphi) = 1.79$ GeV. According to the second and third, the D and F vector mesons should have masses

$$D = 1.94 \text{ GeV} ,$$

$$F = 2.06 \text{ GeV} .$$

The pseudoscalar mesons K, π, and η' do not satisfy a mass relation analogous to that of Eq. (12a) for the vector mesons. Within the model, there are two possible explanations for this. The first is that the quark-antiquark interactions satisfy Eq. (6) in spin-1 states but not in spin-0 states. [Recall that in order to obtain the Gell-Mann–Okubo octet and decimet mass relations, we needed to assume only that the quark-quark interactions satisfied Eq. (6) in spin-1 states.] The second possibility is that the η' contains some admixture of $u\overline{u}$ and $d\overline{d}$ pairs in its wave function in addition to $s\overline{s}$ pairs. In any case, because the mass relation analogous to Eq. (12a) does not hold for pseudoscalar mesons, we cannot predict the masses of the charmed pseudoscalar mesons without further assumptions.

IV. DISCUSSION

As we have remarked in Sec. II, the baryon mass relations we have derived do not allow us to predict the masses of any charmed baryons until at least one of them is discovered to set the scale. However, if the $\psi(3.10)$ is indeed a $\overline{c}c$ state, then we can obtain a rough estimate of charmed baryon masses as follows. In the model the mass difference of about 2 GeV between the ψ and the φ meson arises from the replacement of two strange quarks by two charmed ones in the same configuration, or about 1 GeV for each such replacement. If a similar effect holds for the baryons, then replacing a strange quark by a charmed one in the Σ, for example, would increase its mass by about 1 GeV. Thus, the Σ_{1M} and Σ_{1S} would have masses of about 2.2 and 2.4 GeV, respectively. The mass relations of Eqs. (5), (8), and (9) then tell us that the baryons of the $\underline{120}$ should all have masses below 5 GeV.

In summary, we have extended the usual SU(3) and SU(6) symmetries to the higher symmetries SU(4) and SU(8) in order to incorporate charm, an additional quantum number strongly suggested by the recent discovery of narrow mesons at Brookhaven and SLAC.[1] On the basis of simple additive contributions arising from quark mass differences and different two-body quark-quark interactions, we obtained at different levels of approximation, relations among the masses of the charmed and uncharmed baryons and mesons. We calculated the masses of both the D and F charmed mesons[8] to be about 2 GeV. We would also expect the lowest-mass charmed baryons to have masses of a little over 2 GeV. If a few charmed baryons are indeed found, our mass relations can be used to see whether SU(8) is a worthwhile symmetry scheme, and if so these relations can be used to indicate where other charmed hadrons are to be found.

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APPENDIX

We list here the masses of baryons belonging to the 120-dimensional representation of SU(8) in terms of quark masses and interactions.

$\underline{20}_S$: $\underline{10}_0$: $\Delta = 3m + 3V_1$,

$\Sigma = 2m + m_s + V_1 + 2V_s$,

$\Xi = m + 2m_s + 2V_s + V_{ss}$,

$\Omega = 3m_s + 3V_{ss}$;

$\underline{6}_1$: $\Sigma_1 = 2m + m_c + V_1 + 2V_c$,

$\Xi_1 = m + m_s + m_c + V_s + V_c + V_{sc}$,

$\Omega_1 = 2m_s + m_c + V_{ss} + 2V_{sc}$;

$\underline{3}_2$: $\Xi_2 = m + 2m_c + 2V_c + V_{cc}$,

$\Omega_2 = m_s + 2m_c + 2V_{sc} + V_{cc}$;

$\underline{1}_3$: $\Omega_3 = 3m_c + 3V_{cc}$.

$\underline{20}_M$: $\underline{8}_0$: $N = 3m + \frac{3}{2}V_1 + \frac{3}{2}V_0$,

$\Lambda = 2m + m_s + V_0 + \frac{3}{2}V_s + \frac{1}{2}V_s^a$,

$\Sigma = 2m + m_s + V_1 + \frac{1}{2}V_s + \frac{3}{2}V_s^a$,

$\Xi = m + 2m_s + V_{ss} + \frac{1}{2}V_s + \frac{3}{2}V_s^a$;

$\underline{\bar{3}}_1$: $\Lambda_1 = 2m + m_c + V_0 + \frac{3}{2}V_c + \frac{1}{2}V_c^a$,

$\Xi_1' = m + m_s + m_c + \frac{3}{4}V_c + \frac{1}{4}V_c^a + \frac{3}{4}V_{sc} + \frac{1}{4}V_{sc}^a + V_s^a$;

$\underline{6}_1$: $\Sigma_1 = 2m + m_c + V_1 + \frac{1}{2}V_c + \frac{3}{2}V_c^a$,

$\Xi_1 = m + m_s + m_c + V_s + \frac{1}{4}V_c + \frac{3}{4}V_c^a + \frac{1}{4}V_{sc} + \frac{3}{4}V_{sc}^a$,

$\Omega_1 = 2m_s + m_c + V_{ss} + \frac{1}{2}V_{sc} + \frac{3}{2}V_{sc}^a$;

$\underline{3}_2$: $\Xi_2 = 2m_c + m + V_{cc} + \frac{1}{2}V_c + \frac{3}{2}V_c^a$,

$\Omega_2 = 2m_c + m_s + V_{cc} + \frac{1}{2}V_{sc} + \frac{3}{2}V_{sc}^a$.

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