PHYSICAL REVIEW D VOLUME 12, NUMBER 9 1 NOVEMBER 1975

Chiral invariance in a bag theory*

Alan Chodos and Charles B. Thorn†
Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 11 June 1975)

We study phenomena that involve surface fluctuations of a bag by introducing unconfined $\vec{\pi}$ and σ fields which interact with the bag only at the surface. The resulting theory is chirally symmetric. We find exact classical solutions to the equations of motion, which have the "hedgehog" property that $\vec{\pi}(x) = g(r)\hat{r}$. In addition, we study other classical solutions perturbatively, and estimate the values of the Δ width and of F_π predicted by our model. There are no free parameters except the bag constant B, which sets the over-all scale.

I. INTRODUCTION

The MIT bag model[1] has been remarkably successful in explaining the static properties of the low-lying hadrons. The calculation of these quantities (such as the mass spectra, the magnetic moments, and the charge radii) have been performed in an approximation in which the bag is considered to be a spherical cavity filled with the quanta of colored-quark and vector-gluon fields. The quantitative predictions based on this approximation are quite good, but one is unable to use it to study many interesting phenomena, such as scattering and decay processes, in which the surface of the bag plays an essential dynamical role. In addition one would also like to be able to estimate the size of the error made in neglecting the effect of the surface on the static quantities themselves.

Qualitatively, we picture a strong process as occurring through a fission mechanism analogous to that in the string model.[2] That is, in the classical limit, a bag separates at a point in spacetime into two bags or two bags fuse into one. The requirement that the transition occurs at a point ensures a causal description. To calculate quantum amplitudes one naturally uses the Feynman path-integral formalism and functionally integrates over trajectories which describe fission and/or fusion. The complete calculation requires a detailed understanding of the constraints which relate the surface motion to the constituent fields. Only in two dimensions has this calculation been done.[3] The three-dimensional problem has so far evaded solution.

Even in the string model, however, there is an alternative method for obtaining scattering amplitudes. The idea behind this approach is based on the fission process: The emission of a single bag in a definite state might be well described (in the tree approximation) by the emission of an elementary field excitation coupled locally to the surface of the bag. In the string model this approach gives the exact tree amplitudes for ground-state scattering provided only that the mass of the field is identified with the ground-state mass of the string. In the three-dimensional bag our hope is that this procedure will reasonably approximate the actual emission of a low-lying bag state, e.g., the pion or the empty bag.

In this paper we should like to apply this second method to describe the interaction of pions and empty bags with a single bag. We describe the single bag ("bare hadron") as a cavity filled with quarks, and the pions and empty bags as excitations of phenomenological pion and σ fields interacting with the bare hadron via a local coupling at the surface of the cavity. Our work was motivated in part by a similar attempt by Bardeen and Ellis,[4] and also bears some similarity to recent work by Chang, Ellis, and Lee[5] (who discuss a two-dimensional model) and by Freedman and Jaffe.[6]

The model we choose to study is based on a bare hadron consisting of massless $\mathcal{P}$ and $\mathfrak{N}$ quarks in a spherical cavity of radius R. In such a model, the axial-vector current $\vec{A}_\mu$ is locally conserved inside the cavity,

$$\partial_\mu \vec{A}^\mu = 0 \quad \text{for } r < R ,$$

and conservation of the axial charge is violated only at the boundary,

$$n_\mu \vec{A}^\mu \neq 0 \quad \text{at } r = R ,$$

where n_μ is the normal $\hat{r}$. We introduce an SU(2) $\times$SU(2) multiplet $(\sigma, \vec{\pi})$ of phenomenological fields, which we shall couple to the bare hadron in such a way as to restore the conservation of axial charge. For simplicity, we shall introduce no mass or self-coupling terms for the σ and π fields, and furthermore we shall allow them to couple to the quark degrees of freedom only at the surface of the bag. Since axial charge conservation is violated only at the surface, this procedure is sufficient for our purposes and is also consistent with our intuitive idea that we are trying to account for phenomena such as fission which presumably occur at the surface of the bag. In addition, as we shall see below, our simple choice of Lagrangian allows us to find a family of exact classical solutions to the equations of motion, which will provide us with greater insight into the physical properties of our model.

In the next section we define the model. We then go on to demonstrate our exact classical solutions, and to discuss some of their properties. In Sec. III we discuss a perturbative expansion of other classical solutions. Finally, we present some of the lowest-order phenomenological implications of our model.

II. MODEL AND EXACT CLASSICAL SOLUTIONS

The action describing the bare hadron alone can be written

$$W=\int d^4x\,\theta_R(x)\left[\tfrac{1}{2}i\bar\psi\gamma\cdot\overleftrightarrow{\partial}\psi-B+\partial_\mu(\lambda^\mu\bar\psi\psi)\right]. \quad (2.1)$$

Here $\theta_R(x)$ is unity inside the bag and zero outside; furthermore, $\partial_\mu\theta_R=n_\mu\delta_S$, where n_μ is the inward normal and δ_S is a surface δ function. For the case of a sphere of radius R, $\theta_R(x)=\theta(R-r)$. The field $\lambda_\mu(x)$ is a Lagrange multiplier, which will allow us to impose the desired boundary conditions on the fermion field.[7] This technique, which is treated more fully in Appendix A, gives the same boundary conditions as the original, more cumbersome method of giving the field ψ a mass M outside the bag and letting $M\to\infty$.

In writing the action in the form (2.1), we have clearly isolated the lack of chiral invariance in the term proportional to $\bar\psi\psi$. To restore chiral invariance, we introduce a scalar field σ and an isovector pseudoscalar field $\vec\pi$ and couple them to the fermion fields in the chirally invariant combination $\bar\psi(\sigma+i\vec\tau\cdot\vec\pi\gamma_5)\psi$. Our new action is

$$W=\int d^4x\left\{\theta_R(x)\left[\tfrac{1}{2}i\bar\psi\gamma\cdot\overleftrightarrow{\partial}\psi-B+\partial_\mu(\lambda^\mu\bar\psi(\sigma+i\vec\tau\cdot\vec\pi\gamma_5)\psi)\right]-\tfrac{1}{2}(\partial_\mu\sigma\,\partial^\mu\sigma+\partial_\mu\vec\pi\cdot\partial^\mu\vec\pi)\right\}. \quad (2.2)$$

Notice that we do not confine the π and σ fields inside R, since we are seeking to describe phenomena involving surface fluctuations whose influence is not restricted to the interior of the static bag.

The derivation of the equations of motion and the boundary conditions proceeds exactly as in Appendix A. In this case, the Lagrange multiplier $(n\cdot\lambda)$ is constrained to be $\pm\frac{1}{2}(\sigma^2+\vec\pi^2)^{-1/2}$ and we obtain the equations

$$i\gamma\cdot\partial\psi=0\quad\text{inside } R\,, \quad (2.3a)$$

$$i\,n\cdot\gamma\psi=\pm\frac{1}{(\sigma^2+\vec\pi^2)^{1/2}}(\sigma+i\vec\tau\cdot\vec\pi\gamma_5)\psi \quad (2.3b)$$

at the surface;

$$\partial^2\sigma=\pm\frac{1}{2}\,\frac{1}{(\sigma^2+\vec\pi^2)^{1/2}}\,\bar\psi\psi\delta_S\,, \quad (2.3c)$$

$$\partial^2\vec\pi=\pm\frac{1}{2}\,\frac{1}{(\sigma^2+\vec\pi^2)^{1/2}}\,i\bar\psi\vec\tau\gamma_5\psi\delta_S\,; \quad (2.3d)$$

and the nonlinear boundary condition:

$$n_\mu\partial^\mu[\bar\psi(\sigma+i\vec\tau\cdot\vec\pi\gamma_5)\psi]=\pm2(\sigma^2+\vec\pi^2)^{1/2}B\,. \quad (2.4)$$

The equations of motion (2.3) imply that σ and $\vec\pi$ have discontinuous first derivatives across the boundary. Thus the condition (2.4) is ambiguous. The ambiguity can be resolved by examining the stress-energy tensor:

$$T_{\mu\nu}=\theta_R(x)\,T^{(1)}_{\mu\nu}(x)+[1-\theta_R(x)]\,T^{(2)}_{\mu\nu}(x)\,.$$

The equations of motion guarantee that

$$\partial_\mu T^{\mu\nu(1)}=\partial_\mu T^{\mu\nu(2)}=0$$

and therefore $T_{\mu\nu}$ will be conserved if

$$n_\mu T^{\mu\nu(1)}=n_\mu T^{\mu\nu(2)}$$

at the surface. This condition can be shown to coincide with (2.4) if we resolve the ambiguity in (2.4) by taking

$$\partial_\mu\sigma\equiv\tfrac{1}{2}\left[\partial_\mu\sigma^{(\text{in})}+\partial_\mu\sigma^{(\text{out})}\right] \quad (2.5)$$

and similarly for $\partial_\mu\vec\pi$. The theory possesses invariance under the infinitesimal chiral transformation

$$\begin{aligned}\psi&\to\psi+i\vec\tau\cdot\delta\vec v\gamma_5\psi\,,\\ \sigma&\to\sigma+2\delta\vec v\cdot\vec\pi\,,\\ \vec\pi&\to\vec\pi+2\delta\vec v\sigma\,,\end{aligned} \quad (2.6)$$

and therefore

$$\vec A_\mu=\tfrac{1}{2}\bar\psi\vec\tau\gamma_5\gamma_\mu\psi\,\theta_R(x)+\vec\pi\partial_\mu\sigma-\sigma\partial_\mu\vec\pi \quad (2.7)$$

is conserved.

For the case of a static spherical bag of radius R and static $\vec\pi$ and σ fields, the equations (2.3) reduce to

$$i\gamma\cdot\partial\psi=0\,, \quad (2.8a)$$

$$i\hat r\cdot\vec\gamma\psi=-\frac{1}{(\sigma^2+\vec\pi^2)^{1/2}}(\sigma+i\vec\tau\cdot\vec\pi\gamma_5)\psi\,, \quad (2.8b)$$

$$\nabla^2\sigma=+\frac{1}{2}\,\frac{1}{(\sigma^2+\vec\pi^2)^{1/2}}\,\bar\psi\psi\,\delta(R-r)\,, \quad (2.8c)$$

$$\nabla^2\vec\pi=+\frac{1}{2}\,\frac{1}{(\sigma^2+\vec\pi^2)^{1/2}}\,i\bar\psi\vec\tau\gamma_5\psi\,\delta(R-r)\,. \quad (2.8d)$$

Here we have chosen the arbitrary sign in (2.3) to recover the usual lowest-lying fermion bag states in the limit $\sigma\to\infty$.

The system of equations (2.8) is solved by the following choice for ψ, $\vec{\pi}$, and σ:

$$\psi = \begin{pmatrix} j_0(\omega r)\, v \\ i\vec{\sigma}\cdot\hat{r}\, j_1(\omega r)\, v \end{pmatrix} e^{-i\omega t} , \tag{2.9}$$

$$\vec{\pi} = g(r)\hat{r} , \tag{2.10}$$

$$\sigma = f(r) , \tag{2.11}$$

where v is a constant spinor-isospinor with the property that

$$(\vec{\sigma} + \vec{\tau})\, v = 0 . \tag{2.12}$$

Thus v is proportional to the combination

$$\hat{v} \equiv \frac{1}{\sqrt{2}} \left(|\uparrow; -\rangle - |\downarrow; +\rangle \right) , \tag{2.13}$$

where the arrows refer to ordinary spin and the $\pm$ to isospin. This property of v guarantees that $\bar{\psi}\vec{\tau}\gamma_5\psi$ is proportional to $\hat{r}$:

$$\bar{\psi}\vec{\tau}\gamma_5\psi = i j_0 j_1 [v^\dagger \vec{\tau}\vec{\sigma}\cdot\hat{r}\, v + v^\dagger \vec{\sigma}\cdot\hat{r}\, \vec{\tau}\, v] .$$

But

$$\begin{aligned} v^\dagger \tau_i \sigma_j v &= - v^\dagger \sigma_i \sigma_j v \\ &= -2\delta_{ij} v^\dagger v + v^\dagger \sigma_j \sigma_i v \\ &= -2\delta_{ij} v^\dagger v - v^\dagger \sigma_j \tau_i v \\ &= -2\delta_{ij} v^\dagger v - v^\dagger \tau_i \sigma_j v \end{aligned}$$

so that

$$v^\dagger \tau_i \sigma_j v = -\delta_{ij} v^\dagger v$$

and thus

$$\bar{\psi}\vec{\tau}\gamma_5\psi = -2i j_0 j_1 v^\dagger v \hat{r} . \tag{2.14}$$

With the notation

$$\frac{1}{[g^2(R) + f^2(R)]^{1/2}} = \xi , \tag{2.15}$$

the equations for g and f then become

$$\begin{aligned} f'' + (2/r) f' &= +\tfrac{1}{2}\xi [j_0^2(\omega R) - j_1^2(\omega R)]\, v^\dagger v\, \delta(R - r) \\ &\equiv \alpha\delta(R - r) , \end{aligned} \tag{2.16}$$

$$\begin{aligned} g'' + (2/r) g' - (2/r^2) g &= +\xi j_0(\omega R) j_1(\omega R)\, v^\dagger v\, \delta(R - r) \\ &\equiv \beta\delta(R - r) , \end{aligned} \tag{2.17}$$

which are solved by

$$f(r) = f_0 + \theta(r - R)\, \alpha R^2 \left(\frac{1}{R} - \frac{1}{r} \right) , \tag{2.18}$$

$$g(r) = -\frac{\beta}{3} \left[\theta(R - r)\, r + \theta(r - R) \frac{R^3}{r^2} \right] . \tag{2.19}$$

f_0 is an arbitrary constant which is not determined by (2.16) or by the condition that f be regular at either the origin or infinity.

Armed with the solutions (2.18), (2.19), we now examine the linear boundary condition (2.8b) to obtain an eigenvalue condition for the frequency $\Omega \equiv \omega R$. Equation (2.8b) implies that

$$- j_1(\Omega) v = -\xi [f_0 j_0(\Omega) - g(R) j_1(\Omega)\, \vec{\tau}\cdot\hat{r}\, \vec{\sigma}\cdot\hat{r}]\, v .$$

But

$$\vec{\tau}\cdot\hat{r}\, \vec{\sigma}\cdot\hat{r}\, v = -(\vec{\tau}\cdot\hat{r})^2 v = -v . \tag{2.20}$$

So we have, with $y \equiv j_1(\Omega)/j_0(\Omega)$,

$$(1 - \xi g)\, y = +\xi f_0 ,$$

or, using $\xi^2 = 1/(g^2 + f_0^2)$, we have

$$y = \frac{1 + \xi g}{\xi f_0} . \tag{2.21}$$

We also have the normalization condition (appropriate to the one-fermion sector of the theory) that

$$\int_R d^3x\, \psi^\dagger \psi = 1 , \tag{2.22}$$

which fixes

$$v^\dagger v = \frac{\Omega^2}{R^3} \frac{1}{4\pi(1 - j_0^2(\Omega))} . \tag{2.23}$$

Now $g(R)$ is to be determined as a function of Ω and f_0 from

$$g = \frac{-\beta R}{3} = -\frac{1}{(g^2 + f_0^2)^{1/2}}\, y j_0^2(\Omega) \frac{R}{3}\, v^\dagger v \equiv -\xi d(\Omega) , \tag{2.24}$$

which is a quadratic equation in g^2. This value of g can then be inserted into (2.21), which is a transcendental equation for the eigenvalue Ω as a function of f_0.

In practice, however, it is easier to invert this process and to choose Ω freely and then to calculate both f_0 and the energy as a function of Ω. The energy is given by

$$\begin{aligned} E &= \int d^3x \{ \theta_R(x) [\tfrac{1}{2} i (\psi^\dagger \dot{\psi} - \dot{\psi}^\dagger \psi) + B] \\ &\qquad + \tfrac{1}{2} (\nabla_i \sigma \nabla_i \sigma + \nabla_i \vec{\pi}\cdot \nabla_i \vec{\pi}) \} \\ &\equiv E_\psi + E_{\pi\sigma} + E_{\text{bag}} , \end{aligned} \tag{2.25}$$

where, because of the normalization (2.22),

$$E_\psi = \frac{\Omega}{R} \tag{2.26}$$

and, after integration of the known solutions to the $\vec{\pi}$ and σ fields,

$$E_{\pi\sigma} = \frac{1}{R} [2\pi R^4 (\alpha^2 + \tfrac{1}{3}\beta^2)] . \tag{2.27}$$

Notice that $E_{\pi\sigma}$ (as well as E_ψ) is $1/R$ times a function of the dimensionless parameters Ω and f_0R, and hence, through (2.21), we have

$$E_\psi + E_{\pi\sigma} = \frac{\kappa(\Omega)}{R} . \tag{2.28}$$

On the other hand, $E_{\text{bag}} = 4\pi BR^3/3$, independent of Ω.

At this point, we have a complete classical solution to the problem of a fermion confined to a cavity of fixed radius R with its surrounding pion and σ fields. For each Ω there are two interesting quantities: the energy, represented by $\kappa(\Omega)$, and f_0, which is the value of σ at $r=R$, or equivalently σ_0, which is defined to be the value of σ as $r\to\infty$ and is given by $\sigma_0 = f_0 + \alpha R$ [see Eq. (2.18)]. It is actually more convenient to study the behavior of $F(\Omega) \equiv 1/4\pi(\sigma_0 R)^2$. In Fig. 1 we plot $\kappa(\Omega)$ and $F(\Omega)$ in the first three allowed regions. The regions in which $F(\Omega)$ is negative are unphysical because σ_0 is imaginary there. These regions coincide with the regions where $E_{\pi\sigma}(\Omega) < 0$. The stable points correspond to the local minima of the energy, which for fixed R are the same as the minima of $\kappa(\Omega)$. In the first allowed region, this occurs at $\Omega = 1.68$ and $\kappa = 1.93$.

The bag differs from the cavity in that the radius is no longer independent but becomes a function of Ω through the nonlinear boundary condition (2.4). This is essential to achieve energy-momentum conservation, and corresponds to balancing the pressures due to the fermion and meson fields at the boundary against the universal pressure B. In the unmodified bag theory, this is equivalent to minimizing E with respect to R; if we minimized our expression for E with respect to R keeping Ω fixed, we would obtain the condition

$$4\pi BR^4 = \kappa(\Omega) .$$

However, Eq. (2.4) leads to a different condition:

$$4\pi BR^4 = \tau(\Omega) \tag{2.29}$$

where

$$E_\psi - E_{\pi\sigma} = \tau(\Omega)/R . \tag{2.30}$$

[Compare Eq. (2.28).] Since the radius, and hence τ, must be positive, this leads to an additional restriction on the allowed values of Ω:

$$0 < E_{\pi\sigma} < E_\psi . \tag{2.31}$$

In the case of the cavity, only the leftmost inequality was required. The zeros of τ correspond to places where the pressure of the π and σ fields, together with B, combine to squeeze the bag down to a point. Solving for R in terms of Ω from (2.29), we obtain for the energy [in units of $(4\pi B)^{1/4}$]

$$E(\Omega) = \frac{\kappa + \frac{1}{3}\tau}{\tau^{1/4}} . \tag{2.32}$$

Furthermore, the value of $1/4\pi\sigma_0^2$ picks up addi-

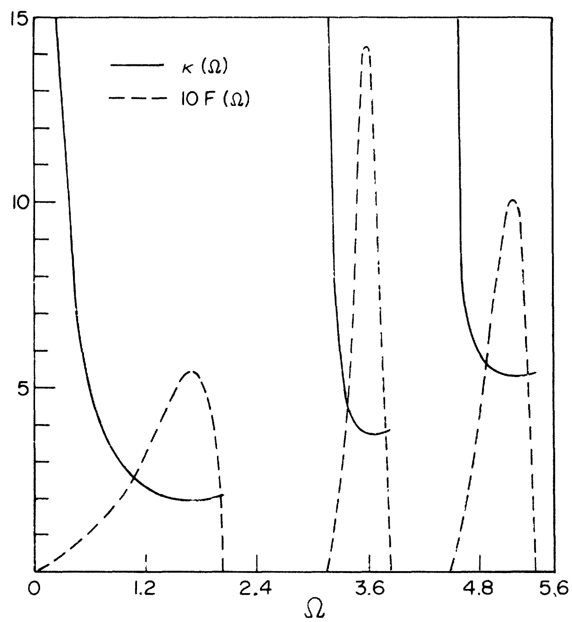

FIG. 1. The energy function $\kappa(\Omega)$ (solid line) and five times $F(x) \equiv 1/4\pi(\sigma_0 R)^2$ (dashed line) plotted against the dimensionless quark frequency Ω for the first three allowed cavity regions.

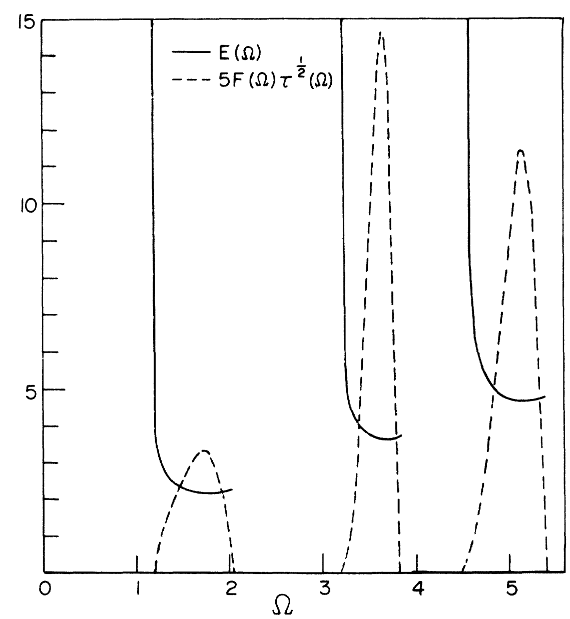

FIG. 2. The energy function $E(\Omega)$ (solid line) and ten times $F_\tau^{1/2} \equiv 1/4\pi\sigma_0^2$ (dashed line) in units of $(4\pi B)^{1/4}$, plotted against the dimensionless quark frequency Ω for the first three allowed bag regions.

tional Ω dependence because R varies with Ω:

$$\frac{1}{4\pi\sigma_0^2}=F(\Omega)\,\tau^{1/2}(\Omega) \tag{2.33}$$

(again in units where $4\pi B=1$).

In Fig. 2 we plot the functions $E(\Omega)$ and $F(\Omega)\tau^{1/2}(\Omega)$. Notice that the allowed regions are smaller than before, as we pointed out above. In the first allowed region the effect is large, since $0<\Omega<1.19$ is now forbidden. In the succeeding regions, the new forbidden strip, which is always at the left end of the allowed region, becomes smaller and smaller and its width approaches zero asymptotically. The position of the energy minimum is also slightly shifted in the bag relative to the cavity. In the first allowed bag region it occurs at $\Omega=1.73$, corresponding to a value of $E(\Omega)$ of 2.195. [There is no real significance in comparing the value of $E(\Omega)$ at its minimum against the corresponding cavity minimum, since in one case the radius is held fixed while in the other it is constrained to vary as a function of Ω.]

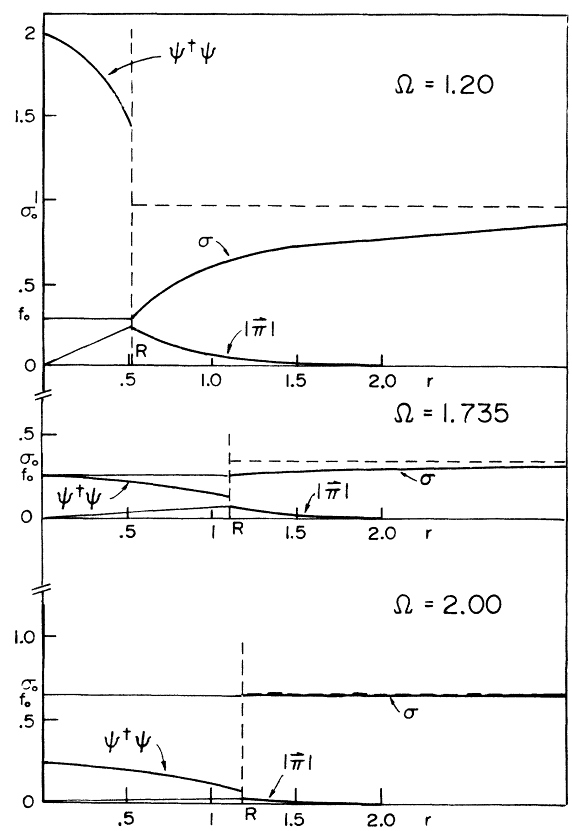

FIG. 3. Behavior of $\psi^\dagger\psi$, σ, $|\vec{\pi}|$ as functions of r for three values of Ω.

In Fig. 3 we illustrate the behavior of the quantities $\psi^\dagger\psi$, σ, and $|\vec{\pi}|$ for three typical values of Ω, in the case of the bag. The first value of Ω (1.2) is near the left end of the first allowed region (see Fig. 2). Here the radius is relatively small and the variation in the π and σ fields relatively large. The second value of Ω (1.735) is the one which minimizes the energy. The third value (2.00) is near the right end of the allowed region, which is where the uncoupled bag solution is recovered at $\Omega=2.0428$. Here the σ field is large but almost constant, and the π field is very small.

In Appendix B, we shall present some of the calculations upon which the above remarks are based. We shall also prove there an additional curious fact which is already evident from an inspection of Figs. 1 and 2: In both the cavity and bag cases, the value of $1/4\pi\sigma_0^2$ reaches its maximum at exactly the same point that the energy achieves its minimum. We shall see in the next section that $1/\sigma_0^2$ enters into the definition of an effective coupling constant, so that if the interaction between the bare hadron and the meson fields were always attractive the energy would be minimized by making the interaction as strong as possible consistent with the boundary conditions. However, we have been unable to obtain any deeper understanding of this intimate connection between the functions $\kappa(\Omega)$ and $F(\Omega)$ in the case of the cavity, and between $E(\Omega)$ and $F(\Omega)\tau^{1/2}(\Omega)$ in the case of the bag.

We conclude this section with the following remarks:

In addition to the arbitrariness due to the parameter f_0, which can be resolved by minimizing the energy, our solution possesses the further arbitrariness that the isospin axes can be rotated relative to the spatial axes. In other words, if we let $R\vec{\tau}$ be any rotation of $\vec{\tau}$, and choose instead of (2.12) a spinor v_R satisfying

$$(\vec{\sigma}+R\vec{\tau})\,v_R=0\ , \tag{2.34}$$

then we continue to have a solution provided we rotate $\vec{\pi}$ in the opposite direction:

$$\vec{\pi}\to\vec{\pi}_R\equiv g(r)\,(R^{-1}\hat{r})\ . \tag{2.35}$$

(ii) Despite this arbitrariness, none of the solutions we have found represents a satisfactory approximation to, say, a $\mathcal{P}$- or $\mathfrak{N}$-quark bag surrounded by pions, for the reason that none of them is a state of definite isospin, and of course we cannot superpose various solutions to obtain a state of definite isospin because the equations we have solved are nonlinear. In fact, our solution has the property that when we insert it into the expression for the total isospin,

$$\vec{\mathrm{I}}=\tfrac{1}{2}\int d^3x\,\psi^\dagger(x)\,\vec{\tau}\,\psi(x)\,\theta_R(x)+\int d^3x\,(\vec{\pi}\times\dot{\vec{\pi}})\ , \tag{2.36}$$

we find $\vec{\mathrm{I}}=0$, because, first, $\dot{\vec{\pi}}=0$, and second, $v^\dagger\vec{\tau}v=0$ as can be explicitly verified from (2.13). In order to get some idea of the properties of other classical solutions, in which, for example, the fermion isospinor is an eigenstate of T_3, we shall turn in the next section to a perturbation expansion in the coupling between the fermion and meson fields.

III. PERTURBATION THEORY

In this section we explore some of the properties of a different set of classical solutions from the ones treated exactly in the previous section. If we represent the σ field as

$$\sigma(x)=\sigma_0+\sigma'(x)\ , \tag{3.1}$$

where σ is taken to vanish as $|x|\to\infty$, then from the full system of equations (2.3) we see that as $\sigma_0\to\infty$ the equations reduce to the bag equations without the pion and σ fields. Our strategy then is to choose, in zeroth order, a solution ψ_0 of the ordinary bag equations, and to develop a systematic expansion in powers of $\lambda\equiv 1/\sigma_0$.

Our experience in Sec. II indicates that λ is not really an independent parameter but should be determined by minimizing the energy. Of course, the first few terms of a perturbation series cannot necessarily be used to determine an accurate minimum. Instead, we shall take λ to be the same as given by the exact solution in order to estimate numerical results.

For our zeroth-order fermion field we shall choose

$$\psi_0=\begin{pmatrix} j_0(\omega_0 r)\,u_0\\ ij_1(\omega_0 r)\,\vec{\sigma}\cdot\hat{r}\,u_0\end{pmatrix}\ , \tag{3.2}$$

where u_0 satisfies $\sigma_3 u_0=\tau_3 u_0=u_0$. The zeroth-order boundary condition determines $\Omega_0\equiv(\omega_0 R)=2.0428$.

The full system of equations is

$$(\omega+i\,\vec{\alpha}\cdot\vec{\nabla})\,\psi=0,\quad r<R \tag{3.3}$$

$$i\hat{r}\cdot\vec{\gamma}\psi=-\frac{1}{(\sigma^2+\vec{\pi}^2)^{1/2}}\,(\sigma+i\,\vec{\tau}\cdot\vec{\pi}\gamma_5)\,\psi,\quad r=R \tag{2.8b$'$}$$

$$\nabla^2\sigma=\frac{1}{2}\,\frac{1}{(\sigma^2+\vec{\pi}^2)^{1/2}}\,\delta(R-r)\,\bar{\psi}\psi\ , \tag{2.8c$'$}$$

$$\nabla^2\vec{\pi}=\frac{1}{2}\,\frac{1}{(\sigma^2+\vec{\pi}^2)^{1/2}}\,\delta(R-r)\,i\bar{\psi}\vec{\tau}\gamma_5\psi\ . \tag{2.8d$'$}$$

We do not include the nonlinear boundary condition (2.4) in our considerations.

One can check that a systematic expansion can be achieved of the following form:

$$\psi=\psi_0+\lambda^2\psi_1+\lambda^4\psi_2+\cdots\ , \tag{3.4a}$$

$$\sigma=(1/\lambda)\,[1+\lambda^4\sigma_1+\cdots]\ , \tag{3.4b}$$

$$\vec{\pi}=\lambda\vec{\pi}_1+\lambda^3\vec{\pi}_2+\cdots\ , \tag{3.4c}$$

$$\omega=\omega_0+\lambda^2\omega_1+\lambda^4\omega_2+\cdots\ . \tag{3.4d}$$

The powers of λ involved in expanding $\vec{\pi}$ and σ are derived from an inspection of (2.8c) and (2.8d). The reason that the correction to σ_0 is $O(\lambda^3)$ is because $\bar{\psi}_0\psi_0=0$ at $r=R$. In lowest nonvanishing order, only $\pi^{(3)}$ contributes since $\bar{\psi}_0\tau^{(\pm)}\gamma_5\psi_0=0$. It turns out to be possible to construct a solution without ever exciting the $\pi^{(\pm)}$ modes. The nonlinear boundary condition may require the introduction of these modes; however, we have ignored them in the interest of simplicity. Henceforth, when we write π we shall mean $\pi^{(3)}$.

Now that we have chosen ψ_0, our next step is to solve for π_1 from

$$\begin{aligned}\nabla^2\pi_1&=\tfrac{1}{2}\,i\,\bar{\psi}_0\gamma_5\psi_0\,\delta(R-r)\\&=-p^2 j_0{}^2\cos\theta\,\delta(R-r)\ ,\end{aligned} \tag{3.5}$$

where $u_0^\dagger u_0=p^2$. The solution is

$$\pi_1=g(r)\cos\theta\ , \tag{3.6}$$

with

$$g(r)=\rho\left[\frac{r}{R}\,\theta(R-r)+\frac{R^2}{r^2}\,\theta(r-R)\right] \tag{3.7}$$

and

$$\rho=g(R)=\tfrac{1}{3}R\,p^2 j_0{}^2(\omega_0 R)\ . \tag{3.8}$$

The equations to be satisfied by ψ_1 are

$$(\omega_0+i\,\vec{\alpha}\cdot\vec{\nabla})\,\psi_1=-\,\omega_1\psi_0,\quad r<R \tag{3.9}$$

and

$$(i\hat{r}\cdot\vec{\gamma}+1)\,\psi_1=-\,i\,\rho\cos\theta\,\gamma_5\psi_0,\quad r=R\ . \tag{3.10}$$

The solution is

$$\psi_1=\omega_1\chi+\zeta\begin{pmatrix} j_2(\omega_0 r)\,\vec{\sigma}\cdot\hat{r}\,u_1\\ -\,i\,j_1(\omega_0 r)\,u_1\end{pmatrix}\ , \tag{3.11}$$

where

$$\chi=\begin{pmatrix} -\,rj_1(\omega_0 r)\,u_0\\ [rj_0(\omega_0 r)-(2/\omega_0)\,j_1(\omega_0 r)]\,i\,\vec{\sigma}\cdot\hat{r}\,u_0\end{pmatrix}, \tag{3.12}$$

$$\zeta=\frac{\Omega_0 p}{9}\,\rho\ , \tag{3.13}$$

$$u_1=\begin{pmatrix} 2\cos\theta\\ -\sin\theta\,e^{i\phi}\end{pmatrix}\ , \tag{3.14}$$

and

$$\omega_1 R \equiv \Omega_1 = -\frac{1}{6}\left(\frac{\Omega_0}{\Omega_0 - 1}\right)\rho . \tag{3.15}$$

The first term in (3.11) is a particular solution to the equation of motion (3.9), and the second term is a solution to the homogeneous equation which is needed to satisfy the boundary condition (3.10).

One can already see from (3.11) one of the essential complications of this type of solution; namely, that with succeeding orders of perturbation theory, increasingly complicated angular dependence is needed to meet the boundary conditions.

To this order, the energy of the fermion fields is

$$E_\psi = \omega_0 + \lambda^2 \omega_1 , \tag{3.16}$$

where we have imposed the normalization condition

$$\int_R d^3x\, \psi^\dagger \psi = 1 \tag{3.17}$$

as in the previous section. The energy of the $\pi\sigma$ system is

$$\begin{aligned} E_{\pi\sigma} &= \tfrac{1}{2}\lambda^2 \int d^3x [(\vec{\nabla}\pi_1)^2] \\ &= 2\pi\rho^2 R\lambda^2 . \end{aligned} \tag{3.18}$$

From (3.8) we see that ρ, which enters into both ω_1 and $E_{\pi\sigma}$, contains p^2 as a factor, and is therefore to be determined from the normalization condition (3.17). To this order, it suffices to determine ρ from

$$\int \psi_0^\dagger \psi_0\, d^3x = 1, \tag{3.17$'$}$$

with the result that

$$4\pi\rho^2 R^2 = -\Omega_1, \tag{3.19}$$

and therefore $E_{\pi\sigma} = -\frac{1}{2}\lambda^2\omega_1$, so that the total energy to this order is

$$E = \omega_0 + \tfrac{1}{2}\lambda^2\omega_1 . \tag{3.20}$$

Observe that the $O(\lambda^2)$ correction to the normalization, which will be of the form

$$\rho \to \rho + \lambda^2 \rho_1 ,$$

produces a change $\lambda^4\omega_1\rho_1/\rho$ in $\lambda^2\omega_1$, and a change

$$4\pi\rho\rho_1 R\lambda^4 = -\lambda^4\omega_1\rho_1/\rho$$

in $E_{\pi\sigma}$, so that the corrections to (3.17$'$) do not affect the total energy until $O(\lambda^6)$.

To calculate the next corrections (ψ_2, π_2, and the $O(\lambda^4)$ term in the energy), one expands Eqs. (3.3) and (2.8) to the required order. It turns out that π_2 has the same $\cos\theta$ angular dependence as π_1, but ψ_2 picks up an additional piece of the form

$$\begin{pmatrix} j_2 u_2 \\ j_3 i\vec{\sigma}\cdot\hat{r}\, u_2 \end{pmatrix},$$

where

$$u_2 = \begin{pmatrix} 3\cos^2\theta - 1 \\ -2\cos\theta\sin\theta\, e^{i\phi} \end{pmatrix}.$$

In the next order, π_3 has angular dependence proportional to $P_3(\cos\theta)$, and σ_1 (which finally enters the calculation) has a piece proportional to $P_2(\cos\theta)$.

One also learns by looking at these higher-order corrections that the effective expansion parameter is not λ^2 but rather the dimensionless combination $\lambda^2\rho$ with ρ given by (3.8).

The calculation of the λ^4 contribution to the energy involves nothing more than some additional algebraic complexity. The result is

$$\begin{aligned} \lambda^4 E_2 &= \frac{1}{R}(\lambda^2\rho)^2 \frac{1}{18}\frac{\Omega_0}{(\Omega_0-1)^2}\left(1 - \tfrac{5}{3}\Omega_0 + \tfrac{2}{3}\Omega_0{}^2 - \frac{1}{4}\frac{1}{\Omega_0 - 1}\right) \\ &= (0.0144)\frac{1}{R}(\lambda^2\rho)^2 . \end{aligned} \tag{3.21}$$

So, to this order, the energy is

$$E = (1/R)[\,2.0428 - 0.1632(\lambda^2\rho) + 0.0144(\lambda^2\rho)^2\,] . \tag{3.22}$$

This function does have a minimum, but it occurs at the unreasonably large value of $\lambda^2\rho = 5.67$.

The value of $\lambda^2\rho$ which gave a minimum of $\kappa(\Omega)$ in the exact cavity solution was

$$\lambda^2\rho = 0.1813 . \tag{3.23}$$

If we regard this as a typical value even for our proton-type quark state and evaluate E we get

$$E = \frac{1}{R}[\,2.014 + O(\lambda^2\rho)^3\,]$$

as compared to $1.93/R$ in the exact solution. There does not seem to be much point in carrying the expansion further. We have learned that reasonable solutions seem to exist, at least to the cavity problem, but that they involve all values of the angular momentum. As λ is increased from zero, the energy decreases as it does in the exact solution. However, since we know only the first few terms in a power-series expansion in λ, it would be sheer speculation to suggest that there exists a stable value of $E(\lambda)$, and that it gives the same value of E as did the minimum of the exact solution in the previous section.

IV. DISCUSSION

One of the interesting features of our phenomenological Lagrangian for pion emission is the absence of any coupling constant. (λ^μ is a dynamical variable fixed by an equation of constraint.) In an exact solution, we could see this explicitly: σ_0, the only free parameter in the solution, was fixed by minimizing the energy. Thus, in principle, the strength of the strong interactions, the forces responsible for hadronic decay, is calculable.

The difficulty we face in estimating this strength is that we only have a perturbative solution with realsitic quantum numbers, and the mechanism is definitely nonperturbative. We can, of course, develop any transition amplitude in a power series in $1/\sigma_0$: For example, the amplitude for decay of a baryon via one-pion emission may be extracted from the Fourier transform of a suitable Green's function:

$$\langle G|\,T[\,:\bar\psi^3(x):\pi_i(y):\psi^3(z):]\,|G\rangle = \int\int \mathfrak{D}\psi\mathfrak{D}\bar\psi\mathfrak{D}\pi\mathfrak{D}\sigma\mathfrak{D}R\,\bar\psi^3(x)\pi_i(y)\psi^3(z) \times \exp\left\{i\int_{-\infty}^{\infty}dt\int d^3x\left[\theta_R(x)(\tfrac{1}{2}i\bar\psi\gamma\cdot\overleftrightarrow{\partial}\psi - B) + \delta_S(x)\frac{\bar\psi(\sigma+i\vec\tau\cdot\vec\pi\gamma_5)\psi}{2(\sigma^2+\vec\pi^2)^{1/2}} - \tfrac{1}{2}[(\partial_\mu\vec\pi)^2+(\partial_\mu\sigma)^2]\right]\right\}.$$

One can calculate this Green's function in the cavity approximation (R a sphere of constant radius) by shifting $\sigma\to\sigma+\sigma_0$ and expanding in a power series in $1/\sigma_0$. σ_0 would then be fixed by requiring

$$\langle\psi|\sigma(x)|\psi\rangle \xrightarrow[|\vec x|\to\infty]{} 0.$$

The amplitude for single-pion emission in lowest order can be obtained more quickly from the equation of motion

$$(\partial^2 - m_\pi^2)\langle B'|\,\pi_i(x)|B\rangle = \frac{i}{2\sigma_0}\delta(r-R)\langle B'|\bar\psi\vec\tau\gamma_5\psi|B\rangle + O(1/\sigma_0^2),$$

which we can use directly in the reduction formula:

$$\langle B'\vec k i\,|B\rangle = -i\int d^4z\,\frac{e^{-ik\cdot z}}{[2\omega_\pi(\vec k)]^{1/2}(2\pi)^{3/2}}(\partial^2-m_\pi^2)\langle B'|\,\pi_i(z)|B\rangle = \frac{R^2}{2\sigma_0}\,\frac{2\pi\,\delta(m_{B'}+m_\pi-m_B)}{[2\omega_\pi(\vec k)]^{1/2}(2\pi)^{3/2}}\int d\Omega\,e^{-i\vec k\cdot\vec z}\langle B'|\bar\psi(z)\tau_i\gamma_5\psi(\vec z)|B\rangle,$$

where k and i are the momentum and isospin labels of the emitted pion. We are neglecting recoil and take the mass difference $m_B - m_{B'}$ from experiment. We calculate $\langle B'|\bar\psi(\vec z)\tau_i\gamma_5\psi(\vec z)|B\rangle$ in the static bag model in the approximation that gluon interactions are neglected. Keeping only the creation and annihilation operators of the quarks in the lowest mode ($\omega\cong\Omega/R$) we have

$$\int d\Omega\,e^{-i\vec k\cdot\hat r R}\bar\psi(R\hat r)\tau_i\gamma_5\psi(R\hat r) = \frac{2x^2 j_0(\Omega)j_1(\Omega)j_1(kR)}{1-j_0^2(\Omega)}\,b_m^\dagger\tau_i b_{m'}u_m^\dagger\vec\sigma\cdot\hat k u_{m'},$$

where u_m is a two-component Pauli spinor.

As an example, we calculate the width for

$$\Delta^{++}(J_z=\tfrac{3}{2})\to p+\pi^+.$$

We obtain

$$\Gamma_\Delta = \frac{k}{6\pi}\left(\frac{x^2 j_0(\Omega)j_1(\Omega)j_1(kR)}{(1-j_0^2(\Omega))\sigma_0 R}\right)^2 \times\sum_{i,a}\left|\sum_{mm'}\langle p|b_m^\dagger\tau_i b_{m'}|\Delta^{++}\rangle\,u_m^\dagger\sigma^a u_{m'}\right|^2 = \frac{k}{3\pi(\sigma_0R)^2}j_1^2(kR)\frac{4x^4j_0^2(\Omega)j_1(\Omega)^2}{(1-j_0^2(\Omega))^2}.$$

From the experimental value of the Δ^{++}-p mass difference, we have

$$k = \Delta M\left[1-\left(\frac{m_\pi^2}{\Delta M}\right)^2\right]^{1/2} \cong 265\text{ MeV}.$$

If we used massless pions k would be 300 MeV. While we have evaluated the transition amplitude in lowest-order perturbation theory, it is amusing to try to take nonperturbative effects into account in the following crude way: We take Ω, R, and $\sigma_0 R$ as evaluated with our exact solution appropriate to a three-quark system. We find

$$\Omega = 1.74,$$

$$\frac{1}{4\pi(\sigma_0 R)^2} = 0.182,$$

$$4\pi BR^4 = 3(1.54).$$

We fix B by fitting the average ΔN mass $\cong 1180$ to obtain

$$(4\pi B)^{1/4} = 236 \text{ MeV}.$$

Inserting all these values into the expression for Γ_Δ we have

$$\Gamma_\Delta \cong 50 \text{ MeV}.$$

The experimental value is 120 MeV. Since we performed a very crude zero-parameter calculation, it is perhaps interesting that the result has the correct order of magnitude.

One can also use our model to estimate the effects of pion emission on various bag-model calculations, for example on the value of g_A. The most important result of these estimates is that the effects seem to be small, which lends support to the neglect of surface fluctuations.

Finally, we can inquire into the value of F_π predicted by our model. Unfortunately, at the level of approximation to which we are working, there does not seem to be an unambiguous way of calculating F_π directly. Nevertheless, we can make the following estimate, which would hopefully be borne out by a more sophisticated calculation.

We observe that in our model we have a conserved axial-vector current, and hence a Goldberger-Treiman relation:

$$F_\pi \propto \frac{g_A}{g_{N\pi}}.$$

Our Δ-width calculation, as compared to the conventional SU(6) calculation using the experimental value of $g_{N\pi}$,[8] tells us that our value of $g_{N\pi}$ is about $1/\sqrt{3} = 0.58$ what it should be. In addition, we know[1] that the bag model predicts a value of $g_A = 1.09$, which is 0.88 what it should be. Thus it is plausible to assume that the value of F_π predicted by our model is of the order of 1.5 times the experimental value.

While this work was in progress we became aware of a report by T. Inoue and Toshide Maskawa[9] in which they consider a chiral-invariant bag model similar to ours but for isoscalar pions. However, they do not study the solutions of that model.

ACKNOWLEDGMENTS

We would like to thank Steve Ellis for stimulating conversations. We would also like to thank our colleagues at the Center for Theoretical Physics for their interest and encouragement.

APPENDIX A

In this appendix we would like to develop a Lagrangian formalism for the full bag theory involving quarks interacting with color gluons. We propose the following action:

$$W = \int_{\text{bag}} d^4x\big[\tfrac{1}{2} i\bar{\psi}\gamma\cdot \overleftrightarrow{D}\psi - \tfrac{1}{4}F_{\mu\nu a}F^{\mu\nu a} - B + \xi_a \partial_\mu (A_a^\mu \bar{\psi}\psi)\big].$$

Since the last term is a total divergence, it only affects the boundary conditions and the field equations are the usual ones. The boundary conditions are

$$-\tfrac{1}{2}n\cdot\gamma\psi + n\cdot\xi_a A_a \psi = 0, \tag{A1}$$

$$n_\mu F_a^{\mu\nu} + \xi_a \bar{\psi}\psi = 0, \tag{A2}$$

$$-\tfrac{1}{4}F_{\mu\nu a}F_a^{\mu\nu} - B + \xi_a\partial_\mu A_a^\mu \bar{\psi}\psi + \xi_a A_a^\mu \partial_\mu \bar{\psi}\psi = 0. \tag{A3}$$

Since $(in\cdot\gamma)^2 = 1$, (A1) implies

$$4(n\cdot A_a \xi^a)^2 = 1,$$

$$\bar{\psi}\psi = 0,$$

from which we can write, choosing $2n\cdot A_a \xi_a = +1$ and n_μ the inward normal,

$$in\cdot\gamma\psi = \psi,$$

$$n_\mu F_a^{\mu\nu} = 0,$$

$$-\tfrac{1}{4}F_{\mu\nu a}F_a^{\mu\nu} - B + \tfrac{1}{2}n\cdot\partial\bar{\psi}\psi = 0,$$

which are the usual bag equations. The only difference is that the gauge must be selected in such a way that

$$2n\cdot A_a\xi_a = +1$$

at the boundary. No physical quantities depend on the value of ξ_a. However, the convention for the sign in the fermion boundary condition is now linked to the choice of gauge. One can of course free oneself from this link by calling $\xi_a A_a^\mu = \lambda^\mu$ a free Lagrange multiplier. If one neglects the gluon coupling and sets $F_a^{\mu\nu} = 0$ one must choose $\lambda^\mu = \partial^\mu\Lambda$ in the former case, λ^μ arbitrary in the latter.

APPENDIX B

The purpose of this appendix is to discuss in somewhat greater detail the formulas for the energy and the radius of the bag for the exact solutions of Sec. II.

As remarked in Sec. II, it is most convenient to treat Ω as the independent parameter, and to calculate both σ_0 (or equivalently f_0) and the energy as functions of Ω. From Eqs. (2.15), (2.21), and (2.24) we have, with $g \equiv g(R)$,

$$g = f_0 y - 1/\xi \tag{B1}$$

and

$$g = -d\xi, \tag{B2}$$

where $\xi = 1/(f_0^2 + g^2)^{1/2}$ and $d \equiv \frac{1}{3} R y j_0^2(\Omega) v^\dagger v$. From (B1),

$$g = -\frac{f_0(1-y^2)}{2y}$$

and

$$\frac{1}{\xi} = f_0 \frac{1+y^2}{2y}.$$

Therefore, from (B2),

$$f_0^2 = \frac{4y^2}{1-y^4} d. \tag{B3}$$

From (2.17),

$$\begin{aligned}\beta^2 &= \frac{9g^2}{R^2} \\ &= \frac{9}{R^2} \frac{1-y^2}{1+y^2} d \\ &= \frac{3}{4\pi R^4} \frac{x^2 j_0^2(\Omega)}{1-j_0^2(\Omega)} \frac{y(1-y^2)}{1+y^2},\end{aligned}$$

where we have used the value (2.23) for $v^\dagger v$. Similarly,

$$\begin{aligned}\alpha^2 &= \tfrac{1}{4} \xi^2 j_0^4 (1-y^2)^2 (v^\dagger v)^2 \\ &= \frac{3}{4} \frac{1}{4\pi R^4} \left(\frac{\Omega^2 j_0^2}{1-j_0^2}\right) \frac{1}{y} \left(\frac{1-y^2}{1+y^2}\right) (1-y^2)^2.\end{aligned}$$

Thus

$$\begin{aligned}E_{\pi\sigma} &= \frac{1}{R}[2\pi R^4(\alpha^2 + \tfrac{1}{3}\beta^2)] \\ &= \frac{1}{8R}\left(\frac{\Omega^2 j_0^2}{1-j_0^2}\right) \frac{1-y^2}{y(1+y^2)} (3 - 2y^2 + 3y^4) \\ &= \frac{1}{8R}(1-y^2) A_1(\Omega) A_2(\Omega),\end{aligned}$$

where

$$\begin{aligned}A_1(\Omega) &\equiv 3 - 2y^2 + 3y^4, \\ A_2(\Omega) &\equiv \frac{\Omega^2 j_0^2}{1-j_0^2} \frac{1}{y(1+y^2)}.\end{aligned}$$

The quantity

$$F(\Omega) \equiv \frac{1}{4\pi\sigma_0^2 R^2}, \quad \text{where } \sigma_0 = f_0 + \alpha R$$

can easily be shown to be

$$F(\Omega) = \frac{12(1-y^2)}{A_1^2(\Omega) A_2(\Omega)}. \tag{B4}$$

We also need to calculate $\tau(\Omega)$. Equation (2.4) can be written as

$$\frac{\partial}{\partial r}[\bar\psi(\sigma + i\vec\tau \cdot \vec\pi\gamma_5)\psi] = -\frac{2B}{\xi}, \tag{B5}$$

where

$$\begin{aligned}\frac{\partial\sigma}{\partial r} &= \frac{1}{2}\left(\frac{\partial\sigma_{\text{in}}}{\partial r} + \frac{\partial\sigma_{\text{out}}}{\partial r}\right) = \tfrac{1}{2}\alpha, \\ \frac{\partial\vec\pi}{\partial r} &= \frac{1}{2}\hat r\left(\frac{\partial g_{\text{in}}}{\partial r} + \frac{\partial g_{\text{out}}}{\partial r}\right) = \tfrac{1}{6}\beta\hat r = -\frac{1}{2}\frac{g}{R}\hat r.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial}{\partial r}[\bar\psi(\sigma + i\vec\tau\cdot\vec\pi\gamma_5)\psi] = (v^\dagger v) j_0^2 \frac{1}{R}\{&[\tfrac{1}{2}\alpha R(1-y^2) - gy] \\ &+ [4f_0(y^2 - \Omega y) \\ &+ 2g\Omega(1-y^2) - 4gy]\},\end{aligned}$$

where the first term in square brackets comes from differentiating the π and σ fields and the second from differentiating the ψ field. Using

$$\frac{\alpha R}{f_0} = \frac{3}{4} \frac{(1-y^2)^2}{y^2}$$

and

$$g = \frac{-f_0(1-y^2)}{2y},$$

the first term becomes

$$\begin{aligned}v^\dagger v j_0^2 \frac{1-y^2}{8R} \frac{f_0}{y^2} A_1(x) &= \frac{1}{4\pi R^4}\left(\frac{\Omega^2 j_0^2}{1-j_0^2}\right)\frac{1}{8}\frac{f_0}{y^2} A_1(\Omega)(1-y^2) \\ &= \frac{1}{2\pi R^4}\frac{1}{\xi}[\tfrac{1}{8}(1-y^2)A_1 A_2].\end{aligned}$$

The second term is

$$\frac{f_0}{y}(1+y^2)[2y - (1+y^2)\Omega]\left(\frac{\Omega^2 j_0^2}{1-j_0^2}\right)\frac{1}{4\pi R^4}.$$

But using the explicit form $y = 1/\Omega - \cot\Omega$ one can show that

$$\frac{\Omega^2 j_0^2}{1-j_0^2}[\Omega(1+y^2) - 2y] = \Omega$$

and therefore the second term is $-(1/2\pi R^4)(\Omega/\xi)$. Therefore, the second boundary condition (B5) is

$$\begin{aligned}4\pi BR^4 &= \Omega - \tfrac{1}{8}(1-y^2)A_1 A_2 \\ &\equiv \tau(\Omega) \\ &= (E_\psi - E_{\pi\sigma})R.\end{aligned}$$

Finally, we shall show that $F(\Omega)$ and $\kappa(\Omega)$ have coincident extrema, as do $F(\Omega)\tau^{1/2}$ and $E(\Omega)$. To this end it is helpful to have the formula

$$\frac{dy}{d\Omega}=\frac{1-j_0^{\,2}}{\Omega^2 j_0^{\,2}}, \tag{B6}$$

which can easily be proved from the explicit form of y. It then follows that

$$\frac{d}{d\Omega}\left[\frac{(1-y^2)^2}{A_1(\Omega)}\right]=-\tfrac{2}{3}F(\Omega). \tag{B7}$$

Now

$$A_2(\Omega)=\frac{12(1-y^2)}{A_1^{\,2}(\Omega)F(\Omega)}$$

from (B4), and therefore

$$\begin{aligned}\kappa(\Omega)&=\Omega+\tfrac{1}{8}(1-y^2)A_1(\Omega)A_2(\Omega)\\&=\Omega+\tfrac{3}{2}\left[\frac{(1-y^2)^2}{A_1}\right]\frac{1}{F(\Omega)}.\end{aligned}$$

So

$$\frac{d\kappa}{d\Omega}=\frac{F'(\Omega)}{F(\Omega)}(\Omega-\kappa), \tag{B8}$$

and therefore κ' and F' vanish together, as required. Furthermore, since $\tau=2\Omega-\kappa$,

$$\frac{d\tau}{d\Omega}=2+\frac{F'}{F}(\Omega-\tau). \tag{B9}$$

The condition that

$$\frac{d}{d\Omega}(F\tau^{1/2})=0$$

is

$$\frac{F'}{F}=-\frac{1}{2}\frac{\tau'}{\tau}. \tag{B10}$$

The condition that

$$\frac{dE}{d\Omega}=\frac{d}{d\Omega}\left[\frac{\kappa+\frac{1}{3}\tau}{\tau^{1/4}}\right]=0$$

is

$$\tau'=\frac{4\tau}{\Omega+\tau}. \tag{B11}$$

But substitution of (B9) into (B11) reveals that (B10) and (B11) are the same condition, which was to be proved.

---

*Work supported in part through funds provided by ERDA under Contract No. AT(11-1)-3069.

†Alfred P. Sloan Foundation Fellow.

[1]A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. D 9, 3471 (1974); A. Chodos, R. L. Jaffe, K. Johnson, and C. B. Thorn, *ibid.* 10, 2599 (1974); T. DeGrand, R. L. Jaffe, K. Johnson, and J. Kiskis, *ibid.* 12, 2060 (1975).

[2]S. Mandelstam, Nucl. Phys. B56, 109 (1973).

[3]C. B. Thorn and M. V. K. Ulehla, Phys. Rev. D 11, 3531 (1975).

[4]W. A. Bardeen and S. D. Ellis, private communication.

[5]S.-J. Chang, S. D. Ellis, and B. W. Lee, Phys. Rev. D 11, 3572 (1975).

[6]B. Freedman and R. L. Jaffe, private communication.

[7]This Lagrange-multiplier technique for reproducing the fermion boundary conditions was also suggested by P. Ramond (private communication through C. Rebbi).

[8]See for example, J. J. J. Kokkedee, *The Quark Model* (Benjamin, New York, 1969), pp. 71–73. The quoted value for Γ_Δ is 80 MeV. However, this includes recoil effects, which we have ignored. If one repeats the calculation completely ignoring recoil, this value changes to 150 MeV, which is the one we use in estimating $g_{NN\pi}$. If we used an experimental width of 120 MeV or 80 MeV our value for f_π would improve.

[9]T. Inoue and T. Maskawa, Kyoto University Report No. KUNS 313, 1975 (unpublished).