Comment on vacuum polarization and the absence of free quarks in four dimensions

S. Gasiorowicz*

School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455

(Received 19 May 1975)

An Abelian model of electric flux line confinement due to Kogut and Susskind is examined, and certain novel features are pointed out. The nonlinear differential equations are studied by means of the *phase-plane method*, which shows that the size of the confining tube cannot be determined by the linear approximation.

In a very interesting paper Kogut and Susskind[1] discuss the connection between vacuum polarization and quark confinement. To avoid the morass of a non-Abelian gauge theory, the authors construct a phenomenological field theory designed to simulate the anomalous dielectric properties of the vacuum in an Abelian theory. Their discussion leads them to the field energy

$$W=\int d^3r\left[\frac{1}{2\epsilon(\phi)}\vec{D}^2+\tfrac{1}{2}(\vec{\nabla}\phi)^2+V(\phi)\right], \tag{1}$$

where D is the electrical induction and[2] ϕ is a scalar field whose sole purpose is to give the desired structure to the vacuum. There is a point charge at the origin, so that for a spherical distribution of flux lines

$$\vec{D}=\frac{g}{4\pi}\frac{\hat{l}_r}{r^2} \tag{2}$$

and

$$W=4\pi\int r^2dr\left[\left(\frac{g}{4\pi}\right)^2\frac{1}{2r^4\epsilon(\phi)}+\frac{1}{2}\left(\frac{d\phi}{dr}\right)^2+V(\phi)\right]. \tag{3}$$

The "dielectric constant" and the potential are to be chosen such that the field energy for the spherical distribution grows with the radius of the volume fast enough ($R^{5/3}$ in the Kogut-Susskind model) to favor a cylindrical distribution. For such a distribution, far from the source, the field energy per unit length for perfectly parallel flux lines, with azimuthal symmetry is

$$W=\int 2\pi\rho\, d\rho\left[\frac{1}{2\epsilon(\phi)}D^2(\rho)+\frac{1}{2}\left(\frac{d\phi}{d\rho}\right)^2+V(\phi)\right]. \tag{4}$$

The constraint of flux conservation reads

$$\int 2\pi\rho\, d\rho\, D(\rho)=g. \tag{5}$$

The minimization of the field energy subject to the constraint leads to

$$\delta\left(W-\lambda\int 2\pi\rho\, d\rho\, D(\rho)\right)=\int 2\pi\rho\, d\rho\left[\left(\frac{D}{\epsilon(\phi)}-\lambda\right)\delta D(\rho)-\frac{1}{2}\frac{D^2}{\epsilon^2}\frac{d\epsilon}{d\phi}\delta\phi-\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\phi}{d\rho}\right)\delta\phi+\frac{dV}{d\phi}\delta\phi\right]=0, \tag{6}$$

from which we read off

$$D=\lambda\epsilon(\phi) \tag{7}$$

and

$$\frac{d^2\phi}{d\rho^2}+\frac{1}{\rho}\frac{d\phi}{d\rho}+\frac{d}{d\phi}\left[\tfrac{1}{2}\lambda^2\epsilon(\phi)-V(\phi)\right]=0. \tag{8}$$

The initial condition

$$\left(\frac{d\phi}{d\rho}\right)_{\rho=0}=0 \tag{9}$$

is necessary to make everything regular. Kogut and Susskind impose the condition that[3]

$$\lim_{\rho\to\infty}\phi(\rho)=0, \tag{10}$$

which then leads to an exponential falloff of flux outside of a small region around the z axis. The "size" of the flux tube is determined by the term linear in ϕ in $(d/d\phi)[\frac{1}{2}\lambda^2\epsilon(\phi)-V(\phi)]$. We argue that *it is not always possible to impose condition (10),* and that any constraints on the size of the flux tube depend on various parameters of the quantity

$$U(\phi)\equiv\tfrac{1}{2}\lambda^2\epsilon(\phi)-V(\phi) \tag{11}$$

in a nonlinear way. The differential equation

$$\phi''+\frac{1}{\rho}\phi'+\frac{dU(\phi)}{d\phi}=0 \tag{12}$$

may, after multiplication by $2\phi'$, be rewritten in the form

$$\frac{d}{d\rho}\left[\phi'^2+2U(\phi)\right]=-\frac{2}{\rho}\phi'^2\leq 0. \tag{13}$$

(Incidentally, in one-dimensional field theories the right-hand side is absent, and that is how one can get solutions in closed form by quadrature.) Thus a solution of the differential equation, when plotted in the (ϕ',ϕ) plane, the *phase plane,*[4] starts on the ϕ axis at "time" $\rho=0$ and proceeds to smaller values of

$$K = \phi'^2 + 2U(\phi). \tag{14}$$

Kogut and Susskind choose $\epsilon(\phi)$ and $V(\phi)$ in the simplest possible manner to disfavor spherical flux distributions, and end up with

$$2U(\phi) = a\phi^4 + b\phi^2. \tag{15}$$

It is clear that stability demands that the coefficient of the highest power of ϕ must be positive, and that the highest power be even. Depending on the sign of b the origin may or may not be a minimum. *Whatever the location of the minimum,* it only is a minimum if

$$\left(\frac{d^2U}{d\phi^2}\right)_{\min} > 0, \tag{16}$$

where the minimum is determined by the condition

$$\left(\frac{dU}{d\phi}\right)_{\min} = 0. \tag{17}$$

Thus the linearized version of the differential equation (12) is, with $\hat{\phi} = \phi - \phi_{\min}$,

$$\hat{\phi}'' + \frac{1}{\rho}\hat{\phi}' + \hat{\phi}\left(\frac{d^2U}{d\phi^2}\right)_{\min} = 0, \tag{18}$$

and the sign condition in (16) shows that the falloff there is not exponential. In the phase plane the solution "spirals" into the minimum point. The size of the flux tube depends on how rapidly the solution approaches the minimum point, and this depends on the "initial condition" $\phi(0)$. With an appropriate rescaling of ϕ and ρ, Eq. (13) may be rewritten in the form

$$\frac{d}{dx}[\psi'^2 + (\psi^2 \pm 1)^2] \leq 0, \tag{19}$$

provided the parameter a in (15) is positive. The upper sign corresponds to $b > 0$ in (15). With that sign the origin is the minimum point. With the choice $b < 0$ of Kogut and Susskind, there are two minimum points, and the solution spirals into one or the other of them, depending on $\psi(0)$. The origin can only be an asymptotic solution for certain discrete $\psi(0)$ (see Ref. 5) and such solutions are highly unstable, since the smallest variation in the initial condition will send the solution into one or the other minima. Such initial values seem to act as boundaries between different "phases" of solutions. It is not the coupling constants that can take "critical" values, but these parameters.[6]

There does not seem to be any difficulty in constructing separate $\epsilon(\phi)$ and $V(\phi)$ such that, for a given $U(\phi)$, the cylindrical geometry is favored. The equation $\delta W/\delta\phi = 0$ for the spherical geometry reads

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} - \frac{\partial V}{\partial\phi} + \frac{1}{2}\left(\frac{g}{4\pi}\right)^2 \frac{1}{r^4}\frac{1}{\epsilon^2(\phi)}\frac{\partial\epsilon}{\partial\phi} = 0. \tag{20}$$

With

$$V(\phi) = \alpha\phi^2 + \beta\phi^4, \qquad \epsilon(\phi) = a\phi^2 + b\phi^4 + \cdots, \tag{21}$$

and the assumed behavior

$$\phi(r) \underset{r\to\infty}{\sim} Ar^{-p}, \tag{22}$$

all one needs to do is find parameters in (21) such that the behavior of W for large R, where R is the radius of the spherical volume over which the integral in (3) is taken, satisfies $W \sim R^k$, $k > 1$. For example, for $a = 0$, with $b\alpha > 0$, $p = \frac{2}{3}$ and $W \sim R^{5/3}$ as was found in Ref. 1.

These comments do not, in any way, invalidate any of the physical arguments behind the Kogut-Susskind model. Our purpose is to correct some details, draw attention to the possibility of different phases, corresponding to the different minima, and to remind those working in this field[7] of the usefulness of the phase-plane method for certain ordinary nonlinear differential equations.[8]

Note added in proof. One can take the point of view that condition (5) stabilizes the solutions, so that (10) is indeed satisfied. In that case one finds the interesting possibility of the existence of different families of flux tubes characterized by different values of the energy per unit length.[5] I wish to thank P. Kaus for a discussion on this subject.

*Work supported in part by ERDA Contract No. AT(11-1) 1764.

[1]J. Kogut and L. Susskind, Phys. Rev. D 9, 3501 (1974).

[2]Kogut and Susskind use the combination $\phi - \phi_0$ wherever we have ϕ.

[3]Kogut and Susskind write the condition as $\lim_{\rho\to\infty}\phi(\rho) = \phi_0$ asymptotically. The important thing is that $\epsilon(\phi) \to 0$ appropriately, so that the cylindrical geometry is preferred.

[4]R. Finkelstein, R. LeLevier, and M. Ruderman, Phys. Rev. 83, 326 (1951).

[5]These are the "particle-like" solutions of Ref. 4.

[6]The choice $b < 0$, interesting because of the possibility of different "phases," unfortunately leads to difficulties with the convergence of (5), when (7) is used.

[7]While on the subject of reminders, I would like to draw attention to a much neglected paper on the quantum corrections to nonlinear field equations by D. Yennie, Phys. Rev. 88, 527 (1952).

[8]The model of M. Creutz [Phys. Rev. D 10, 1749 (1974)] is also inconsistent before the "bag" limits are taken. There is no difficulty if the quark degrees of freedom are kept.