Reggeon field theory on a lattice*

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We construct an analog model of interacting Ising spins on a lattice which has the same critical behavior as Reggeon field theory, in the physical number of dimensions. At the critical point the total and elastic cross sections have asymptotic behaviors $\sigma_{\text{tot}} \propto (\ln s)^{-\gamma}$ and $\sigma_{\text{el}} \propto (\ln s)^{-2\gamma-z}$. By studying the properties of the fixed point of an explicit nonlinear renormalization-group transformation on the lattice, we show that $-\gamma \leq z \leq 2$, so that both the Froissart bound and the constraint $\sigma_{\text{el}} < \sigma_{\text{tot}}$ are satisfied.

I. INTRODUCTION

The Reggeon calculus[1] provides us with a model of diffraction scattering which explicitly satisfies the constraints of crossed-channel unitarity. In this model the behavior of the scattering amplitude at high energies and small momentum transfer is related to the infrared behavior of a field theory of self-interacting quasiparticles moving in two space dimensions. Recently[2,3] there has been considerable progress in determining this behavior using the techniques of the renormalization group and the ϵ expansion, the number of space dimensions being generalized to $D=4-\epsilon$.

Before the interaction is taken into account, the quasiparticles have a dispersion relation of the form $E=-\Delta+\alpha' k^2$, where the "bare" trajectory is taken to be $\alpha(t)=1+\Delta+\alpha' t$. According to the ϵ expansion, there exists a critical value of Δ for which the Green's functions of the interacting theory scale in the infrared region. This leads to the scaling behavior of the elastic-scattering amplitude $F(s,k^2)$ at large s, small k^2:

$$F(s,k^2)=is(\ln s)^{-\gamma} f(k^2(\ln s)^z)\,, \tag{1.1}$$

which implies that the total and elastic cross sections behave as

$$\sigma_{\text{tot}} \propto (\ln s)^{-\gamma}\,, \quad \sigma_{\text{el}} \propto (\ln s)^{-2\gamma-z}\,. \tag{1.2}$$

This behavior occurs when the renormalized Δ is zero, and there is no energy gap in the spectrum. On the other hand, if Δ is less than its critical value Δ_c, $F(s,k^2)$ has an exponential falloff in $\ln s$ characteristic of a simple Regge pole with intercept below one.

The scaling behavior (1.1) is a signal that the theory undergoes a phase transition at $\Delta=\Delta_c$, and it is profitable to look at the problem from this viewpoint. In fact, Δ plays the role of inverse temperature in the sense that the system has a finite correlation length for $\Delta<\Delta_c$, which becomes infinite at the critical temperature.

The exponents γ and z have been calculated to $O(\epsilon^2)$ (see Refs. 4 and 5), but these expressions show no great sign of converging for $\epsilon=2$. The ϵ-expansion approach therefore leaves several questions unanswered:

(1) Is there a phase transition in $D=2$ dimensions? It is well known that dimensionality is crucial in such considerations.

(2) What is the nature of the ordered state? How do the correlation functions behave for $\Delta>\Delta_c$? It is important to decide whether short-range correlations again appear (a renormalized pole below one), or whether the Froissart bound becomes saturated for $\Delta \gg \Delta_c$, as suggested by some approximate calculations.[6]

(3) Do the exponents γ and z in $D=2$ satisfy the most elementary constraints of direct channel unitarity, namely, the Froissart bound ($-\gamma \leq 2$), and the condition $\sigma_{\text{el}} < \sigma_{\text{tot}}$ ($-\gamma - z \leq 0$)?

This last question is particularly interesting, since, while the Reggeon calculus explicitly satisfies crossed-channel unitarity, there is no explicit direct-channel unitarity manifested, although, with a suitable direct-channel interpretation of the bare Reggeon, it is clear that all possible absorptive effects are included in the model. These conditions are therefore crucial "experimental" tests of the strong-coupling solution of the Reggeon calculus.

An overwhelming problem in discussing these equations directly in $D=2$ is the large number of degrees of freedom of the system. Even if one replaces the space-time continuum by a discrete lattice (a process which does not change the infrared behavior) one is still faced with an infinite number of degrees of freedom at each lattice site. In this paper we show that this number can be reduced to two, namely that the critical behavior of the field theory is identical with that of a system of Ising spins interacting via certain well-defined interactions. This forms the content of Sec. II of this paper. The first step is to write down the generating functional for the Green's functions of the theory. Going onto the lattice, this functional integral becomes a conventional multiple integral. The crucial fact which allows us to evalute partially this integral is that as the dimensionless renormalized coupling g_R tends to the zero g_1 of the Gell-Mann–Low function $\beta(g)$, with Δ fixed at its critical value, then the bare coupling r tends to infinity.

This result, which was proved for the case of an infinite cutoff in Ref. 7, requires more careful analysis in our case, since the discrete nature of the lattice already implies a finite cutoff in the theory. This analysis is carried out in Appendix A, where we also show that the critical value of Δ is large when r is large. The significance of this result is that, by taking $r\to\infty$ and holding Δ at its critical value, we explore the scaling behavior of the theory. This maneuver has the effect of focusing our attention on values of the energy and momentum much less than the scale set by r^2 and α'.

For large values of Δ and r the multiple integral over the fields at each lattice site can be performed by the method of steepest descent. We find that there are two saddle points in the integration at each site. We label these by a variable $s=\pm1$, and integrate out the remaining variables. The kinetic terms in the Lagrangian then link values of s at neighboring lattice sites. We find that we have simplified the generating function of the field theory to the generating function for a system of Ising spins on a lattice with certain specific interactions. These interactions prove to have the same symmetry properties as the original Lagrangian, indeed, as a Lagrangian with arbitrarily complicated interactions; so, according to the folklore that it is the symmetry properties of a system, rather than the number of degrees of freedom, which determine its critical behavior, it is not surprising that our analog model is a good representation of the Reggeon field theory.

Having constructed this analog model, we may hope that it will succumb to the well-developed methods for dealing with lattice spin systems, and provide answers to the questions listed above. However, we have, so far, been unable to obtain a general proof of the existence of a phase transition in $D=2$, and must rely on approximate methods. In Sec. III, we construct an explicit nonlinear renormalization-group transformation on the lattice, which we shall assume has a fixed point with the general features of that suggested by the ϵ expansion. We then discuss the scaling laws implied by the existence of this fixed point, and the consequences for the exponents γ and z.

We find that the exponents calculated by this method will automatically satisfy the constraints of direct channel unitarity mentioned in (3) above. This is the main result of this paper.

II. THE ANALOG MODEL

We begin with the Reggeon field theory defined by the Lagrangian density

$$\mathcal{L}=\tfrac{1}{2}\psi^\dagger\overleftrightarrow{\partial}_t\psi+\alpha'|\nabla\psi|^2+V(\psi^\dagger,\psi), \tag{2.1}$$

where the potential $V(\psi^\dagger,\psi)$ is given by

$$V(\psi^\dagger,\psi)=(-\Delta)\psi^\dagger\psi-\tfrac{1}{2}ir\psi^\dagger\psi(\psi^\dagger+\psi). \tag{2.2}$$

As usual, we have included only a constant triple-Pomeron coupling in V, assuming all other couplings to be irrelevant in the scaling region. It will become apparent later that such an assumption is unnecessary, although we retain it initially to simplify the algebra. In (2.2) the bare Pomeron intercept is equal to $1+\Delta$; we shall work throughout with bare quantities, rendering the theory finite by a suitable cutoff in momentum. We are interested in the infrared behavior of the two-point function

$$\langle 0|T[\psi(t,x)\psi^\dagger(0,0)]|0\rangle,$$

which is proportional to the Fourier transform of the elastic amplitude, evaluated at rapidity t and impact parameter x.[8]

It is more convenient to write (1) and (2) in terms of the real and imaginary parts of $\psi=\varphi+i\chi$. Then, dropping a total time derivative,

$$\mathcal{L}=2i\varphi\partial_t\chi+\alpha'(\nabla\varphi)^2+\alpha'(\nabla\chi)^2+V, \tag{2.3}$$

where

$$V=-(\Delta+ir\varphi)(\varphi^2+\chi^2). \tag{2.4}$$

The generating functional for the Green's functions is

$$Z[J]=\int\delta\varphi\,\delta\chi\exp\left[-\int dt\,d^Dx(\mathcal{L}+J^\dagger\psi+J\psi^\dagger)\right]. \tag{2.5}$$

We shall initially take $J=0$. With the φ and χ contours running along the real axis, the integral in (2.5) is defined only for $\Delta<0$. However, with only a bare triple-Reggeon interaction, it can be shown that $\Delta_c>0$.[7] To continue into $\Delta>0$ we first distort the φ contour, as shown in Fig. 1 (solid curve). The new contour intersects the imaginary axis at $\varphi=ic$ $(c>0)$ and is asymptotic to lines making angles $\alpha<\pi/6$ with the real axis. We can now continue in Δ up to any value less than cr, for then $\mathrm{Re}(\Delta+ir\varphi)<0$ everywhere on the contour, and $\mathrm{Re}(ir\varphi^3)\to-\infty$ as $|\varphi|\to\infty$. The χ integral is kept along the real axis.

We now replace the continuum (t, x) space by a cubic lattice with spacing a_x and a_t in the space and time directions, respectively. Equation (2.5) is then replaced by

$$Z = \int \prod_i d\varphi_i d\chi_i \exp\left[-a_t a_x^D \sum_i \mathcal{L}(\varphi_i, \chi_i)\right], \tag{2.6}$$

where i labels the lattice points, and the derivatives ∂_t, ∇ are replaced by finite difference operators on the lattice (divided by a_t and a_x, respectively). We shall continue, however to use the same notation for convenience.

As explained in the Introduction, we explore the scaling region by taking the mathematical limit $\Delta, r \to \infty$. The integrals in (2.6) can then be evaluated by the method of steepest descent. To do this, we temporarily ignore the kinetic terms in $\mathcal{L}$, and search for saddle points in the φ, χ integrations at each lattice point.

Solving the equations $\partial V/\partial\varphi = 0$, $\partial V/\partial\chi = 0$ we find four candidates:

(a) $\varphi = 0, \quad \chi = 0$;

(b) $\varphi = 2i\Delta/3r, \quad \chi = 0$;

(c) $\varphi = i\Delta/r, \quad \chi = +\Delta/r$;

(d) $\varphi = i\Delta/r, \quad \chi = -\Delta/r$.

The point (a) is obviously dominant in the limit $\Delta \to -\infty$. However, when $\Delta > 0$ the points (a) and (b) are outside the domain of convergence in φ of the χ integral, and consequently the hypercontour cannot be distorted through them. The points (c) and (d) are, however, accessible, and it is a matter of straightforward algebra to show that the φ contour can be distorted in such a way (dashed line in Fig. 1) that, keeping χ on the real axis, the contribution from any portion of the contour is exponentially smaller than that from the region of the saddle points.

We then write, at each point i of the lattice,

$$\varphi_i = \frac{i\Delta}{r} + \varphi_i' , \tag{2.7}$$

$$\chi_i = \frac{\Delta}{r} s_i + \chi_i' , \tag{2.8}$$

where we have introduced the dichotomic variable $s_i = \pm 1$. Since the second derivatives of V at the saddle points are $O(\Delta)$, we can restrict $\varphi_i', \chi_i' \lesssim O(\Delta^{-1/2})$. Substituting (2.7) and (2.8) in both the kinetic and potential terms of $\mathcal{L}$ and retaining only second-order terms in φ_i' and χ_i', we obtain a multiple Gaussian integral which can be explicitly evaluated. This is carried out in Appendix B.

Keeping only the leading terms as $r, \Delta/r \to \infty$, and dropping an irrelevant constant, we obtain

$$Z = \sum_{s_i = \pm 1} \exp\Big\{-a_t a_x^D \sum_i \Big[\frac{\Delta^2\alpha'}{r^2}(\nabla s_i)^2 + \frac{2\Delta\alpha'}{r^2}(s_i \nabla^2 s_i)(\partial_t s_i)\Big]\Big\}. \tag{2.9}$$

It is fairly easy to see where the terms in (2.9) come from. The first term is just the leading contribution to the term $(\nabla\chi)^2$ in (2.3). However, the leading contribution to the term $2i\varphi\partial_t\chi$ is proportional to $\partial_t s$, which vanishes when summed over the lattice (with periodic boundary conditions). The first nonvanishing term involving $\partial_t s$ comes from diagonalizing the φ_i' and χ_i' integrations, which mixes up space and time. There is no term like $(\partial_t s)^2$, which is why the model is not like a conventional Ising model. This difference is traceable to the linear time derivative in (2.1), i.e., the linear dependence of the bare propagator on energy E.

We now want to interpret s_i as a spin at each site of the lattice, and Z as the partition function of the system. Denoting each lattice point i by coordinates $(\hat{t}, \hat{x})$ (which take integer values) we have (taking $D = 1$ for clarity)

$$(\nabla s)^2 = a_x^{-2}(s_{\hat{t},\hat{x}} - s_{\hat{t},\hat{x}+1})^2 = \text{const} - 2a_x^{-2} s_{\hat{t},\hat{x}} s_{\hat{t},\hat{x}+1} \tag{2.10}$$

$$(s\nabla^2 s)(\partial_t s) = \tfrac{1}{2}a_t^{-1}a_x^{-2} s_{\hat{t},\hat{x}}(s_{\hat{t},\hat{x}+1} - 2s_{\hat{t},\hat{x}} + s_{\hat{t},\hat{x}-1}) \times (s_{\hat{t}+1,\hat{x}} - s_{\hat{t}-1,\hat{x}}) \tag{2.11}$$

$$\simeq \tfrac{1}{2}a_t^{-1}a_x^{-2} s_{\hat{t},\hat{x}}(s_{\hat{t},\hat{x}+1} + s_{\hat{t},\hat{x}-1}) \times (s_{\hat{t}+1,\hat{x}} - s_{\hat{t}-1,\hat{x}}), \tag{2.12}$$

where in (2.12) we have dropped a term which

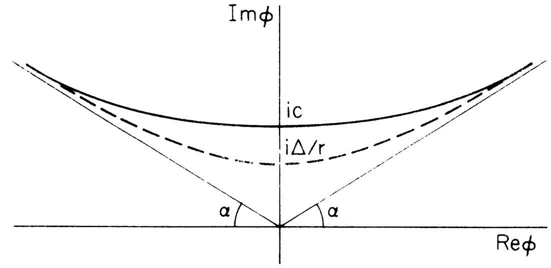

FIG. 1. Distortions of the φ contour. The heavy solid line is the contour for continuing into $\Delta > 0$. The dashed line is the contour through the saddle point.

sums to zero over the lattice. In writing (2.10) and (2.11) we have been careful to respect the symmetry properties of the terms, under spin flip (S), time reversal (T), and space reversal (P). Ignoring again a multiplicative constant, the partition function now takes the standard form

$$Z=\sum_{s_i} e^{-\mathcal{H}}, \tag{2.13}$$

with

$$\mathcal{H}=-K\sum_{\langle ij\rangle} s_i s_j \pm L\sum_{\langle ijk\rangle} s_i s_j s_k\,, \tag{2.14}$$

where the first term represents a sum over nearest-neighbor pairs in the space directions, and the second over groups of three neighboring spins in any of the configurations shown in Fig. 2, those of the first two taking the plus sign, and the second two the minus sign. The couplings K and L are given (when $D=2$) in terms of the original parameters

$$K=\frac{2\Delta^2\alpha'}{r^2}a_t\,,\quad L=\frac{\Delta\alpha'}{r^2}\,. \tag{2.15}$$

The form (2.14) for the Hamiltonian may seem somewhat arbitrarily dependent on our initial Lagrangian (2.1) and the mechanics of the saddle-point integration. However, this is not so, for it is easily seen that $\mathcal{L}$ is invariant under the transformation ($t\to -t$, $\chi\to-\chi$), which is equivalent to the transformation TS on our spin model. Also the only time-dependent term in $\mathcal{L}$ changes sign under T alone. Of course $\mathcal{L}$ is also invariant under $P(x\to -x)$. A little thought then shows that (2.14) is the simplest nearest-neighbor interaction between simple Ising spins which has the same symmetry properties as $\mathcal{L}$. In fact, any Reggeon field theory with a linear bare trajectory has these symmetry properties; no matter how complicated the couplings. To the extent that the symmetry properties, rather than the number of degrees of freedom, characterize the critical behavior of a system, we may then believe that our spin system has the same critical behavior as an arbitrary Reggeon field theory. It addition, our saddle-point method gives a proof of this connection when only a constant triple-Pomeron coupling is present, and other couplings can be shown to be irrelevant in the ϵ expansion.

Having established this connection for Z we turn to the Green's functions. In this paper we shall just consider the two-point function

$$\langle 0|T[\psi(t,x)\psi^\dagger(0,0)]|0\rangle$$

which (at least in the ϵ expansion) gives the leading contribution to the elastic amplitude when particles are coupled to the theory. Calculating this directly is complicated since our Lagrangian in (5) is not normal-ordered and there are tadpole-cancellation terms in φ to be added. In addition φ takes on a nonzero vacuum expectation value when we perform the saddle-point calculation. This can be avoided by observing that, for $t\geq 0$,

$$\begin{aligned}\langle 0|\chi(t,x)\chi(0,0)|0\rangle &= -\tfrac{1}{4}\langle 0|[\psi^\dagger(t,x)-\psi(t,x)]\\ &\qquad\times[\psi^\dagger(0,0)-\psi(0,0)]|0\rangle\\ &=\tfrac{1}{4}\langle 0|\psi(t,x)\psi^\dagger(0,0)|0\rangle\,,\end{aligned} \tag{2.16}$$

since ψ annihilates the vacuum. Now the left-hand side is equal to

$$Z^{-1}\int \delta\varphi\,\delta\chi\,\chi(t,x)\chi(0,0)\exp\left[-\int dt\,d^Dx\,\mathcal{L}(\varphi,\chi)\right], \tag{2.17}$$

and so we can perform the same saddle-point calculation, keeping only the leading terms in $\chi(t,x)$ and $\chi(0,0)$ to give

$$\langle 0|T[\psi(t,x)\psi^\dagger(0,0)]|0\rangle \propto \langle s_{t,x}s_{0,0}\rangle_{\mathcal{H}}\,. \tag{2.18}$$

We have therefore reduced the system with a doubly infinite number of degrees of freedom at each lattice site to a system with just two; this considerably simplifies the construction and approximate evaluation of an explicit renormalization-group transformation.

III. THE RENORMALIZATION GROUP ON THE LATTICE

In this section we discuss the construction of an explicit renormalization-group transformation on the lattice, and its relation to the critical exponents of the Reggeon field theory. Our discussion follows that of Niemeyer and van Leeuwen[9] for

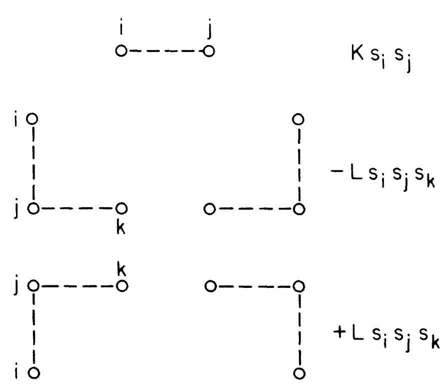

FIG. 2. Couplings on the lattice, shown in $D=1$ for simplicity. Space runs across, and time up, the page.

the conventional Ising model, which is an explicit realization of the ideas of Kadanoff[10] and Wilson.[11]

We group the sites into cells, which consist of a finite number of sites, and have the same periodic structure as the original lattice. For the moment we shall take these cells to be cubes (in $D=2$), each containing eight sites. With each cell we associate a spin s' which takes the values ± 1, according to whether the sum of all the site spins in the cell is positive or negative. The configurations when the sum is zero are assigned either to $s'=+1$ or $s'=-1$ in such a way that changing all site spins transforms a configuration with $s'=+1$ into one with $s'=-1$.[12] The site spins can now be relabeled by the cell spin s' plus an internal cell variable σ (which in our case will take 2^7 values) in such a way that changing s' while keeping σ fixed corresponds to flipping all site spins in the cell. We now define a new Hamiltonian $\mathcal{H}'$ for the lattice of cells by summing over the internal variables σ. This is determined by requiring that the partition function be unchanged. Labeling cells by i', we have explicitly

$$e^{-\mathcal{H}'(s')-Ng}=\sum_{\sigma_i} e^{-\mathcal{H}(s)}\ . \tag{3.1}$$

The factor Ng, which is proportional to the total number of cells and is independent of the s', always appears and can be identified with the self-energy of each cell. The transformed Hamiltonian $\mathcal{H}'$ will depend on new couplings K', L' and inevitably long-range couplings (next-nearest neighbor, etc.) will be induced. If we are careful to respect the symmetry properties of $\mathcal{H}$ in defining the transformation then these new couplings will have the same symmetry properties as K and L. In the analysis below we shall ignore them, although they may be included, following the general analysis given in Ref. 8, without changing our results. In numerical work, they must be included.

The infrared behavior of the theory is determined by the fixed points $\mathcal{H}^*$ of the transformation (3.1), and the approach to these points. What type of fixed points do we expect to find in our model? Evidently there is one at $L^*=0$, $K^*\neq 0$, which is simply the two-dimensional Ising fixed point. This is not of interest here. Recalling that the renormalized Pomeron intercept is equal to one when Δ takes on a critical value (proportional to r^2/α'), we expect that the fixed point of interest will have some value $L^*\neq 0$. This we expect to be unstable in L, so that Δ has to be chosen exactly equal to its critical value for scaling to occur. However, we do not want to be forced to set K equal to a critical value, since it depends on the arbitrary parameter a_t. The fixed point of interest must then be stable in K. There are three possibilities: $K^*=0$, K^* finite, $K^*=\infty$. The last case would not be a sensible result since there is then no fixed point at finite L^*, and the model would have infinitely long-range correlations in x at fixed t. This would correspond to the scattering amplitude being nonanalytic at $k^2=0$ at finite energy. Such a behavior is inconsistent with the input ideas of the Reggeon calculus, and, moreover, does not arise in the ϵ expansion. We therefore dismiss this possibility, bearing in mind that it should be checked by direct computation.

We shall initially consider the case $K^*=0$, and proceed to discuss the consequences for the Reggeon field theory and its exponents. To do this, it is convenient to introduce a magnetic field term $H\sum_i s_i$ into $\mathcal{H}$. The fixed-point value of H is $H^*=0$, by symmetry. Near the fixed point (K^*, L^*, H^*) we assume, as is customary, that the renormalization-group transformation is linear. Since this transformation commutes with S and T, and the three couplings (K, L, H) have distinct symmetry properties under S and T,

$$\begin{aligned} &SK=K\,, \quad TK=K\,,\\ &SL=-L\,, \quad TL=-L\,,\\ &SH=-H\,, \quad TH=H\,, \end{aligned} \tag{3.2}$$

we see that the three couplings are mapped separately near the fixed point. So we can write

$$\begin{aligned} &K'=2^{z_1}K\,,\\ &L'-L^*=2^{z_2}(L-L^*)\,,\\ &H'=2^{z_3}H\,, \end{aligned} \tag{3.3}$$

where the numbers 2^{z_i} are the derivatives of the transformation at the fixed point. If the fixed point is stable in K then $z_1<0$. We expect the fixed point to be unstable in L and H, so that $z_2, z_3>0$.

To obtain the scaling behavior of the correlation function we take (3.1) and expand to second order in H on either side, identifying terms proportional to s_i', s_j', where i' and j' are two cells which are far apart relative to the lattice spacing,

$$H'^2 s_{i'}' s_{j'}'\, e^{-\mathcal{H}'(s')-Ng}=H^2\sum_{\sigma_k'}\left(\sum_{i'} s\right)\left(\sum_{j'} s\right)e^{-\mathcal{H}(s)}\,, \tag{3.4}$$

where $\sum_{i'} s$ denotes the sum over all the spins in the cell i'. The right-hand side of (3.4) is a sum of 2^6 terms, which are asymptotically equal for cell separations much greater than the site spacing. So

$$2^{2z_3-6}s'_{i'}s'_{j'}e^{-\mathcal{H}'(s')-Ng}=\sum_{\sigma'_k} s_{i'}^{(1)}s_{j'}^{(1)}e^{-\mathcal{H}(s)}\,, \tag{3.5}$$

where $s_{i'}^{(1)}$ denotes a typical site spin in the cell i'. If we now sum both sides over all values of the s', and divide by Z, we obtain the scaling law for the two-point function

$$\hat{G}(\hat{t},\hat{x},\mathcal{H})=2^{2z_3-6}\hat{G}\left(\frac{\hat{t}}{2},\frac{\hat{x}}{2},\mathcal{H}'\right)\,, \tag{3.6}$$

where $(\hat{t},\hat{x})$ are integers labeling points on the lattice. Repeating the transformation an arbitrary number of times and taking $L=L^*$, $H=0$, (3.6) generalizes to[13]

$$\hat{G}(\hat{t},\hat{x},K)=\lambda^{2z_3-6}\hat{G}\left(\frac{\hat{t}}{\lambda},\frac{\hat{x}}{\lambda},\lambda^{z_1}K\right) \tag{3.7}$$

for arbitrary λ. Equation (3.7) is still not the scaling law we require as it involves different values of K on either side. We can circumvent this difficulty by referring back to the field theory. The two-point function G of the field theory depends on $t=a_t\hat{t}$, $x=a_x\hat{x}$, the parameters α', r, and a momentum cutoff which we choose to be a_x^{-1}. The important point is that no cutoff is necessary in the energy integrations, and the theory is finite in the limit $a_t\to 0$. However, changing a_t cannot affect the infrared behavior of G, so we can write the following relations:

$$\frac{\Delta^2}{r^2}\hat{G}(a_t^{-1}t,a_x^{-1}x,K)=G(t,x,\alpha',r,a_x,a_t)$$
$$\sim G(t,x,\alpha',r,a_x,0)\,, \tag{3.8}$$

the second relation being valid in the large (t,x) region. Recalling that $K\propto a_t$, we see that (3.8) implies that $\hat{G}(\hat{t},\hat{x},K)$ is a function of just $\hat{t}K$ and $\hat{x}$. We can then rewrite (3.7) as[14]

$$\hat{G}(\hat{t},\hat{x},K)=\lambda^{2z_3-6}\hat{G}\left(\frac{\hat{t}}{\lambda^{1-z_1}},\frac{\hat{x}}{\lambda},K\right)\,, \tag{3.9}$$

which is equivalent to the relation for G (dropping parameter dependence)

$$G(t,x)=t^{(2z_3-6)/(1-z_1)}f\left(\frac{x^2}{t^{2(1-z_1)^{-1}}}\right)\,. \tag{3.10}$$

Comparing with (1.1) we find

$$-\gamma=\frac{2z_3-4}{1-z_1}\,, \tag{3.11}$$

$$z=\frac{2}{1-z_1}\,. \tag{3.12}$$

Since $z_1\leqslant 0$, we see that $z\leqslant 2$. Also, since G, being a spin-spin correlation function, is bounded by unity, we must have $2z_3-6\leqslant 0$, which by (3.11) implies that $-\gamma\leqslant 2$, i.e., the Froissart bound is satisfied. The condition that $\hat{G}$ should be bounded at $t\to\infty$ at fixed x can be interpreted as the statement that the elastic amplitude should be bounded as the energy becomes large, at fixed impact parameter. This implies that the ratio of the elastic to the total cross section is bounded, and will in fact tend to zero if $z_3< 3$. If $z_3=3$ we can always choose the couplings to the external particles small enough so that the elastic cross section is less than the total cross section. These results can be extended without difficulty to D dimensions, provided that the fixed point has the same character.

In deriving the scaling relation (3.10) we chose to keep L at its critical value L^*. By allowing it to differ from L^*, we can derive a generalized scaling relation which governs the behavior of G as Δ tends to its critical value Δ_c:

$$G(t,x,\Delta)=(\Delta_c-\Delta)^{\alpha}$$
$$\times f\left(t(\Delta_c-\Delta)^{\gamma_1},\; x^2(\Delta_c-\Delta)^{\gamma_2}\right)\,, \tag{3.13}$$

where

$$\alpha=\frac{6-2z_3}{z_2}\,,\quad \gamma_1=\frac{1-z_1}{z_2}\,,\quad \gamma_2=\frac{2}{z_3}\,. \tag{3.14}$$

We now consider the case when $K^*\neq 0$, but is finite. The behavior of K under the renormalization group is then determined by an exponent $z'_1<0$:

$$K'-K^*=2^{z'_1}(K-K^*)\,. \tag{3.15}$$

The scaling equation (3.9) becomes

$$\hat{G}(\hat{t},\hat{x},k)=\lambda^{2z_3-6}\hat{G}\left(\frac{\hat{t}}{\lambda}\,\frac{(K^*+\lambda^{z'_1}(K-K^*))}{K}\,,\;\frac{\hat{x}}{\lambda},K\right)\,, \tag{3.16}$$

which reduces asymptotically to

$$\hat{G}(\hat{t},\hat{x},K)\sim\lambda^{2z_3-6}\hat{G}\left(\frac{\hat{t}}{\lambda},\frac{\hat{x}}{\lambda},K\right)\,, \tag{3.17}$$

so in this case we have

$$-\gamma=2z_3-4\,, \tag{3.18}$$

$$z=2\,, \tag{3.19}$$

where, once again, $2z_3-6\leqslant 0$ so that $-\gamma\leqslant 2$. Which of the possibilities ($z<2$ or $z=2$) is correct depends on a detailed investigation. Evidently the ϵ-expansion results correspond to the former case.

IV. DISCUSSION

In this paper we have restricted ourselves to describing the derivation of the analog model, and showing that, if it has the type of phase transition suggested by the ϵ expansion, the exponents will satisfy the simplest constraints of direct-channel unitarity. This result comes about as a consequence of the appearance of the three-spin coupling L which links space and time in a commensurate way and naturally leads to the result $z=2$. This is only modified by the coupling K, to $z \leq 2$. Otherwise K would be driven to infinity and no phase transition would occur. In addition, the result that the scaled Reggeon field $r\psi/\Delta$ is bounded leads to the bound $-\gamma - z \leq 0$ on the two-point function. This in turn implies that the opacity at fixed impact parameter does not increase with energy, and so the elastic is less than the total cross section. Finally, these two results together imply the Froissart bound $-\gamma \leq 2$.

We conclude by remarking that, since our analog model is an abstraction of the already abstract Reggeon field theory, it would be desirable to obtain a more direct interpretation in terms of high-energy scattering.

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While this paper was in preparation we heard that R. C. Brower, J. Ellis, R. Savit, and W. J. Zakrzewski have considered an analog model, which is slightly different from ours in that the real part of the Reggeon field is also represented by an Ising spin.

APPENDIX A

We now demonstrate the relationship between the limits $g_R \to g_1$ and $r, \Delta/r \to \infty$ for a theory with a finite cutoff, Λ. We take Λ to have dimension t^{-1} so, for example, it could be given by $1/a_t$ or $\alpha'/a_x^{\,2}$. Following Ref. 3 we normalize the propagator and vertex function at points determined by a parameter E_N which also has dimension t^{-1}. g_R is given by

$$g_R = Zr/\alpha'^{D/4} E_N{}^{\epsilon/4} \equiv Zg, \tag{A1}$$

where Z is a dimensionless renormalization constant which can be expressed as a function of the dimensionless variables g_R and

$$x = E_N/\Lambda\,, \tag{A2}$$

or alternatively g and x.

Following the approach of Ref. 15 we write

$$\beta_E = E_N \frac{\partial}{\partial E_N} g_R \bigg|_{r,\alpha',\Lambda} = -\frac{\epsilon}{4} g_R + g_R \beta_E \frac{\partial \ln Z}{\partial g_R} + g_R\, x\, \frac{\partial \ln Z}{\partial x}\,, \tag{A3}$$

$$\beta_\Lambda = \Lambda \frac{\partial}{\partial \Lambda} g_R \bigg|_{r,\alpha',E_N} = g_R \beta_\Lambda \frac{\partial \ln Z}{\partial g} - g_R\, x\, \frac{\partial \ln Z}{\partial x}\,. \tag{A4}$$

Using the fact that $Z(g_R = 0, x) = 1$, we can integrate (A3) and (A4) and obtain

$$Z(g_R, x) = \exp\left\{ \int_0^{g_R} dg_R' [1/g_R' + \epsilon/4\beta(g_R', x)] \right\}, \tag{A5}$$

where

$$\beta(g_R, x) = \beta_E(g_R, x) + \beta_\Lambda(g_R, x)\,. \tag{A6}$$

An infrared-stable fixed point corresponds to a zero of β of the form

$$\beta(g_R, x) \underset{g_R \to g_1}{\sim} \beta'(x)[g_R - g_1(x)]\,, \tag{A7}$$

with $\beta'(x) > 0$. Then,

$$\ln Z(g_G, x) \underset{g_R \to g_1}{\sim} [\epsilon/4\beta'(x)] \ln[g_1(x) - g_R]\,, \tag{A8}$$

and we see from (A1) that g_R can be driven to g, either by taking E_N to zero or r to infinity. Substituting (A8) into (A3) we see that $\beta'(x)$ must be independent of x in order for β_E to be free of singularities at $g_R = g_1$. By applying a similar argument to the other renormalization constants in the theory one can show that the critical indices are independent of the cutoff.

Let us now consider the behavior of Δ_c for large r. From Ref. 7 we know that

$$\Delta_c = \int_0^\infty dE_N [Z_3(g, x)^{-1} - 1] = (r^2/\alpha')^{2/\epsilon} \times \int_0^\infty dt\, t^{-2} \left[Z_3\left(t, \frac{(r^2/\alpha')^{2/\epsilon}}{\Lambda t}\right)^{-1} - 1 \right], \tag{A9}$$

where Z_3 is the wave-function renormalization constant and

$$t = (r^2/\alpha')^{2/\epsilon} E_N{}^{-1}\,. \tag{A10}$$

The integral in (A9) is easily estimated provided $(r^2/\alpha')^{2/\epsilon} < \Lambda$. For $t < 1$ we can use the perturbation expansion for Z_3. In two dimensions we find a leading contribution of order $(r^2/\alpha')\ln(\alpha'\Lambda/r^2)$. For $t > 1$ we expand $Z_3(g, x)$ in a power (or asymptotic) series in x. The resulting integrals converge (provided the critical index $\gamma < 1$) because the large-t behavior of Z_3 is controlled by the fixed point. This range of t integration therefore yields a contribution to Δ_c of order $(r^2/\alpha')^{2/\epsilon}$. Thus,

$$\Delta_c/r \xrightarrow[r\to\infty]{} \infty\,. \tag{A11}$$

The contributions to Δ_c from large and small values of t are positive, and we expect that in general $\Delta_c > 0$. This can be verified explicitly in the ϵ expansion.[7] In fact there cannot be a phase transition for large negative values of Δ, since then the functional integrals are dominated by the saddle point at $\varphi = \chi = 0$ and the first few terms in the perturbation series give a good approximation for the Green's functions.

APPENDIX B

We show how to evaluate the integral (2.6) by the saddle-point method outlined in the text. Since the field has essentially two components it is convenient to use a matrix notation. We write

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \quad \Psi^T = (\varphi \quad \chi)\,. \tag{B1}$$

The Lagrangian then has the form

$$\mathfrak{L} = \Psi^T \Xi \Psi + V\,, \tag{B2}$$

where

$$\Xi = \begin{pmatrix} -\alpha'\nabla^2 & i\partial_t \\ -i\partial_t & -\alpha'\Delta^2 \end{pmatrix}\,. \tag{B3}$$

Writing, as in Eqs. (2.7) and (2.8),

$$\Psi = \Psi_0 + \Psi'\,, \tag{B4}$$

where

$$\Psi_0 = \frac{\Delta}{r}\begin{pmatrix} i \\ s \end{pmatrix}, \tag{B5}$$

the potential V can be written, to second order in Ψ',

$$V = \Psi'^T A \Psi'\,, \tag{B6}$$

where A is just the matrix of second derivatives of V at Ψ_0:

$$A = \Delta\begin{pmatrix} 2 & -is \\ -is & 0 \end{pmatrix}\,. \tag{B7}$$

Note that A has positive eigenvalues, justifying integrating Ψ' along the real axis. Equation (A2) can now be written, completing the square, as

$$\mathfrak{L} = [\Psi'^T + \Psi_0^T \Xi(\Xi + A)^{-1}]\,(\Xi + A)[\Psi' + (\Xi + A)^{-1}\Xi\Psi_0] + \Psi_0^T \Xi\Psi_0 - \Psi_0^T \Xi(\Xi + A)^{-1}\Xi\Psi_0\,. \tag{B8}$$

Performing the Gaussian integration over Ψ', we have, apart from a constant factor,

$$Z = \sum_{s_t} \{\det(\Xi + A)\}^{-1/2} \exp\Big(-a_t a_x^D \sum_t \{\Psi_0^T \Xi \Psi_0 - \Psi_0^T \Xi(\Xi + A)^{-1}\Xi\Psi_0\}\Big)\,. \tag{B9}$$

This can be simplified further in the limit $\Delta \to \infty$, since $A = O(\Delta)$ while $\Xi = O(1)$. The determinant is then simply a constant, independent of the s_t, and the leading term in the exponent is

$$\Psi_0^T \Xi \Psi_0 = \frac{\Delta^2}{r^2}(i \quad s)\begin{pmatrix} -\alpha'\Delta^2 & i\partial_t \\ -i\partial_t & -\alpha'\Delta^2 \end{pmatrix}\begin{pmatrix} i \\ s \end{pmatrix} = -\frac{\Delta^2\alpha'}{r^2}\, s\nabla^2 s\,, \tag{B10}$$

which is equivalent to the first term $(\nabla s)^2$ in (2.9). To this order the time dependence disappears, so we must include the next term

$$-\Psi_0^T \Xi A^{-1} \Xi \Psi_0 = -\frac{\Delta}{r^2}(i \quad s)\begin{pmatrix} -\alpha'\nabla^2 & i\partial_t \\ -i\partial_t & -\alpha'\nabla^2 \end{pmatrix}\begin{pmatrix} 0 & is \\ is & 2 \end{pmatrix}\begin{pmatrix} -\alpha'\nabla^2 & i\partial_t \\ -i\partial_t & -\alpha'\nabla^2 \end{pmatrix}\begin{pmatrix} i \\ s \end{pmatrix}. \tag{B11}$$

Multiplying out the matrices we obtain

$$-\frac{\Delta}{r^2}\, s(-\alpha'\partial_t s\nabla^2 + \alpha'\nabla^2 s\partial_t + 2\alpha'^2\nabla^4)s\,. \tag{B12}$$

The first two terms are equivalent and give the second term in (2.9). The ∇^4 term represents a next-nearest-neighbor coupling in the space direction, which we ignore compared with (B10).

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[14]This manipulation is similar to that one could use to solve an Ising model with unequal couplings J_1, J_2 in the two directions. On passing to the continuum limit it becomes clear that we can make them equal by rescaling the coordinates. Our case is slightly more complicated since, in effect, J_1/J_2 is not invariant under the renormalization-group transformation.

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