Semiclassical bound states in an asymptotically free theory*

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We have carried out a semiclassical calculation of the particle spectrum in the Gross-Neveu model. It is a two-dimensional model with N species of fermions interacting through a symmetrical scalar-scalar interaction and is renormalizable, asymptotically free, and exhibits spontaneous symmetry breaking. We find a rich spectrum of particles which can be interpreted as fermion-antifermion bound states and multifermion bound states. These states fall into supermultiplets whose origin we do not completely understand. The binding mechanism is a vacuum-polarization effect rather than the direct interaction between particles. A general method for handling fermions in semiclassical calculations is developed.

I. INTRODUCTION

The problem of finding the particle spectrum (e.g., bound states) of a quantum field theory has been a long-standing one. Recently, we[1,2] and several other groups[3-6] have been developing semiclassical methods for attacking this problem. For the sine-Gordon equation a field-theoretic version of the WKB method appears to give the spectrum (including the "elementary particle," numerous bound states, and solitons) exactly.[2] It is no doubt an accident that WKB is exact for this system. However, these sine-Gordon results do suggest that semiclassical methods may become a useful tool for attacking the bound-state problem in field theory.

So far, work on semiclassical methods has been deficient in at least three respects: (i) Most of the work has been on systems in two-dimensional space-time; (ii) the theories have usually been super-renormalizable; (iii) it has not been clear how to incorporate fermions in a systematic way.

In this paper we study a model which answers a number of questions raised by points (ii) and (iii). We will have nothing directly to say about (i) as we will still be in two dimensions. However, it now appears to us that the problems of going from two to four dimensions are technical rather than conceptual. Of course, a program can be as easily halted by severe technical problems as by conceptual ones.

Specifically, we use a WKB method to compute the particle spectrum of the Gross-Neveu[7] model. It is in two-dimensional space-time and is defined by the Lagrangian

$$\mathcal{L} = \sum_{k=1}^{N} i\bar{\psi}^{(k)} \not{\partial} \psi^{(k)} + \frac{g^2}{2}\left(\sum_{k=1}^{N} \bar{\psi}^{(k)} \psi^{(k)}\right)^2 . \tag{1.1}$$

The model thus contains N fermions coupled symmetrically through a scalar-scalar interaction. We will generally suppress the particle-type indices k and use the notation

$$i\bar{\psi}\not{\partial}\psi = \sum_k i\bar{\psi}^{(k)} \not{\partial} \psi^{(k)} ,$$
$$\bar{\psi}\psi = \sum_k \bar{\psi}^{(k)} \psi^{(k)} . \tag{1.2}$$

The model is renormalizable (g is dimensionless), γ_5 invariant, and formally scale invariant. For large N one can sum the leading sets of diagrams and establish that in this limit the model is asymptotically free. Gross and Neveu[7] also found that $\bar{\psi}\psi$ develops a vacuum expectation value so that γ_5 invariance is spontaneously broken. In the process the dimensionless coupling constant g is traded for an arbitrary dimensional parameter $g\langle\bar{\psi}\psi\rangle$ and disappears from the theory. The end result is that the theory contains no dimensionless parameter other than the number of fermions N. Consequently, any physical dimensionless quantity such as the ratio of two particle masses can depend only on N. This rather striking phenomenon, whose ultimate origin is the renormalization group, will be present in our WKB calculations. We can take this as an indication that semiclassical methods are compatible with renormalization-group ideas.

Following Gross and Neveu, we find it useful to replace (1.1) by

$$\mathcal{L} = \bar{\psi} i\not{\partial}\psi - g\sigma\bar{\psi}\psi - \tfrac{1}{2}\sigma^2 , \tag{1.3}$$

where we have used the notation of (1.2) and introduced a neutral scalar field σ. Using the equation of motion

$$\sigma = -g\bar{\psi}\psi \tag{1.4}$$

the Lagrangian in (1.3) becomes equivalent to that in (1.1). Our WKB method is based upon the evaluation of certain functional integrals by a station-

ary-phase approximation. It is not obvious how to use a stationary-phase method when there are integrations over anticommuting fermion fields. The advantage of the Lagrangian in (1.3) is that the fermion fields enter bilinearly and can be integrated out of the problem leaving an effective action containing only the boson field σ. We then do the σ integration by stationary phase. To do this we must find space-time-dependent fields σ around which the effective action is stationary. This effective action is nonlocal and highly nonlinear, but it turns out to be possible to find stationary points. The first such example was found by Callan, Coleman, Gross, and Zee.[8] It is analogous to the kink in the φ^4 theory[1-3,5] or the soliton[2,3] in the sine-Gordon equation, i.e., it is a particlelike solution which is time-independent in its rest frame and which has a peculiar topology. We have found a large number of further stationary points of the effective action. In particular, we find solutions which are particlelike but have a nontrivial time dependence in the rest frame. The WKB method then quantizes these classical solutions producing a spectrum of particle masses.

The kinklike solutions produce an exotic sort of particle which probably has no counterpart in four dimensions. However, the vast majority of our solutions are not kinks. They correspond to less exotic objects such as the original fermion, fermion-antifermion bound states, or multifermion bound states. Such states surely exist in four-dimensional theories, and we would conjecture that in four, as well as in two dimensions, there is a correspondnece between classical field configurations and particle states. Assuming this to be so, it remains to be seen if such a correspondence can be effectively exploited.

Below we will describe the particle spectrum of the model as given by our WKB calculation. To interpret this spectrum we will need to know something about the symmetries of the model. The Gross-Neveu model has an obvious U(N) internal symmetry. Actually it has an O($2N$) symmetry of which U(N) is a subgroup. This may be seen as follows. Choose a Majorana representation[9] for the γ matrices $\gamma^0=\sigma^y$, $\gamma^1=i\sigma^x$ and write

$$\psi^{(k)}=\psi_1^{(k)}+i\psi_2^{(k)}\,, \tag{1.5}$$

where $\psi_1^{(k)}$ and $\psi_2^{(k)}$ are Hermitian two-component spinors. The Lagrangian then takes the form

$$\mathcal{L}=\sum_k i\left(\psi_1^{(k)}\frac{\partial}{\partial t}\psi_1^{(k)}+\psi_2^{(k)}\frac{\partial}{\partial t}\psi_2^{(k)}+\psi_1^{(k)}\sigma_z\frac{\partial}{\partial x}\psi_1^{(k)}+\psi_2^{(k)}\sigma_z\frac{\partial}{\partial x}\psi_2^{(k)}\right)-g\sigma\sum_k(\psi_1^{(k)}\sigma_y\psi_1^{(k)}+\psi_2^{(k)}\sigma_y\psi_2^{(k)})-\frac{\sigma^2}{2}\,, \tag{1.6}$$

which is Hermitian and nonvanishing because the ψ's anticommute. When written in the form (1.6), it is clear that the Lagrangian is invariant under orthogonal transformations on the $2N$-component vector $\psi_j^{(k)}$ ($k=1,2,\ldots,N$), $j=1,2$. The fermion number operator $Q\equiv\int\psi^\dagger\psi\,dx$ has nontrivial commutation relations with other generators of O($2N$). Therefore, a nontrivial representation of O($2N$) will contain states with more than one value of Q. Hence we may expect, for example, that some fermion-antifermion states will be degenerate with fermion-fermion states.[10] The σ field is an O($2N$) scalar while ψ is an O($2N$) vector. The only other O($2N$) representations which we will encounter are the totally antisymmetric O($2N$) tensors of rank $n_0<N$. The number of states in a multiplet corresponding to such a tensor is $(2N)!/n_0!(2N-n_0)!$. Scalars and O($2N$) vectors are special cases of completely antisymmetrical tensors of rank $n_0=0$ and $n_0=1$, respectively.

Because of our inability to evaluate certain Gaussian functional integrals we have not been able to carry through a complete WKB calculation in the Gross-Neveu model. What we have been able to do is a sort of zeroth-order calculation which, in ordinary potential theory, is analogous to using the quantization rule $\oint p\,dq=2n\pi$ rather than the more accurate $\oint p\,dq=(2n+1)\pi$. [In the sine-Gordon equation the analogous approximation is equivalent to setting $\gamma'=(\lambda/m^2)(1-\lambda/8\pi m^2)^{-1}\approx\lambda/m^2$.] Even with this approximation our results should become exact in the limit of large N and are probably *qualitatively* correct for any N greater than 2 or 3.

We find the particle spectrum shown in Fig. 1. There is a large, unexpected degeneracy beyond

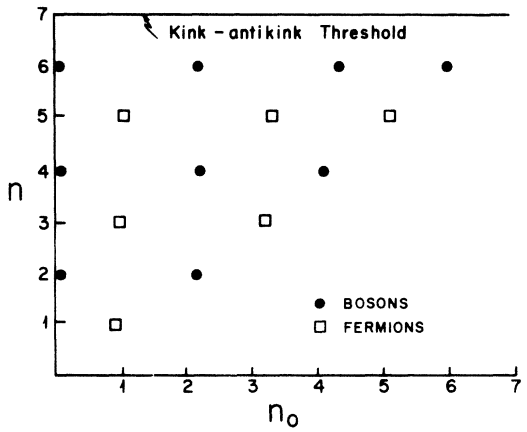

FIG. 1. The particle spectrum for $N=7$. The integer n, running vertically, labels a supermultiplet, each of whose members has a mass $M_n=g\sigma_0(14/\pi)\sin(n\pi/14)$. The horizontally running integer, n_0, is an O($2N$) quantum number. The degeneracy of each dot (boson) or box (fermion) is $(14)!/n_0!(14-n_0)!$.

that required by O(2N) symmetry. This degeneracy might be real or it may be an artifact of our zeroth-order calculation. There are supermultipliets listed by a "principal quantum number" $n = 1, 2, \ldots < N$. The common mass of the states in the nth supermultiplet is

$$M_n = g\sigma_0 \frac{2N}{\pi} \sin\left(\frac{n}{N}\frac{\pi}{2}\right), \quad n = 1, 2 \ldots < N, \tag{1.7}$$

where σ_0 is the vacuum expectation value of σ.[11] We see that ratios of masses are independent of g as they should be. If n is odd the supermultiplet is composed of fermions and contains O(2N) representations corresponding to all completely antisymmetrical tensors of rank $n_0 = 1, 3, 5 \ldots \leq n$. For example, the $n = 1$ state is a fermion belonging to a vector representation of O(2N). This is the "elementary particle" of the theory. For large N,

$$M_1 \approx g\sigma_0, \tag{1.8}$$

which agrees with the result of Gross and Neveu. The $n = 3$ supermultiplet contains an O(2N) vector which is some kind of excited state of the elementary particle and a completely antisymmetrical O(2N) tensor of rank 3. The latter is a bound state of three fermions and/or antifermions. The supermultiplets with n even are composed of bosons and contain O(2N) antisymmetrical tensors of rank $n_0 = 0, 2, 4 \ldots \leq n$. For example, $n = 2$ contains an O(2N) scalar and an antisymmetric tensor of second rank. The tensor is a set of two-body bound states with fermion-fermion, antifermion-antifermion, and fermion-antifermion quantum numbers. The O(2N) scalar is a different sort of object. It may be thought of as a particle associated with the σ field. At the $n = 4$ level there is an excited σ, a state which can be thought of as an excitation of the second-rank tensor at $n = 2$ and a new state corresponding to a completely antisymmetrical tensor of rank 4. This new object is a bound state of 4 fermions and/or antifermions analogous to the 2- and 3-particle states found at levels $n = 2$ and 3. The pattern continues in the same way for $n = 5, 6, \ldots$ on up to N.

The quantum numbers of the states in our spectrum are not unexpected. In the limit of large N the leading *exchange* is the sum of bubbles shown in Fig. 2(a). In the nonrelativistic limit, this exchange produces an attractive δ-function potential. Such a potential will produce bound states only in channels where the spatial wave function is completely symmetrical. For fermions this means that the O(2N) wave function must be completely antisymmetrical, i.e., an O(2N) antisymmetric tensor.

For large N the bubble exchange is weak[7] and a Schrödinger equation calculation is valid. One computes a δ-function potential by summing the diagrams in Fig. 2(a) at zero-momentum transfer and then solves the Schrödinger equation as in Ref. 2. In this way one finds a binding energy which agrees with that computed from (1.7)

$$|E_B| = nM_1 - M_n = M_1\left(\frac{\pi}{2N}\right)^2 (n^3 - n) + O\left(\frac{1}{N^3}\right) \tag{1.9}$$

to the indicated order in N^{-1}. These nonrelativistic bound states correspond to the states with $n_0 = n$. They are the lowest states with given O(2N) quantum numbers and are consequently stable. Equation (1.9) is valid only if n/N is small. For n and N both large the binding energy per particle is, in units of M_1,

$$\frac{nM_1 - M_n}{nM_1} = 1 - \frac{2N}{\pi n} \sin\left(\frac{n}{N}\frac{\pi}{2}\right), \tag{1.10}$$

which for $n/N \sim 1$ shows binding by a finite fraction of the rest mass. Thus, strong binding can occur even for large N.

The bubble exchanges in Fig. 1(a) are not the only important interactions for large N. For fermion-antifermion interactions in an O(2N) single state the annihilation bubbles in Fig. 2(b) are dominant. The sum of these bubbles leads to an interaction which is marginally attractive. In leading order in N, Gross and Neveu[7] found a σ bound state at the fermion-antifermion threshold. It is presumably the $n = 2$, O(2N) singlet state discussed above. We find that it is bound in the next order in N^{-2}. This disagrees with a detailed diagrammatic calculation by Schonfeld[12] who finds that the bound state remains at threshold to this order. We do not understand the origin of this

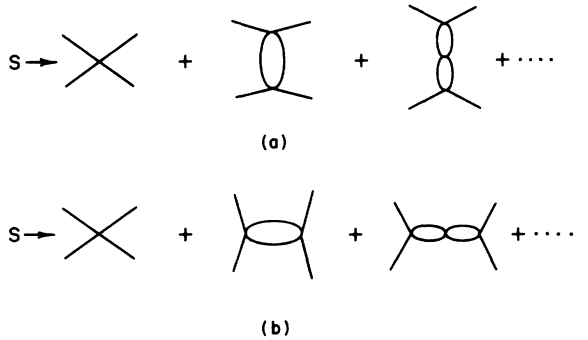

FIG. 2. (a) The leading exchanges in the large-N limit. The s channel contains the bound state. (b) The annihilation bubbles which dominate in the O(2N) singlet channel.

discrepancy. In any case there is a weak attraction between fermion-antifermion pairs in an $O(2N)$ singlet state. One might therefore imagine that the particles in the model will be made up of a number of fermions and antifermions paired into $O(2N)$ singlet states plus further "valence" fermions and antifermions in an antisymmetrical tensor state. Our particle spectrum is consistent with such a picture.

The particle spectrum ends at $n=N$ where the mass is $M_N = 2Ng\sigma_0/\pi$. The mass of the Callan-Coleman-Gross-Zee kink is (in our zeroth-order approximation) $M_{\text{kink}} = Ng\sigma_0/\pi$. Thus the Nth state is just at the kink-antikink threshold. Higher-mass states would be unstable against decay into kink-antikink pairs.[13]

There is a striking similarity between the sine-Gordon equation and the Gross-Neveu model. In the zeroth-order WKB approximation the particle spectrum of the sine-Gordon theory is given by $M_n \approx (m2\xi/\pi)\sin(\pi n/2\xi)$ where $\xi = 8\pi m^2/\lambda$, plus a soliton at mass $M_{\text{soliton}} \approx m\xi\pi$. With the identification $N \to \xi$ the energy levels are identical to those of the Gross-Neveu model. The particle content of the levels is, of course, very different in the two theories. There is no doubt an underlying reason for this correspondence between the theories but we do not know what it is. However, we can use this correspondence to try to guess what would happen if we could do a complete WKB calculation. In the sine-Gordon equation the result of the complete calculation is simply to replace λ/m^2 in the zeroth-order formula by $(\lambda/m^2)(1-\lambda/8\pi m^2)^{-1}$ which is equivalent to making the replacement $\xi \to \xi - 1$. The analogous replacement in the present model would be to replace N by $N-1$ in Eq. (1.7) and in the formula for the kink mass. The theory would then be singular at $N=1$. One expects such a singularity since at $N=1$ the Gross-Neveu model can be Fierz-transformed to the usual Thirring model which contains a single massless fermion. Our zeroth-order calculation is certainly not valid for N as small as 1.

If it were to turn out that a full WKB calculation differs from the present one only by changing N to $N-1$, then the extra degeneracy in the mass spectrum would presumably be real and a consequence of some underlying dynamical symmetry. Another possibility is that in a complete WKB calculation the masses within a supermultiplet will be split by terms of order N^{-2}. If this happens, the $n=2$ singlet state might remain at threshold to order N^{-2}, in agreement with Schonfeld.

While the finer details of our approximate semiclassical spectrum are clearly not to be taken too seriously, the qualitative picture of a rich spectrum organized into some kind of supermultiplets is almost certainly correct. This unexpected wealth of particle states seems to be a consequence of the asymptotic freedom of the theory. The detailed form of the classical σ field which corresponds to a quantum bound state suggests that the binding mechanism is not a direct interaction between the bound fermions but rather is some kind of vacuum-polarization effect. The fact that the theory is unstable in the infrared is most likely the reason for this.

The paper is not meant to be self-contained. The reader is expected to be somewhat familiar with our previous papers,[1] especially that on the sine-Gordon equation,[2] and with the original paper of Gross and Neveu.[7] In Sec. II and Appendix A we work out a general formalism for handling fermions. This method will work in four as well as two dimensions. In Sec. III we find all the time-independent stationary points of the effective action for σ. To do this we use some techniques from scattering theory. Time-dependent stationary points are obtained in Sec. IV. We find them by guessing the form of the solution and explicitly verifying that they are stationary points.

II. FERMIONS IN THE SEMICLASSICAL METHOD

Our semiclassical method, which was suggested by ideas of Gutzwiller, Maslov, and Keller, is based on a stationary-phase approximation to the functional integral for $\operatorname{tr} e^{-iHT}$. When fermion fields are present one is integrating over anticommuting as well as commuting variables, and it is not obvious what a stationary-phase approximation means. Below we will show how the fermion fields can be integrated out explicitly leaving an integral over commuting fields only. The final integral over the commuting field is then done by stationary phase in the usual way.

The way in which we use an approximate representation for $\operatorname{tr} e^{-iHT}$ to compute the particle spectrum of a theory was illustrated in our previous papers. Having obtained a semiclassical formula for $\operatorname{tr} e^{-iHT}$ the succeeding calculations are the same for fermions and bosons. We will not repeat the details here.

A. Integrating out the fermions

Using the Lagrangian in (1.3) the functional-integral representation for $\operatorname{tr} e^{-iHT}$ is

$$\operatorname{tr} e^{-iHT} = \int [d\psi][d\bar{\psi}][d\sigma] \times \exp\left[i \int_0^T dt \int_{-\infty}^{\infty} dx \left(\frac{-\sigma^2}{2} + \bar{\psi}(i\partial\!\!\!/ - g\sigma)\psi \right) \right], \tag{2.1}$$

where integration runs over periodic fields σ, i.e.,

$$\sigma(x, t+T)=\sigma(x, t)\,. \tag{2.2}$$

It turns out that the boundary condition on the fermion fields is that they are antiperiodic $\psi(x, t+T)=-\psi(x, t)$. This is discussed in Appendix A.

The integral over ψ and $\bar{\psi}$ in (2.1) is of the Gaussian type which can be evaluated exactly. It is worked out in Appendix A. Here we simply state the result in terms of certain Floquet indices α_i which are defined as follows. For a given σ, consider the Dirac equation for ψ

$$(i\partial\!\!\!/ - g\sigma)\psi=0\,, \tag{2.3}$$

which is a linear equation with periodic (in time) coefficients. The general theory of such equations is well known. One looks for solutions to (2.3) such that

$$\psi_i(x, t+T)=e^{-i\alpha_i}\psi_i(x, t)\,, \tag{2.4}$$

which defines the Floquet indices α_i. In many ways the α_i are analogous to the eigenvalues of the more familiar Dirac equation in a time-independent potential. If we put the system in a finite spatial box the α_i are discrete as are the energy levels for a time-independent σ. As the size of the box goes to infinity the α_i form a continuum for $|\alpha|>g\sigma_0 T$, where σ_0 is the vacuum expectation value of σ. For these continuum "states," ψ approaches plane waves at infinity as it does for an ordinary scattering process. There can also be discrete "bound-state" indices in the interval $0\leq|\alpha_i|<g\sigma_0 T$. For these discrete indices ψ falls exponentially at spatial infinity in analogy with the behavior of an ordinary bound-state wave function.

In Sec. IV we will give an explicit, nontrivial example of discrete and continuum indices α. They also appeared in our work on the sine-Gordon equation, where, in a different context, we called them stability angles.

The origin of the analogy between Floquet indices and energy eigenvalues is not hard to find. A special case of a σ periodic with period T is a time-independent field $\sigma(x)$. Setting $\psi=e^{-i\omega t}U(x)$ in (2.3) leads to the usual eigenvalue problem for ω. If ω_i is an eigenvalue, the corresponding index is clearly $\omega_i T$ and we have the correspondence

$$\alpha_i \to \omega_i T \quad \text{(time-independent } \sigma)\,. \tag{2.5}$$

The Dirac equation in an external scalar field is invariant under charge conjugation. It follows from this that the α_i come in plus-minus pairs with equal magnitude but opposite signs. We will use this fact to simplify a number of formulas. Our equations will be written in terms of the positive α_i only. The reader should keep in mind that *in what follows we will take the symbol α_i to mean a positive index and all sums or products over the indices α_i are to be taken over positive indices only.*

With this convention that α_i refers to a positive index, it is shown in Appendix A that

$$\int[d\psi]\,d[\bar{\psi}]\exp\left(i\int_0^T dt\int_{-\infty}^{\infty}dx\,\bar{\psi}(i\partial\!\!\!/ - g\sigma)\psi\right)=e^{iN\varphi}\prod_i(1+e^{-i\alpha_i})^{2N},\quad \varphi=\sum_i\alpha_i\,, \tag{2.6}$$

where as usual N is the number of species of fermions. We can write

$$\prod_i(1+e^{-i\alpha_i})^{2N}=\sum_{\{n\}}C(N,\{n\})\exp\left(-i\sum_i n_i\alpha_i\right),\quad C(N,\{n\})=\prod_i\frac{(2N)!}{(2N-n_i)!\,n_i!}\,, \tag{2.7}$$

where the sum is over all (finite) sets of integers n_i such that $0\leq n_i\leq 2N$. Inserting (2.6) and (2.7) into (2.1) yields

$$\operatorname{tr}e^{-iHT}=\sum_{\{n\}}C(N,\{n\})\int[d\sigma]\exp\left(i\int_0^T\int_{-\infty}^{\infty}\frac{-\sigma^2}{2}\,dt\,dx+iN\varphi(\sigma)-i\sum_i n_i\alpha_i(\sigma)\right), \tag{2.8}$$

where we have indicated that φ and the α_i depend functionally on σ and the integration is over all σ such that $\sigma(x, t+T)=\sigma(x, t)$ and $|\sigma(x, t)|\to|\sigma_0|$ as $|x|\to\infty$. Equation (2.8) is exact. Our semiclassical approximation will be to do the integration over σ by stationary phase.

B. Renormalization

The argument of the exponential on the right-hand side of (2.8) is divergent and must be renormalized. We do this as follows.

Starting in a finite box of length L and with an ultraviolet cutoff Λ we subtract a constant $N\varphi(0)$ from the action and supply a multiplicative renormalization factor Z in front of the $\sigma^2/2$ term in the Lagrangian. With these modifications (and using the notation $\sigma_0=\langle\sigma\rangle_{\text{vac}}$) we can write the argument of the exponential in

(2.8) in the form

$$i\left(-\int_0^T\int_{-\infty}^{\infty}\frac{Z}{2}(\sigma^2-\sigma_0^2)dx\,dt+N[\varphi(\sigma)-\varphi(\sigma_0)]-\sum_i n_i\alpha_i(\sigma)\right)+i\left(LTZ\frac{-\sigma_0^2}{2}+N[\varphi(\sigma_0)-\phi(0)]\right), \tag{2.9}$$

where the first term in large parentheses contains the differences $\sigma^2-\sigma_0^2$ and $\varphi(\sigma)-\varphi(\sigma_0)$ and the second contains the corresponding differences between $\sigma=\sigma_0$ and $\sigma=0$. We renormalize the second term which contains only constant fields according to the prescription of Gross and Neveu.[7] That is, we choose Z such that the second term in (2.9) is a maximum at σ_0, an arbitrary dimensional number which we choose as the vacuum expectation value of σ. The calculation is identical to that done by Gross and Neveu except that we choose to use a noncovariant cutoff procedure where the spatial momenta k_1 are cut off at Λ. Having thus determined Z, we can then ignore the second term in (2.9) which is just a c number. In the first term in (2.9) we find that there is a logarithmic divergence in $N[\varphi(\sigma)-\varphi(\sigma_0)]$ which precisely cancels against the divergence in $\frac{1}{2}Z(\sigma^2-\sigma_0^2)$. The integrand in (2.8) then becomes finite.

Explicit examples of this renormalization procedure are given in Secs. III and IV.

C. Occupied fermion states

In a previous paper, we introduced the notion of occupied fermion states. The ground state of a quantized fermion field in a time-independent external field σ is the configuration in which all states in the negative-energy sea are filled. Excited states corresponding to particles or scattering states are obtained by occupying some of the positive-energy states. Looking at Eqs. (2.6), (2.7), and (2.8), it is clear that we can extend this notion to the idea of occupied Floquet index states. The term in Eq. (2.8) for which all the $n_i=0$ corresponds to the vacuum sector of the theory in which no positive-α states are occupied. For this term the factor $e^{iN\varphi(\sigma)}$ in the integrand corresponds, according to (2.6), to the contribution to the action of the filled states in a "negative α sea." The terms in (2.8) for which the n_i are nonzero can be thought of as configurations in which ith positive index state is occupied by n_i fermions. According to (2.6), n_i must be less than $2N$. This is in agreement with the Pauli principle. We cannot put two identical fermions in the same index state so that with N species of fermions and N species of antifermions one can put at most $2N$ fermions in a given state.

When we do the stationary-phase integration over σ to obtain the energy levels of the system in the semiclassical approximation, each term in the sum over $\{n\}$ will yield a distinct set of quantum states. Therefore we can think of the set of integers $\{n\}$ as being quantum numbers which label states. In our actual calculations this labeling of states will be equivalent to classifying states according to irreducible representations of $O(2N)$.

It is intuitively clear that to find the particle spectrum of the model we can restrict ourselves to those terms in (2.8) in which only discrete index states are occupied. Those terms in which α's in the continuum are occupied correspond to scattering states of the theory.

In practice we will find that all the σ's which satisfy the stationary-phase condition have the property that there is a single discrete index state α_0 and we will be able to occupy it by a number $0\leq n_0\leq 2N$ of fermions or antifermions. For such σ's, the general formula in (2.8) reduces to

$$\mathrm{tr}\,e^{-iHT}\sum_{n_0=0}^{2N}\frac{(2N)!}{n_0!(2N-n_0)!}\int[d\sigma]\exp\left(-\int_{-0}^{T}\int_{-\infty}^{\infty}\frac{Z}{2}(\sigma^2-\sigma_0^2)dt\,dx+iN[\varphi(\sigma)-\varphi(\sigma_0)]-in_0\alpha_0(\sigma)\right), \tag{2.10}$$

where we have renormalized the integrand as in (2.9). The particle states which we obtain from (2.10) will evidently carry the quantum number n_0. We can identify this quantum number with $O(2N)$ representations as follows. The combinatorial factor $(2N)!/n_0!(2N-n_0)!$ is clearly the degeneracy of a state labeled by n_0. The number of independent components of a completely antisymmetrical $O(2N)$ tensor of rank n_0 is easily found to be $(2N)!/n_0!(2N-n_0)!$. Therefore we can identify the terms in (2.10) as coming from states which are completely antisymmetrical $O(2N)$ tensors of rank n_0. The special cases $n_0=0$ and $n_0=1$ are $O(2N)$ scalars and vectors, respectively.

D. The stationary-phase condition

To evaluate (2.10) in the stationary-phase approximation we need to find the points where the phase of the integrand is stationary. These are fields σ satisfying the functional equation

$$\frac{\delta}{\delta\sigma(x,t)}\left(-\int_{-0}^{T}\int_{-\infty}^{\infty}\frac{Z}{2}\{[\sigma(x',t')]^2-\sigma_0^2\}\,dt'\,dx'+N[\varphi(\sigma)-\varphi(\sigma_0)]-n_0\alpha_0(\sigma)\right)=0\ . \tag{2.11}$$

and the boundary conditions $\sigma(x,t+T)=\sigma(x,t)$, $|\sigma(x,t)|\to|\sigma_0|$ as $|x|\to\infty$. In Appendix A we show that for a nondegenerate α_i which is not equal to an integral multiple of π

$$\frac{\delta}{\delta\sigma(x,t)}\alpha_i=g\bar{\psi}_i(x,t)\psi_i(x,t)\,, \tag{2.12}$$

where ψ_i satisfies (2.3) and (2.4) and is normalized such that

$$\int_{-\infty}^{\infty}\psi_i^*\psi_i\,dx=1\,. \tag{2.13}$$

Using this result we can write (2.11) as

$$-\frac{1}{g}Z\sigma(x,t)=-N\sum_{i=0}^{\infty}\bar{\psi}_i(x,t)\psi_i(x,t)+n_0\bar{\psi}_0(x,t)\psi_0(x,t)\,, \tag{2.14}$$

which is valid unless $\alpha_0=\pi, 2\pi, 3\pi\ldots$. In practice we will find that our solutions have $\alpha_0=\pi$. A slight modification of (2.14) is then required. This is discussed in Appendix A and Sec. IV.

Evidently our formalism leads to a kind of time-dependent generalization of the Hartree-Fock method. The ψ_i are solutions to the Dirac equation in the scalar field σ which is in turn determined self-consistently by (2.14). The first term on the right in (2.14) is the contribution to σ of the negative-α sea and the second term is the contribution of the additional n_0 fermions and/or antifermions in the state α_0. While Eqs. (2.3), (2.4), (2.13), and (2.14) look like Hartree-Fock equations our semiclassical approximation should not be confused with the usual Hartree-Fock approximation. Our equations refer to time-dependent fields whereas the Hartree-Fock procedure for approximating an energy level is based on time-independent equations.

In Sec. IV we will display explicit time-dependent solutions to Eqs. (2.3), (2.4), (2.13), and (2.14), suitably modified to take account of the fact that α_0 will be π. Before doing that we will find a set of time-independent stationary-phase points. For a time-independent σ we have [see (2.5)] $\alpha_i=\omega_i T$ where the ω_i are the positive energy levels of the Dirac equation in the time-independent external field σ. The analog of (2.11) is then easily seen to be

$$\frac{\delta}{\delta\sigma(x)}\left(-\int_{-\infty}^{\infty}\frac{Z}{2}\{[\sigma(x')]^2-\sigma_0^2\}\,dx'+N\sum_{i=0}^{\infty}[\omega_i(\sigma)-\omega_i(\sigma_0)]-n_0\omega_0(\sigma)\right)=0\,. \tag{2.15}$$

In the next section we will see how to solve (2.15) directly without recourse to the analog of (2.14).

E. Calculating the particle spectrum

In Sec. IV we will find a solution to (2.3), (2.4), (2.13), and (2.14) for each given value of T and n_0. These are the stationary-phase points of the functional integrals in (2.10). To carry out the full WKB method, we should then perform a Gaussian functional integral around each stationary-phase point. We have not been able to see how to compute these Gaussian integrals. In our semiclassical approximation the particle spectrum depends only on the phase of the Gaussian integrals. This phase is generally small compared to the main phase which is the phase of the integrand at the stationary-phase point. Therefore, as a first approximation we can neglect the phase of the Gaussian integrals and still hope that our result will be qualitatively correct. We will certainly be getting the correct answer for large N. For large N the phase of the integrand is of order N while the phase of the relevant Gaussian integrals will be of order 1.

In our previous papers we have shown how to take translational invariance into account and how to obtain the correct energy-momentum relation $E=(p^2+m^2)^{1/2}$ for our particle states. We will not go through the same calculations here. Our classical σ fields will correspond to particles at rest. The energies of states will then be the masses of particles.

Except for this neglect of small phases from the Gaussian integrals and the restriction to the rest frame, the calculation of the particle spectrum will proceed as in the sine-Gordon equation. Let σ_{T,n_0} be a solution to (2.3), (2.4), (2.13), and (2.14) for given T and n_0. We can define an effective action

$$S_{n_0}(T)\equiv\int_0^T\int_{-\infty}^{\infty}\left[-\frac{Z}{2}(\sigma_{T,n_0}{}^2-\sigma_0^2)\right]dt\,dx+N[\varphi(\sigma_{T,n_0})-\varphi(\sigma_0)]-n_0\alpha_0(\sigma_{n_0,T})\,. \tag{2.16}$$

In practice we will find that $\alpha_0=\pi$ for all n_0 and T and that the sum of the first two terms on the right of (2.16) does not depend on n_0 so that

$$S_{n_0}(T)=S(T)-n_0\pi \tag{2.17}$$

in an obvious notation. We will go through the quantization for this particular case. The extension to more general situations is straightforward.

We define $T(E)$ by

$$\frac{dS(T(E))}{dT} = -E \tag{2.18}$$

and make a Legendre transformation

$$W(E) = S(T(E)) + ET(E) . \tag{2.19}$$

In the semiclassical approximation the energy levels will then be the roots of

$$W(E) = 2k\pi + n_0\pi , \qquad k = 0, 1, 2, \ldots , \quad k \neq 0 \text{ for } n_0 = 0 . \tag{2.20}$$

Note that the energy levels will depend only on the "principal quantum number" n defined by

$$n = 2k + n_0 . \tag{2.21}$$

If n is even (odd) the energy level E_n contains all $O(2N)$ antisymmetrical tensors of even (odd) rank up to n. The states with n odd are fermions while those with n even are bosons. The specific calculations of Sec. IV produce the spectrum discussed in the Introduction.

For time-independent σ's we do not need the full WKB quantization formalism. If σ_{n_0} is a solution to (2.15) then the energy of the corresponding quantum state is the "classical" energy

$$E_{n_0} = + \int_{-\infty}^{\infty} \frac{Z}{2} (\sigma_{n_0}{}^2 - \sigma_0{}^2) dx - N \sum_{i=0}^{\infty} [\omega_i(\sigma_{n_0}) - \omega_i(\sigma_0)] + n_0\omega_0(\sigma_{n_0}) . \tag{2.22}$$

It will turn out that most of the energy levels obtained from time-independent σ's in the next section are the subset of levels obtained from the general formula (2.20) with $k = 0$. In these cases the time-dependent solutions degenerate into time-independent fields. There are additional time-independent solutions in which $\sigma(+\infty) = -\sigma(-\infty) = \pm\sigma_0$. These are analogous to kinks in the φ^4 theory or solitons in the sine-Gordon equation.

III. TIME-INDEPENDENT SOLUTIONS

We want to use the semiclassical functional method on the model Lagrangian

$$\mathcal{L}_\psi = \bar{\psi}(i\partial\!\!\!/)\psi + \tfrac{1}{2} g^2 (\bar{\psi}\psi)^2 , \tag{3.1}$$

where ψ is an N-component massless fermion field, a model originally studied by Gross and Neveu. As explained in Sec. II, this amounts to finding, for a time-independent auxiliary scalar field $\sigma(x)$, the solution to

$$\frac{\delta}{\delta\sigma(x)} \left[\int_{-\infty}^{+\infty} \left(-\frac{Z}{2} \{ [\sigma(x')]^2 - \sigma_0{}^2 \} \right) dx' + N \sum_{i=0}^{\infty} [\omega_i(\sigma) - \omega_i(\sigma_0)] - n_0\omega_0(\sigma) \right] = 0 , \tag{3.2}$$

which determines the stationary-phase points of Eq. (2.8). We will show how to solve this equation directly, by using the results of Appendix B, without knowing a particular classical solution $\sigma(x)$. Classical solutions can be reconstructed by the inverse scattering method. The strategy is the following: Equation (3.2) is reexpressed as a function of the scattering data of an associated Schrödinger scattering problem. This is arranged in such a way that the following are true.

(a) $\sigma(x)$ is expressed as a function of asymptotic scattering data

$$\sigma(x) = F\{r(k) \ (-\infty < k < +\infty); \ k_l \ (l = 1, 2, \ldots, n)\} , \tag{3.3}$$

where $r(k)$ is the reflection coefficient and the set $-k_l{}^2$ are the bound-state eigenvalues.

(b) Varying functionally with respect to $\sigma(x)$ in Eq. (3.2) becomes separate variations on $r(k)$ and k_l.

(c) An algebraic expression for the bound-state energy levels is obtained without finding an explicit classical solution to the scattering problem.

(d) Knowing the levels, $\sigma(x)$ and $\psi(x)$ can be reconstructed by inverse scattering methods.

This will be shown in detail below.

Consider the Dirac equation for the time-independent potential $\sigma(x)$

$$(i\partial\!\!\!/ - g\sigma)\psi(x) = 0 . \tag{3.4}$$

Choose the representation

$$\gamma_1 = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \sigma_z , \quad \gamma_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Then setting $\partial_t{}^2 = \omega^2$ and multiplying on the left of Eq. (3.4) by $(i\partial\!\!\!/ + g\sigma)$ gives

$$(\omega^2 + \partial_x{}^2 - g^2\sigma^2 - g\sigma'\sigma_z)\psi = 0 \tag{3.5}$$

or

$$[\omega^2 + \partial_x{}^2 - g^2(\sigma^2 - \sigma_0{}^2) - g\sigma'\sigma_z - g^2\sigma_0{}^2]\psi = 0 . \tag{3.6}$$

Put this in the form of a Schrödinger equation

$$\psi'' - [g^2(\sigma^2 - \sigma_0{}^2) + g\sigma'\sigma_z]\psi = -(\omega^2 - g^2\sigma_0{}^2)\psi \tag{3.7}$$

and make the identification

$$k^2 \equiv (\omega^2 - g^2\sigma_0{}^2) , \tag{3.8}$$

$$u(x) \equiv g^2(\sigma^2 - \sigma_0{}^2) + g\sigma'\sigma_z = g^2(\sigma^2 - \sigma_0{}^2) \pm g\sigma' , \tag{3.9}$$

with the boundary condition

$$\sigma(x) \to \sigma_0 \ \text{ as } \ |x| \to \infty . \tag{3.10}$$

We now have a problem expressed in the Schrödinger formalism of Appendix B and can write the potential $u(x)$ as a function of reflection coef-

ficients and bound-state eigenvalues of the asymptotic scattering problem. The reader is advised to examine Appendix B at this point in order to follow the argument.

Consider the trace identity for the coefficient C_1, Eq. (B16), which by Eq. (3.9) and Eq. (3.10) implies

$$\frac{-iC_1}{g^2} = \int_{-\infty}^{+\infty} \tfrac{1}{2}(\sigma^2 - \sigma_0^2)dx, \tag{3.11}$$

which is the first term in Eq. (3.2). By using Eq. (B13)

$$C_{2j+1} = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} k^{2j}\ln[1-|r(k)|^2]\,dk - \frac{2}{2j+1}(ik_0)^{2j+1} \tag{3.12}$$

for $j=0$, we get an expression for C_1 directly as a function of scattering data:

$$\frac{+iC_1}{g^2} = \frac{1}{2\pi g^2}\int_{-\infty}^{+\infty}\ln[1-|r(k)|^2]\,dk + \frac{2}{g^2}k_0 = \int_{-\infty}^{+\infty} -\frac{(\sigma^2-\sigma_0^2)}{2}\,dx. \tag{3.13}$$

We have kept only one bound-state eigenvalue in the above formulas, since keeping the sum over them leads to linearly additive energies, and therefore to no new bound states, as the reader can verify. The astute reader can now see the strategy. The variation $\delta/\delta\sigma(x)$ in Eq. (3.2) is being replaced by separate variations on $r(k)$ and k_l, the independent scattering data.

By carefully counting modes in a box, as in previous papers[1,2] one sees that

$$N\left(\sum_{i=0}^{\infty}[\omega_i(\sigma)-\omega_i(\sigma_0)]\right) = N\left(-\int\delta\frac{d\omega}{\pi} + \omega_0(\sigma) - g\sigma_0\right), \tag{3.14}$$

where the phase shift δ is given by

$$\delta = \frac{a^*(k)}{a(k)} = \frac{1}{2\pi}\mathrm{P}\int_{-\infty}^{+\infty}\frac{\ln[1-|r(q)|^2]}{k-q}\,dq + 2\tan^{-1}\frac{k_0}{k}. \tag{3.15}$$

Here P stands for taking the principal value and we have used Eq. (B11), keeping only one bound state, k_0. From Eq. (3.8) one immediately gets

$$\int_0^{\infty}\delta\,d\omega = \frac{1}{2\pi}\int\frac{k\,dk}{(k^2+g^2\sigma_0^2)^{1/2}}\mathrm{P}\int_{-\infty}^{+\infty}\frac{\ln[1-|r(q)|^2]}{k-q}\,dq + 2\int\frac{k\,dk}{(k^2+g^2\sigma_0^2)^{1/2}}\tan^{-1}\frac{k_0}{k}. \tag{3.16}$$

Having done this we can assemble the pieces of the action in Eq. (3.2), written as a function of scattering data, and carry out the variations,

$$\frac{S}{TN} = \frac{Z}{N}[I_1(\sigma(r,k_0))] - \frac{1}{\pi}[I_2(\sigma(r,k_0))] + \omega_0(k_0) - g\sigma_0 - \frac{n_0}{N}\omega_0(k_0), \tag{3.17}$$

where I_1 is the representation of $\int_{-\infty}^{+\infty}[-\frac{1}{2}(\sigma^2-\sigma_0^2)]\,dx$,

$$I_1 = \frac{1}{2\pi g^2}\int_{-\infty}^{+\infty}\ln[1-|r(k)|^2]\,dk + \frac{2}{g^2}k_0, \tag{3.18}$$

and I_2 is the representation of $\int\delta\,d\omega$,

$$I_2 = \frac{1}{2\pi}\int_0^{+\infty}\frac{k\,dk}{(k^2+g^2\sigma_0^2)^{1/2}}\mathrm{P}\int_{-\infty}^{+\infty}\frac{\ln[1-|r(q)|^2]}{k-q}\,dq + 2\int_0^{\infty}\frac{k\,dk}{(k^2+g^2\sigma_0^2)^{1/2}}\tan^{-1}\frac{k_0}{k}. \tag{3.19}$$

To find the stationary point, or to solve Eq. (3.2), first vary with respect to $r(k)$, which implies $r(k)=0$. The stationary point is one for which the potential $u(x)$ is reflectionless. Here we see why previous applications of the semiclassical functional method[1,2] always seemed to give reflectionless potentials. It is simply one piece of the stationary-phase condition for the integrand of the path integral representing $\mathrm{tr}\,e^{-iHt}$, which is clear only when one has converted the problem to one involving asymptotic scattering data directly.

After some algebra, the second term I_2, Eq. (3.19), can be written as

$$\int_0^{\infty}\frac{k\,dk}{(k^2+g^2\sigma_0^2)^{1/2}}\tan^{-1}\frac{k_0}{k} = k_0 - \frac{\pi}{2}g\sigma_0 + k_0\int_0^{\Lambda}\frac{dk}{(k^2+g^2\sigma_0^2)^{1/2}} + (g^2\sigma_0^2-k_0^2)^{1/2}\tan^{-1}\frac{(g^2\sigma_0^2-k_0^2)^{1/2}}{k_0}, \tag{3.20}$$

where Λ is a cutoff momentum. Defining $k_0 = g\sigma_0\sin\theta$ implies $\omega_0 = g\sigma_0\cos\theta$, and noting that

$$(g^2\sigma_0^2-k_0^2)^{1/2}\tan^{-1}\frac{(g^2\sigma_0^2-k_0^2)^{1/2}}{k_0} = g\sigma_0\cos\theta\left(\frac{\pi}{2}-\theta\right) \tag{3.21}$$

gives

$$\int_0^\infty \frac{k\,dk}{(k^2+g^2\sigma_0^2)^{1/2}} \tan^{-1}\frac{k_0}{k} = g\sigma_0\cos\theta\left(\frac{\pi}{2}-\theta\right) + g\sigma_0\sin\theta\int_0^\Lambda \frac{dk}{(k^2+g^2\sigma_0^2)^{1/2}} + g\sigma_0\sin\theta - \frac{\pi}{2}g\sigma_0 . \tag{3.22}$$

As explained in Sec. I B, the renormalization constant Z is defined to be such that the constant field $\sigma=\sigma_0$ satisfies the variational equation (2.11). In that constant σ, one simply has a free massive fermion and Eq. (2.14) gives the value of Z

$$Z = g^2\frac{N}{\pi}\int_0^\Lambda \frac{dk}{(k^2+g^2\sigma_0^2)^{1/2}} . \tag{3.23}$$

One then sees that the divergent parts of I_2, coming from Eqs. (3.20) and (3.22), exactly cancel $ZN^{-1}I_1$ and the remaining finite terms in Eq. (3.17) are, after some cancellation,

$$-\frac{S}{NT} = \frac{2}{\pi}g\sigma_0\sin\theta - \frac{2}{\pi}g\sigma_0\theta\cos\theta + \frac{n_0}{N}g\sigma_0\cos\theta . \tag{3.24}$$

Varying with respect to θ gives the quantization condition

$$\left(\frac{2}{\pi}\theta - \frac{n_0}{N}\right)\sin\theta = 0 , \tag{3.25}$$

which implies

$$\theta = \frac{\pi}{2}\frac{n_0}{N} . \tag{3.26}$$

Using the definition of the energy E in Sec. II and using the above value for θ gives the energy spectrum

$$E_{n_0} = \frac{2}{\pi}Ng\sigma_0\sin\theta = \frac{2}{\pi}g\sigma_0 N\sin\left(\frac{\pi}{2}\frac{n_0}{N}\right), \tag{3.27}$$

where n_0 is the number of occupied fermion states.

Knowing that $u(x)$ is reflectionless and knowing the value for θ allow us to reconstruct $\sigma(x)$ by inverse scattering methods.[14,15] The result is

$$\sigma = \sigma_0 + \sigma_0 y\tanh\left(g\sigma_0 yx - \tfrac{1}{4}\ln\frac{1+y}{1-y}\right) - \sigma_0 y\tanh\left(g\sigma_0 yx + \tfrac{1}{4}\ln\frac{1+y}{1-y}\right), \tag{3.28}$$

where $y=\sin\theta$.

For y close to 1, σ looks like a well-separated kink-antikink pair: The presence of many fermions in the kink and the antikink prevent them from collapsing on each other.

So far we have been looking for a time-independent $\sigma(x)$ satisfying the boundary condition $\sigma(x) \to \sigma_0$; $|x|\to\infty$. There is another type of solution with boundary condition $\sigma(+\infty) = -\sigma(-\infty) = \pm\sigma_0$. This is the kink originally discovered by Callan, Coleman, Gross, and Zee. For such a $\sigma(x)$

$$C_1 = -\frac{1}{2i}\int u(x)dx = \frac{1}{2i}\int g^2(\sigma^2-\sigma_0^2)dx \pm \frac{g}{i}\sigma_0 , \tag{3.29}$$

which adds on an extra term in Eq. (3.24). The calculation is similar to the one above and one finds that for any n_0 the only consistent solution is $k_0=1$ or $\omega_0=0$. Therefore there is only a single kink which is the same hyperbolic-tangent solution as was found in the ϕ^4 theory.[1] In this kink, the Dirac equation has a zero-energy bound state ψ_0, with $\bar\psi_0\psi_0=0$. We can put any number $n_0<N$ of fermions in this state without increasing the energy, and since $\bar\psi_0\psi_0=0$, the trapped fermions do not react back on $\sigma(x)$, leaving the simple hyperbolic tangent an exact solution for any $n_0<N$. The mass of a kink is found to be

$$M_{\text{kink}} = g\sigma_0\frac{N}{\pi} , \tag{3.30}$$

independent of n_0.

IV. TIME-DEPENDENT SOLUTIONS

In this section, we exhibit nontrivial time-dependent, periodic solutions to the functional Eq. (2.11). After semiclassical quantization, as explained in Refs. 1 and 2, they will give us additional particlelike states.

Time-dependent solutions to the fermion-loop functional are more difficult to find than time-independent ones. This is because the formalism for inverse scattering in a time-dependent potential has not been developed yet; hence, we have no general method analogous to Sec. III and Appendix B; instead we have to guess the correct form. The very simple analytic form of the results of this section leads us to suspect that there must exist some easily tractable general formalism, at least for potentials which are periodic in time, and reflectionless.

Considering the close relationship of the results of Sec. III with those of the sine-Gordon Lagrangian, we look for periodic σ fields which would be analytically related to the sine-Gordon doublet in the same fashion as the kink of Sec. III is related to the sine-Gordon soliton. After contemplating for some time the analytic form of the sine-Gordon doublet and of the small oscillations around

it, as given in Ref. 2, we are led to try, in units of σ_0, setting $g=1$,

$$\sigma = 1 + \xi f_2 + \eta f_4 , \tag{4.1}$$

where

$$f_2 = D^{-1}\cos\Omega t ,$$

$$f_4 = D^{-1} = (\cosh Kx + a\cos\Omega t + b)^{-1} ,$$

with t and x expressed in units of $(g\sigma_0)^{-1}$. K, Ω, ξ, η, a, and b are constants to be determined later by consistency.

For the solutions of the Dirac equation[16]

$$(i\not{\partial} - \sigma)\psi = 0 \tag{4.2}$$

with σ as in (4.1), we try the form

$$\psi(k) = e^{i(kx-\omega t)}\begin{pmatrix} u_1 \\ d_1 \end{pmatrix} , \tag{4.3}$$

with

$$\begin{aligned} u_1 &= \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \alpha_4 f_4 , \\ d_1 &= \beta_0 + \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3 + \beta_4 f_4 , \end{aligned} \tag{4.4}$$

where we have introduced the functions

$$f_1 = D^{-1}\sinh Kx , \quad f_3 = D^{-1}\sin\Omega t . \tag{4.5}$$

It is then straightforward to compute the α's and β's of (4.4). One finds, up to an over-all phase,

$$\begin{aligned} u_1 &= \frac{1}{A}\left(1 + \frac{iK}{2k} f_1 + \frac{ia}{k} - \frac{a\Omega}{2k}\,\frac{(1-ik)}{\omega} f_3 - \eta\,\frac{i}{2k} f_4\right) , \\ d_1 &= \frac{1}{A}\left(\frac{1-ik}{\omega} + \frac{K}{2k}\,\frac{i+k}{\omega} f_1 - \frac{a}{k}\,\frac{i+k}{\omega} f_2 \right. \\ &\qquad \left. + \frac{a\Omega}{2k} f_3 + \frac{\eta}{2k}\,\frac{i+k}{\omega} f_4\right) , \end{aligned} \tag{4.6}$$

where A is a normalization factor, which is found to be

$$\frac{1}{A} = \frac{1}{\sqrt{L}}\,\frac{k\sqrt{2}}{(4k^2+K^2)^{1/2}} . \tag{4.7}$$

One also finds a set of relations among the parameters of σ:

$$\begin{aligned} &\xi = -2a, \quad \eta = -\tfrac{1}{2}bK^2, \quad \Omega^2 + K^2 = 4 , \\ &\eta^2 + K^2(1-b^2) + \Omega^2 a^2 = 0 , \\ &\omega^2 - k^2 = 1 . \end{aligned} \tag{4.8}$$

By analogy with the sine-Gordon doublet, we write

$$\Omega = \frac{2}{(1+\epsilon^2)^{1/2}} , \quad K = \frac{2\epsilon}{(1+\epsilon^2)^{1/2}} . \tag{4.9}$$

One can then consider ϵ and b as free parameters at this stage.

We can now compute the contribution of the continuum states given by (4.6) to the right-hand side of Eq. (2.14). The result is

$$\bar{\psi}(k)\psi(k) = \frac{N}{L\omega}\sigma - \frac{2\epsilon^2 b(k^2+1)^{-1/2}D^{-1}N}{L(1+\epsilon^2)[k^2(1+\epsilon^2)+\epsilon^2]} . \tag{4.10}$$

The sum over the continuum modes then gives

$$\sum_k \bar{\psi}(k)\psi(k) = \frac{N}{\pi}\sigma\int_0^\Lambda \frac{dk}{(k^2+1)^{1/2}} - \frac{\epsilon b N D^{-1}}{1+\epsilon^2}\left(1 - \frac{2}{\pi}\tan^{-1}\epsilon\right) , \tag{4.11}$$

where Λ is an ultraviolet cutoff, in units of $g\sigma_0$. Considering the value of the renormalization constant Z in (2.14), we then see that the first term of the right-hand side of (4.11) is exactly equal to the left-hand side of (2.14). We will have achieved our task of finding a time-dependent solution to (2.14) if the second term of the right-hand side of (4.11) cancels exactly with the contribution of the bound states, which we now compute.

The bound-state solutions to the Dirac equation (4.2) involve linear combinations of the functions

$$\begin{aligned} \varphi_1 &= D^{-1}\cos\tfrac{1}{2}\Omega t\cosh\tfrac{1}{2}Kx , \\ \varphi_2 &= D^{-1}\sin\tfrac{1}{2}\Omega t\sinh\tfrac{1}{2}Kx , \\ \varphi_3 &= D^{-1}\sin\tfrac{1}{2}\Omega t\cosh\tfrac{1}{2}Kx , \\ \varphi_4 &= D^{-1}\cos\tfrac{1}{2}\Omega t\sinh\tfrac{1}{2}Kx . \end{aligned} \tag{4.12}$$

By identification in Eq. (4.2), one finds the bound-state wave functions to be

$$\begin{aligned} u_0 &= \lambda\left(\varphi_1 + \frac{\epsilon}{(1+\epsilon^2)^{1/2}}\,\frac{1-a-b}{a-\frac{1}{2}\eta}\,\varphi_4\right) \\ &\quad + \mu\left(i\epsilon\,\frac{1-b-\frac{1}{2}\eta}{a-\frac{1}{2}\eta}\varphi_2 - \frac{i}{(1+\epsilon^2)^{1/2}}\,\frac{1+b-a}{1+b+\frac{1}{2}\eta}\,\varphi_3\right) , \end{aligned} \tag{4.13}$$

$$\begin{aligned} d_0 &= \lambda\left(-i\epsilon\,\frac{1-b-\frac{1}{2}\eta}{a-\frac{1}{2}\eta}\,\varphi_2 - \frac{i}{(1+\epsilon^2)^{1/2}}\,\frac{1+b-a}{1+b+\frac{1}{2}\eta}\,\varphi_3\right) \\ &\quad + \mu\left(\varphi_1 - \frac{\epsilon}{(1+\epsilon^2)^{1/2}}\,\frac{1-a-b}{a-\frac{1}{2}\eta}\,\varphi_4\right) , \end{aligned} \tag{4.14}$$

where λ and μ are free parameters; λ/μ being arbitrary, this means that there are two bound states, both of which have their Floquet index equal to $\pm\pi$. Because of this, one must be careful in applying the reasoning of Appendix A to compute their contribution to σ: One must apply first-order degenerate perturbation theory. This is most easily done by noticing that a perturbation $\delta\sigma$ of the type considered in Appendix A is diagonal in the basis

$$\lambda_1 = \mu_1 ,$$
$$\lambda_2 = -\mu_2 . \tag{4.15}$$

In such a perturbation, the Floquet indices move away from $\pm\pi$. The amount by which they move gives the value of $\bar{\psi}_0\psi_0$ which will enter Eq. (2.14). Performing this calculation and normalizing the bound-state wave functions gives the remarkably simple result

$$\bar{\psi}_0\psi_0 = \frac{\epsilon D^{-1}}{(1+\epsilon^2)^{1/2}} . \tag{4.16}$$

Combining this with Eq. (4.11), we see that we have found a solution to the functional Eqs. (2.11)–(2.14) by choosing

$$b = \frac{N-n_0}{N} \frac{(1+\epsilon^2)^{1/2}}{1-(2/\pi)\tan^{-1}\epsilon} . \tag{4.17}$$

To compute the action around one period $S_{n_0}(T)$ according to Eq. (2.16), we follow the related calculation of the sum of stability angles in the sine-Gordon equation as described in Appendix C of Ref. 2. We do not reproduce the details, but simply give the result:

$$S_{n_0}(T) = 2N(\tan^{-1}\epsilon - \epsilon) - n_0\pi \tag{4.18}$$

with

$$T = \frac{2\pi}{\Omega} = \pi(1+\epsilon^2)^{1/2} . \tag{4.19}$$

Hence the classical energy, according to (2.18),

$$E = \frac{2N}{\pi} \frac{\epsilon}{(1+\epsilon^2)^{1/2}} . \tag{4.20}$$

By defining

$$\theta = \tan^{-1}\epsilon , \tag{4.21}$$

we find the quantization condition to be

$$\theta_n = \frac{\pi}{2}\frac{n}{N} , \tag{4.22}$$

where n is defined as in (2.20)–(2.21). Hence the value of the energy levels

$$E_n = \frac{2N}{\pi} \sin\frac{\pi n}{2N} . \tag{4.23}$$

This is, in units of $g\sigma_0$, the result announced in the Introduction by Eq. (1.7). The physical interpretation of the level structure thus obtained is also discussed in the Introduction. As for the values of the various parameters of the solution, we find

$$b = \frac{N-n_0}{N-n} \frac{1}{\cos\theta_n} . \tag{4.24}$$

Going back to Eq. (4.8), we find that the existence of a requires $n \geq n_0$, and, furthermore, that $a = 0$ at $n = n_0$; this means that after some computing we recover as a particular case the time-independent solutions of Eq. (3.27). This also means that we have found analytically a particular small oscillation of those time-independent solutions, which looks like a vibrational excitation. This is a first step in the calculation of the quantum corrections to the time-independent solutions. We also see that in Eq. (4.9) we can choose ϵ pure imaginary, $\epsilon = i/v$, and still have a real σ in Eq. (4.1). Such a σ will then describe the scattering of two fermion-containing kinks of Sec. III with velocities $\pm v$ in the center-of-mass system.

APPENDIX A: Tr e^{-iHT} FOR FERMION FIELDS

We derive Eqs. (2.6) and (2.12). For an anticommuting fermion field ψ the boundary condition in the functional integral for $\mathrm{tr}\, e^{-iHt}$ could either be that it is periodic $\psi(t+T) = \psi(t)$ or that it is antiperiodic $\psi(t+T) = -\psi(t)$. By means of a simple example we will see that the latter is the correct condition. Consider a simple fermionic oscillator defined by the Lagrangian

$$L = ia^\dagger(t)\dot{a}(t) - \omega a^\dagger(t)a(t) . \tag{A1}$$

The system has two energy levels ϵ_0 and ϵ_1 corresponding to the single fermion state being empty or occupied. Since we have not specified the quantum-mechanical ordering of the Lagrangian we know the energy levels only up to an additive constant. However, $\epsilon_1 - \epsilon_0$ must be ω, so we have

$$\epsilon_0 = E_0 - \tfrac{1}{2}\omega ,$$
$$\epsilon_1 = E_0 + \tfrac{1}{2}\omega , \tag{A2}$$

where E_0 is arbitrary. It is clear that

$$\mathrm{tr}\, e^{-iHT} = e^{-iE_0T}(e^{i\omega T/2} + e^{-i\omega T/2}) . \tag{A3}$$

Let us now compute $\mathrm{tr}\, e^{-iHT}$ using the functional-integral method. According to the rules for integrating over fermion fields, one has

$$\mathrm{tr}\, e^{-iHT} = \int [da][da^\dagger] \exp\left(i\int_0^T L\, dt \right)$$
$$= 2e^{-iE_0T} \frac{\mathrm{Det}[id/dt - \omega]}{\mathrm{Det}[id/dt]}$$
$$= 2e^{-iE_0T} \prod_n \frac{\epsilon_n(\omega)}{\epsilon_n(0)} , \tag{A4}$$

where the normalization is set by

$$\mathrm{tr}\, e^{-iHT}\big|_{\omega=0} \equiv 2e^{-iE_0T} \tag{A5}$$

and the eigenvalues $\epsilon_n(\omega)$ are defined by

$$i\frac{d}{dt} f_n(t) - \omega f_n(t) = \epsilon_n(\omega) f_n(t) ,$$
$$f_n(t+T) = \pm f_n(t) , \tag{A6}$$

where the + and − signs refer to periodic and antiperiodic boundary conditions, respectively. For periodic conditions

$$\epsilon_n = -\frac{2n\pi}{T} - \omega, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{A7}$$

while for the antiperiodic case

$$\epsilon_n = -\frac{(2n+1)\pi}{T} - \omega, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{A8}$$

It is easy to verify that the product of eigenvalues in (A4) will reproduce (A3) if we choose antiperiodic boundary conditions for which

$$\prod_{n=-\infty}^{\infty} \frac{\epsilon_n(\omega)}{\epsilon_n(0)} = \prod_{n=-\infty}^{\infty} \left(1 + \frac{\omega T}{(2n+1)\pi}\right) = \cos\frac{\omega T}{2}. \tag{A9}$$

Any free-field theory of fermions can be reduced to a sum of Lagrangians of the form (A1). Therefore antiperiodic boundary conditions are correct in free-field theory. We will assume that this is general and take our fermion fields to be antiperiodic in time.

Having established the boundary conditions, the calculation of the functional integral in (2.6) proceeds along the lines of Appendix A of Ref. 2. Recalling that ψ really stands for a set of N two-component fermion fields $\psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(N)}$ we write the functional integral in Eq. (2.6) in the more explicit form, setting $g = 1$ for simplicity.

$$I_N(\sigma) = \int \prod_{j=1}^{N} [d\psi^{(j)}][d\bar{\psi}^{(j)}] \times \exp\left(i \sum_{k=1}^{N} \bar{\psi}^{(k)}(i\partial\!\!\!/ - \sigma)\psi^{(k)}\right), \tag{A10}$$

which defines $I_N(\sigma)$. It is immediate that

$$I_N(\sigma) = [I_1(\sigma)]^N, \tag{A11}$$

and according to the rules for integrating over fermion fields I_1 is given by

$$I_1(\sigma) = I_1(0)\frac{\mathrm{Det}[\gamma^0(i\partial\!\!\!/ - \sigma)]}{\mathrm{Det}[\gamma^0 i\partial\!\!\!/]} = I_1(0)\prod \frac{\epsilon(\sigma)}{\epsilon(0)}, \tag{A12}$$

where the eigenvalues ϵ are defined by

$$\gamma^0(i\partial\!\!\!/ - \sigma)\xi = \epsilon\xi \tag{A13}$$

and

$$\xi(x, t+T) = -\xi(x, t), \quad \xi(x+L, t) = \xi(x, t), \tag{A14}$$

where we have imposed periodic boundary conditions in a large box of length L. The eigenvalue can be obtained from the Floquet indices as follows. If we have a solution to the Dirac equation

$$(i\partial\!\!\!/ - \sigma)\psi_k = 0 \tag{A15}$$

satisfying

$$\psi_k(x, t+T) = e^{-i\alpha_k}\psi_k(x, t), \quad \psi_k(x+L, t) = \psi_k(x, t), \tag{A16}$$

then the function

$$\xi_{n,k} = \exp\left(i(2n+1)\frac{\pi t}{T} + i\alpha_k\frac{t}{T}\right)\psi_k \tag{A17}$$

satisfies

$$\xi_{n,k}(x, t+T) = \xi_{n,k}(x, t), \quad \xi_{n,k}(x+L, t) = \xi_{n,k}(x, t), \tag{A18}$$

and

$$\gamma^0(i\partial\!\!\!/ - \sigma)\xi_{n,k} = -\left(\frac{(2n+1)\pi}{T} + \frac{\alpha_k}{T}\right)\xi_{n,k}. \tag{A19}$$

We now have the eigenvalues and, from (A12),

$$I_1(\sigma) = I_1(0)\prod_k \prod_{n=-\infty}^{\infty}\left(1 + \frac{\alpha_k}{(2n+1)\pi}\right) = I_1(0)\prod_k \cos\frac{\alpha_k}{2}. \tag{A20}$$

Comparing with free-field theory shows that, up to a phase which represents an ambiguity in the vacuum energy, $I_1(0) = \Pi_k[2]$ so that

$$I_1(\sigma) = \prod_k (e^{i\alpha_k/2} + e^{-i\alpha_k/2}). \tag{A21}$$

As mentioned in the text, for the Dirac equation in a scalar external field charge-conjugation invariance implies that the α_k come in plus-minus pairs with equal magnitude and opposite signs. Using this fact we can rewrite $I_1(\sigma)$ as

$$I_1(\sigma) = \prod_{\alpha_k > 0} (e^{i\alpha_k/2} + e^{-i\alpha_k/2})^2 = \prod_{\alpha_k > 0} e^{i\alpha_k}(1 + e^{-i\alpha_k})^2, \tag{A22}$$

which yields for I_N

$$I_N(\sigma) = [I_1(\sigma)]^N = e^{iN\varphi}\prod_{\alpha_k > 0}(1 + e^{-i\alpha_k})^{2N}, \quad \varphi = \sum_{\alpha_k > 0}\alpha_k. \tag{A23}$$

This is equation (2.6) where in the text we used

the convention that all α's are positive.

We now turn to the derivation of Eq. (2.12). It is based on the fact that α_k is essentially an eigenvalue of $\gamma^0(i\partial\!\!\!/ - \sigma)$. Using ordinary first-order perturbation theory for the eigenvalue problem in (A19) leads to

$$\frac{1}{T}\frac{\delta\alpha_k}{\delta\sigma(x,t)} = \frac{\bar{\xi}_{n,k}(x,t)\xi_{n,k}(x,t)}{\int_0^T\int_{-\infty}^{\infty}\xi^*_{n,k}\xi_{n,k}\,dx\,dt}\,. \tag{A24}$$

We can simplify (A24) by using

$$\bar{\xi}_{n,k}(x,t)\xi_{n,k}(x,t) = \bar{\psi}_k(x,t)\psi_k(x,t) \tag{A25}$$

and

$$\int_0^T\int_{-\infty}^{\infty}\psi_k^*\psi_k\,dx\,dt = T\int_{-\infty}^{\infty}\psi_k^*\psi_k\,dx\,, \tag{A26}$$

where to get (A26) one uses the Dirac equation to show that $\int|\psi|^2\,dx$ is time-independent. Combining (A24), (A25), and (A26) yields

$$\frac{\delta\alpha_k}{\delta\sigma(x,t)} = \frac{\bar{\psi}_k(x,t)\psi_k(x,t)}{\int_{-\infty}^{\infty}|\psi_k|^2dx}\,, \tag{A27}$$

which is Eq. (2.12) for $g=1$ with the normalization convention (2.13). Equation (A25) can be incorrect if the eigenvalues in (A19) are degenerate. One then has to use degenerate perturbation theory. This point is relevant because we find σ's which have discrete α's at $\pm\pi$. The eigenvalue in (A19) corresponding to $\alpha=-\pi$ and $n=1$ is degenerate with that corresponding to $\alpha=\pi$ and $n=0$. When such a circumstance arises it is straightforward to work out the generalization of (A25). This is done explicitly in Sec. IV.

One can easily extend these techniques to more general cases. Consider the functional integral

$$I = \int[d\psi][d\bar{\psi}]\exp[i\bar{\psi}(i\partial\!\!\!/ - A)\psi] \tag{A28}$$

over antiperiodic ψ's. The matrix A is any combination of c-number functions multiplying γ matrices and/or internal symmetry matrices with the periodicity property $A(t+T)=A(T)$ and γ^0 Hermitian.

As before, we can define Floquet indices by

$$\begin{aligned}&(i\partial\!\!\!/ - A)\psi = 0\,,\\ &\psi(x,T+t) = e^{-i\alpha_i}\psi(x,t)\,.\end{aligned} \tag{A29}$$

A calculation identical to that done above yields

$$I = \prod_k (e^{i\alpha_k/2} + e^{-i\alpha_k/2})\,.$$

In general, the α_k will not come in $\pm$ pairs, but we can always write I as

$$I = \exp\left(\frac{i}{2}\sum_k|\alpha_k|\right)\prod_k(1+e^{-i|\alpha_k|}), \tag{A30}$$

We now have the filled sea contributing a phase $\frac{1}{2}\sum_k|\alpha_k|$ and each factor in the product represents either an empty (1) or occupied ($e^{-i|\alpha_k|}$) state.

APPENDIX B: THE ASSOCIATED SCATTERING PROBLEM AND TRACE IDENTITIES

In Sec. III we will use some formulas from the scattering problem for a Schrödinger operator. These essentially reexpress the potential (time-independent) as certain functions of the scattering asymptotics, i.e., boundary conditions as $|t|\to\infty$. We could simply list the definitions and expressions we will need, but the reader would have no insight as to why the scattering problem or the trace identities have anything to do with computing the stationary-phase point of a functional integral. So, we will begin with a brief discussion of the strategy in a finite dimensional setting where the technical difficulties of scattering theory do not obscure the method.

The aim of inverse scattering methods is to solve a nonlinear evolution equation by linear techniques. A directly related problem is to find a procedure to tell when such an equation has a large, perhaps infinite number of conservation laws and to write them down. Lax[17] in a classic paper on the Korteweg-de Vries equation showed how to associate a linear eigenvalue problem, whose eigenvalues are timeindependent, with the original nonlinear equation. These eigenvalues are constants of the motion for the original problem. Zakharov and Faddeev[15] gave an interpretation of moments of these constants as regularized traces of a singular boundary-value problem. These are the so-called trace identities. We will give a simple example of these notions for a system on a lattice, i.e., the eigenvalues are assumed discrete.

A more elaborate discussion for a particular lattice case can be found in a paper by Flashka.[18] Lax's basic idea is to take a space of $N\times N$ symmetric matrices which are the values of a linear operator $L(t)$ and ask under what conditions are the eigenvalues of $L(t)$ independent of time. A sufficient condition is that $L(t)$ be unitarily equivalent to $L(0)$. Since any unitary operator obeys an equation of the form

$$\dot{U} = BU \text{ where } B = -B^\dagger\,, \tag{B1}$$

unitary equivalence $U^{-1}(t)\mathfrak{L}(t)U(t) = L(0)$ for all t implies what is known as the Lax representation

$$\dot{L} = [B, L]\,. \tag{B2}$$

In other words, unitary equivalence implies Eq. (B2). The converse is also true: Given some skew Hermitian B, if Eq. (B2) holds then $\mathfrak{L}(t)$ is

unitarily equivalent to $\mathcal{L}(0)$. This is the key to the inverse method, since if one rewrites a nonlinear evolution equation by guessing an $\mathcal{L}$ and a B so that Eq. (B2) holds, then the spectral problem for L determines all the constants of the motion. In our case L will be the Schrödinger operator and its eigenvalues are expressed in terms of asymptotic scattering data. In Sec. IV we find a solution to the stationary-point problem essentially by guessing it. Faddeev has suggested that our guess is the solution for an inverse scattering problem for a Dirac operator containing a time-dependent potential. If true, this means that theoretically we could have solved the time-dependent case in the same way as in Sec. III, but using a different associated $\mathcal{L}$.

There is more information in Lax's representation than one would suspect at first glance. One can extract all the conservation laws for the system without solving Eq. (B2). Given a set of eigenvalues $\lambda_1 \ldots, \lambda_n$ of $\mathcal{L}$ one defines moment functions $M_1 = \mathrm{Tr}\lambda_2$, $M_2 = \mathrm{Tr}\lambda_i^2$, etc. The reason for this is twofold: first the expression for $\lambda_i = f(q_1 \cdots q_n)$ in the original variables is both complicated and unknown; second, one can find the M_i, which are conserved, by a method due to Faddeev and Zakharov.

One notices right away that the M_i are coefficients in the expansion of an invariant function. Namely, $\mathrm{Tr}(L-\lambda)^{-1} = \sum_i (1/\lambda_i - \lambda)$ which is t independent for any λ where it exists. Then $\mathrm{Tr}(L-\lambda)^{-1}$ generates $M_i = \mathrm{Tr}L^i$, since

$$\mathrm{Tr}(L-\lambda)^{-1} = -\frac{1}{\lambda}\sum_{i=0}^{\infty}\frac{1}{\lambda^i}(\mathrm{Tr}L^i). \tag{B3}$$

These are the famous trace identities, and although they are trivial here, they become important for the Schrödinger problem, where $\mathrm{Tr}(L-\lambda)^{-1}$ is replaced by the coefficient of a scattering matrix, and the expansion is an asymptotic one in inverse powers of the continuous eigenvalue.

After these lengthy preliminary remarks we just state some definitions and results for the Schrödinger case which we will need, and hope that now the relevance of these expressions will not be so surprising. What follows, especially the particular form of the trace identities, is due to Faddeev and Zakharov,[15] whose work we merely summarize. The reader can refer to their paper for derivations and details. The two ingredients we will need are a dispersion relation for the phase shift and the set of trace identities, all expressed as functions of scattering asymptotics.

The Schrödinger equation for a potential $U(x)$

$$-\psi_{xx} + U(x)\psi = k^2\psi, \tag{B4}$$

subject to a suitable bound on the integral of the potential, has a twofold positive continuous spectrum, and a finite number of negative eigenvalues $-k_l^2$ $(l=1, 2, \ldots, n)$. The function $r(k)$, the reflection coefficient, is defined through the asymptotic form of the solution $\psi(x, k)$ of (B4) as follows.

$$\begin{aligned}&\psi(x,k) \sim e^{ikx} + r(k)e^{-ikx} \quad (x \to -\infty),\\ &\psi(x,k) \sim t(k)e^{ikx} \quad (x \to +\infty).\end{aligned} \tag{B5}$$

If $\psi_l(x)$ are the eigenfunctions of the discrete spectrum, normalized by the condition $\psi_l(x) \sim e^{k_l x}$ $(x \to -\infty)$ and d_l are the corresponding normalization factors, $d_l = (\int_{-\infty}^{+\infty}\psi_l^2 dx)^{-1}$, then the set $S = \{r(k), k_l, d_l\}$ is the asymptotic scattering data for the problem (B4). It is well known that the mapping $U(x) \to S$ of the set of potentials $U(x)$ into the scattering data is uniquely invertible. Knowing S, one can recover $U(x)$ completely. The way to do this is by the method of inverse scattering and has been rigorously investigated by Faddeev.[14]

We will need some more definitions, which are the standard ones. The Schrödinger equation (B1) has a pair of solutions $f(x, k)$, $g(x, k)$, defined uniquely for all real k by the conditions

$$\begin{aligned}&f(x,k) \sim e^{ikx} \quad (x \to +\infty),\\ &g(x,k) \sim e^{-ikx} \quad (x \to -\infty).\end{aligned} \tag{B6}$$

Also, the coefficients $a(k)$, $b(k)$ are defined by

$$\begin{aligned}&f(x,k) = b(k)g(x,k) + a(k)g(x,-k),\\ &g(x,k) = -b(-k)f(x,k) + a(k)f(x,-k),\end{aligned} \tag{B7}$$

where $a(k)$, $b(k)$ are related through

$$|a|^2 = 1 + |b|^2. \tag{B8}$$

In terms of $f(x, k)$ and $g(x, k)$, Eq. (B4) is invertible, giving

$$a(k) = \frac{1}{2ik}\{f, g\}, \tag{B9}$$

where $\{f, g\} \equiv f_x g - g_x b$. The coefficient $a(k)$ can be defined in terms of the coefficient of reflection by

$$r(k) = \frac{b(k)}{a(k)} \text{ and } 1 - |r|^2 = \frac{1}{|a|^2}. \tag{B10}$$

In order to express the phase shift as a function of scattering data, we will need the following dispersive relation for $a(k)$

$$a(k) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln[1-|r(q)|^2]}{k-q}\,dq\right\} \times \prod_{l=1}^{m}\frac{k-ik_l}{k+ik_l} \quad (\mathrm{Im}\,k \neq 0). \tag{B11}$$

The other main formulas we will need are the trace identities. Here they are simply two ways of writing a representation for the coefficients C_n in the asymptotic expansion of $\ln a(k)$, $|k| \to \infty$, in inverse powers of k

$$\ln a(k) = \sum_{n=1}^{\infty} \frac{C_n}{k^n} , \tag{B12}$$

where it can be shown that

$$C_{2j} = 0 ,$$

$$C_{2j+1} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} k^{2j} \ln[1 - |r(k)|^2]\, dk - \frac{2}{2j+1} \sum_{l=1}^{m} (ik_l)^{2j+1} , \tag{B13}$$

but also

$$C_{2j+1} = -\left(\frac{1}{2i}\right)^{2j+1} \int_{-\infty}^{+\infty} \sigma_{2j+1}(x) dx , \tag{B14}$$

where $\sigma_n(x)$ obey the recursion formula

$$\sigma_n(x) = -\frac{d}{dx}\, \sigma_{n-1}(x) - \sum_{k=1}^{n-1} \sigma_{n-k-1}(x)\sigma_k(x) , \qquad n = 2, \ldots \tag{B15}$$

and

$$\sigma_1(x) = U(x) .$$

In particular, Eqs. (B14) and (B15) imply

$$C_1 = -\frac{1}{2i} \int_{-\infty}^{+\infty} U(x) dx ,$$

$$C_3 = -\frac{1}{8i} \int_{-\infty}^{+\infty} U^2(x) dx , \tag{B16}$$

etc.

These formulas will be sufficient to find the time-independent stationary-phase points of Eq. (2.8).

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