PHYSICAL REVIEW D VOLUME 12, NUMBER 8 15 OCTOBER 1975

Nonspherical deformations of hadronic bags*

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(Received 2 June 1975)

A Hamiltonian formalism is developed to describe the classical motion of a bag-type model for small oscillations of the boundary around a static configuration and to give an approximate quantization of the system. It is shown that in the case of the one-dimensional bag the method reproduces the exact form of the mass-squared operator, which is known from the light-cone quantization. The formalism is then applied to the study of a three-dimensional bosonic bag and in particular to the analysis of the P-wave excitations.

I. INTRODUCTION

In the relativistic bag model, as proposed in Ref. 1, the hadronic fields are confined within a finite region of space, the bag, by the action of a uniform external pressure. This pressure is introduced in the theory by adding to the potential-energy term in the Lagrangian a term proportional to the volume of the bag, of the form $-BV$. It is easy to show that the dynamical system so defined is invariant under Lorentz transformations.[1] The phenomenological parameter B determines the scale of hadron masses.

The model appears to have a rich dynamical structure and useful phenomenological applications,[2-5] but the nonlinearity of the equations of motion for the boundary makes the study of the exact form of the solutions of the theory extremely difficult, even in the case where one assumes that the fields are uncoupled in the interior of the bag. The only classical solutions which are known up to now correspond to motions of the system with a static spherical boundary. A semiclassical quantization of these solutions has been used in Refs. 1, 2, and 5 for a very interesting phenomenological analysis of the S-wave states of the bag.

In this article we try to go beyond the static-boundary solutions by considering the motion of the bag in the limit of small boundary oscillations. From the Lagrangian of the system we derive an approximate Hamiltonian, which becomes exact in the limit where the amplitude of the mode with a static boundary becomes much larger than the amplitudes of the other modes of the field. In this sense our Hamiltonian describes the small oscillations of the bag about the static-boundary solution.

We shall use the approximate Hamiltonian to deduce information on the excited states of the quantum system. We shall derive an expression for the invariant masses of the quantum levels, which we expect to become exact only in the limit of large occupation numbers of the static-boundary mode, but which we conjecture to give good approximate results also for the lower levels of the spectrum. This conjecture is supported by the analysis of the one-dimensional bag: This system can be solved exactly by the use of light-cone variables, and we shall see that the expansion of the mass-squared operator for small boundary oscillations in an ordinary spacelike frame reproduces the correct result, known from the light-cone quantization.

In this article we study a bag model for a single complex scalar field. Even if nonsuitable for phenomenological purposes, this system has the advantage of an algebraic structure much simpler than a set of spinor fields, and we find it convenient to illustrate our method.

Notice that the Lagrangian of the bag is singular, in the sense that no independent kinetic-energy term is associated with the motion of the boundary. The passage from the Lagrangian to the Hamiltonian is then not straightforward, and involves the use of Dirac's method to eliminate the boundary degrees of freedom.[6] This in turn requires a careful definition of ratios of infinite factors, so that the whole procedure becomes quite complicated. In the approximation of small boundary oscillations, however, the constraints expressing the dependence of the boundary variables on the field degrees of freedom linearize, and the boundary variables can be eliminated by a linear canonical transformation. The final result is the appearance in the Hamiltonian of a term which couples the normal modes of the expansion of the fields and is strongly reminiscent of the term responsible for collective excitations in nuclear physics. This Hamiltonian can be diagonalized, and one obtains a new set of normal modes, which describe the small oscillation of the whole system about the static solution.

The plan of the article is as follows. In Sec. II we study the small oscillations of a one-dimensional bag. The problem is academic, since the exact solution of the equations of motion is known, but it is a useful test of the method. In Sec. III we apply the formalism to the three-dimensional bag with a massless charged scalar field. We show that the different partial waves decouple, and we derive the form of the Hamiltonian that describes their evolution. In Sec. IV we diagonalize the Hamiltonian for the P waves. This sector is particularly interesting, because it contains the first excited levels which cannot be obtained from a semiclassical quantization of the static-boundary solution, and because it contains an expected zero-frequency mode, representing a translation of the whole system. In Sec. V we present a few conclusions.

II. ONE-DIMENSIONAL BAG

The system we want to study is defined by the Lagrangian[1]

$$L=\int_{x_1(t)}^{x_2(t)} dx\left[\left(\frac{\partial\phi^*}{\partial t}\frac{\partial\phi}{\partial t}-\frac{\partial\phi^*}{\partial x}\frac{\partial\phi}{\partial x}\right)-B\right], \quad (2.1)$$

where $\phi(x,t)$ is a complex scalar field subject to the constraints

$$\phi(x_1(t),t)=\phi(x_2(t),t)=0, \quad (2.2)$$

and $x_1(t)$, $x_2(t)$ are the boundary variables, i.e., the end points of the bag.

The boundary conditions [Eqs. (2.2)] represent two holonomic constraints: It is possible to obtain them as equations of motion starting from a different Lagrangian,[7] but, for our purposes, it is more convenient to assume them in the definition of the system.

The Lagrangian of Eq. (2.1) implies the following equation of motion for the field in the interior of the bag:

$$\frac{\partial^2}{\partial t^2}\phi(x,t)-\frac{\partial^2}{\partial x^2}\phi(x,t)=0, \quad (2.3)$$

and, together with the boundary constraints [Eqs. (2.2)], it also implies the two additional nonlinear boundary conditions

$$\begin{aligned}\left(\frac{\partial\phi^*}{\partial x}\frac{\partial\phi}{\partial x}\bigg|_{x=x_1}\right)\left[1-\left(\frac{dx_1}{dt}\right)^2\right]&=B,\\ \left(\frac{\partial\phi^*}{\partial x}\frac{\partial\phi}{\partial x}\bigg|_{x=x_2}\right)\left[1-\left(\frac{dx_2}{dt}\right)^2\right]&=B.\end{aligned} \quad (2.4)$$

Equations (2.2)–(2.4) look rather difficult to solve, but it is easy to find a set of static-boundary solutions, characterized by constant values of x_1 and x_2. Indeed, with x_1 and x_2 constant, Eqs. (2.2) and (2.3) admit as general solution

$$\phi(x,t)=\sum_{m=-\infty}^{\infty}\phi_m\exp\left(\frac{i\pi m t}{x_2-x_1}\right)\sin\left(\pi m\frac{x-x_1}{x_2-x_1}\right). \quad (2.5)$$

Substituting this expression in Eqs. (2.4), we find that the coefficients of all time-dependent terms in the product $(\partial\phi^*/\partial x)(\partial\phi/\partial x)$ at $x=x_1$ and $x=x_2$ must be zero. This can be achieved by taking just one of the normal-mode coefficients ϕ_m different from zero. Inserting

$$\phi(x,t)=\phi_m\exp\left(\frac{i\pi mt}{x_2-x_1}\right)\sin\left(\pi m\frac{x-x_1}{x_2-x_1}\right) \quad (2.6)$$

in the left-hand side of Eqs. (2.4) we find the condition

$$\left(\frac{\pi m}{x_2-x_1}\right)^2\phi_m^*\phi_m=B, \quad (2.7)$$

so that

$$\phi(x,t)=\sqrt{B}\,\frac{x_2-x_1}{\pi m}e^{i\theta}\times\exp\left(\frac{i\pi mt}{x_2-x_1}\right)\sin\left(\pi m\frac{x-x_1}{x_2-x_1}\right) \quad (2.8)$$

is a solution of the classical equations of motion of the bag.

Notice that, for each value of m, Eq. (2.8) gives a one-parameter class of solutions, the relevant parameter being the length of the bag,

$$l=x_2-x_1. \quad (2.9)$$

As a matter of fact, we should consider as free parameters in Eq. (2.8) also the arbitrary phase θ and the over-all position of the bag in space, determined for instance by the coordinate of its center

$$x_0=\tfrac{1}{2}(x_1+x_2). \quad (2.10)$$

These are, however, cyclic variables.

The purpose of this section is to study motions of the system which can be thought of as small oscillations about the static-boundary solutions of lowest frequency, corresponding to $m=1$. We expand around these solutions because a semiclassical method of quantization associates with them the state of lowest energy. We shall obtain a Hamiltonian, quadratic in a set of canonical variables that describe the small oscillations of the system and depending on a "large" action variable, related to the amplitude of the static-boundary solution.

Before proceeding, we find it useful to summarize briefly the properties of the exact solutions of the system, which, as we have mentioned in the Introduction, can be obtained by a suitable canonical transformation in the light-cone frame.[1]

According to Ref. 1, it is convenient to introduce light-cone variables

$$x^{\pm} = (t \pm x)/\sqrt{2} \tag{2.11}$$

and to consider x^+ as the evolution variable. In terms of these variables the equations of motion and boundary conditions become

$$\frac{\partial^2}{\partial x^+ \partial x^-} \phi(x^+, x^-) = 0, \tag{2.12}$$

$$\phi(x_i^-(x^+), x^+) = 0 \tag{2.13}$$

and

$$2\left(\frac{\partial \phi^*}{\partial x^-} \frac{\partial \phi}{\partial x^-}\bigg|_{x^- = x_i^-}\right) \frac{dx_i^-}{dx^+} = B, \tag{2.14}$$

where x_i^-, $i = 1, 2$, are the end points of the bag.

The crucial point is now that Eq. (2.12) is invariant under changes of parametrization of the form

$$x^- = x^-(\xi). \tag{2.15}$$

By choosing a new variable in such a way that

$$\frac{dx^-}{d\xi} = \frac{2}{B} \frac{\partial \phi^*}{\partial \xi} \frac{\partial \phi}{\partial \xi} \tag{2.16}$$

[which is always possible since $(\partial \phi^*/\partial \xi)(\partial \phi/\partial \xi)$ is a non-negative-definite quantity], the nonlinear boundary condition (2.14) reduces to

$$\frac{d\xi_i}{dx^+} = 1, \quad i = 1, 2 \tag{2.17}$$

and can be satisfied by taking

$$\xi_1 = x^+, \quad \xi_2 = x^+ + A, \tag{2.18}$$

where A is a constant parameter.

Equations (2.12) and (2.13) now give

$$\frac{\partial^2}{\partial x^+ \partial \xi} \phi = 0 \tag{2.19}$$

and

$$\phi(x^+, x^+) = \phi(x^+, x^+ + A) = 0. \tag{2.20}$$

The most general solution to these equations is

$$\phi = \sum_{n=-\infty}^{\infty} \phi_n \left[\exp\left(-\frac{2\pi i n x^+}{A}\right) - \exp\left(-\frac{2\pi i n \xi}{A}\right)\right]. \tag{2.21}$$

It is useful to parametrize the field in terms of a variable σ with constant range: $0 \leq \sigma \leq 1$. One defines

$$\Phi(\sigma, x^+) = \phi(x^+, x^+ + A\sigma). \tag{2.22}$$

Equation (2.21) can now be written as

$$\Phi(\sigma, x^+) = \sum_{n=1}^{\infty} \left[\frac{\Phi_{n,+}}{\sqrt{2n}} (e^{-2\pi i n \sigma} - 1) e^{-2\pi i n x^+/A} + \frac{\Phi^*_{n,-}}{\sqrt{2n}} (e^{2\pi i n \sigma} - 1) e^{2\pi i n x^+/A}\right], \tag{2.23}$$

where a new set of expansion coefficients has been introduced, so that the canonical light-cone Poisson brackets

$$\left\{\frac{\partial \phi(\sigma)}{\partial \sigma}, \phi(\sigma')\right\} = 0, \tag{2.24}$$

$$\left\{\frac{\partial \phi(\sigma)}{\partial \sigma}, \phi^*(\sigma')\right\} = \frac{i}{2} \delta(\sigma - \sigma') \tag{2.25}$$

imply harmonic-oscillator Poisson brackets for the variables $\Phi_{n,i}$ ($n = 1, 2, \ldots$; $i = +, -$)

$$\{\Phi_{m,i}, \Phi_{n,j}\} = 0, \tag{2.26}$$

$$\{\Phi_{m,i}, \Phi^*_{n,j}\} = i\delta_{m,n}\delta_{i,j}. \tag{2.27}$$

It is apparent from Eq. (2.23) that the infinitesimal generator of x^+ evolution is given by

$$H \equiv P^- = \left(\frac{2\pi}{A}\right) \sum_{n=1}^{\infty} \sum_{i=\pm} n \Phi^*_{n,i} \Phi_{n,i}. \tag{2.28}$$

On the other hand, the total P^+ momentum carried by the field is given by

$$\begin{aligned} P^+ &= 2\int_{x_1^-}^{x_2^-} dx^- \frac{\partial \phi^*}{\partial x^-} \frac{\partial \phi}{\partial x^-} \\ &= \int_{\xi_1}^{\xi_2} B\, d\xi \\ &= BA. \end{aligned} \tag{2.29}$$

Substituting in Eq. (2.28) one finds[1]

$$P^- = \frac{2\pi B}{P^+} \sum_{n=1}^{\infty} \sum_{i=\pm} n \Phi^*_{n,i} \Phi_{n,i} \tag{2.30}$$

or

$$\begin{aligned} M^2 &= 2P^+P^- \\ &= 4\pi B \sum_{n=1}^{\infty} \sum_{i=\pm} n \Phi^*_{n,i} \Phi_{n,i}. \end{aligned} \tag{2.31}$$

This equation becomes the following equation for the mass-squared operator of the quantum system:

$$M^2 = 4\pi B \left(\sum_{n=1}^{\infty} \sum_{i=\pm} n \Phi^\dagger_{n,i} \Phi_{n,i} + c\right), \tag{2.32}$$

where one allows for the presence of an additive constant that cannot be determined from the classical limit.

We see that the mass-squared operator is given by a sum of terms proportional to the occupation numbers of an infinite set of uncoupled harmonic oscillators, labeled by an internal quantum number n and by an index $i=+,-$ associated with the two possible charge states of the complex field ϕ.

Let us now analyze the evolution of the system in an ordinary spacelike frame. It is convenient to map the region $x_1(t)\leq x\leq x_2(t)$ occupied by the bag into a constant domain

$$0\leq\sigma\leq 1 \tag{2.33}$$

by

$$x(\sigma)=x_1+\sigma(x_2-x_1) = x_0+(\sigma-\tfrac{1}{2})l\,. \tag{2.34}$$

We define

$$\Phi(\sigma,t)=\phi(x(\sigma),t). \tag{2.35}$$

In terms of Φ the Lagrangian becomes

$$L=\int_0^1 d\sigma\left\{l\left[\dot\Phi-\frac{\dot x_0+\dot l(\sigma-\frac{1}{2})}{l}\Phi'\right]^*\left[\dot\Phi-\frac{\dot x_0+\dot l(\sigma-\frac{1}{2})}{l}\Phi'\right]-\frac{\Phi'^*\Phi'}{l}\right\}-Bl\,, \tag{2.36}$$

where we have introduced the notation $\dot f$ and f' for $\partial f/\partial t$ and $\partial f/\partial\sigma$, respectively.

The appearance in L of the combination

$$\dot\Phi-\frac{\dot x_0+\dot l(\sigma-\frac{1}{2})}{l}\Phi'$$

is due to the fact that the Lagrangian contains the time derivative of the fields at fixed x, whereas $\dot\Phi$ is the time derivative at fixed σ.

From the Lagrangian L we can derive the expressions for the momenta $\Pi(\sigma)$, $\Pi^*(\sigma)$, p_0, and p_l conjugate to the dynamical variables $\Phi^*(\sigma)$, $\Phi(\sigma)$, x_0, and l. We find

$$\Pi(\sigma)=\frac{\delta L}{\delta\dot\Phi^*(\sigma)} = l\,\dot\Phi(\sigma)-[\dot x_0+\dot l(\sigma-\tfrac{1}{2})]\,\Phi'(\sigma), \tag{2.37}$$

$$p_0=-\frac{1}{l}\int d\sigma[\Pi^*(\sigma)\Phi'(\sigma)+\Pi(\sigma)\Phi'^*(\sigma)]\,, \tag{2.38}$$

and

$$p_l=-\frac{1}{l}\int d\sigma(\sigma-\tfrac{1}{2})[\Pi^*(\sigma)\Phi'(\sigma)+\Pi(\sigma)\Phi'^*(\sigma)]\,. \tag{2.39}$$

We see that the momenta are not all independent, but that two constraint equations are implied by by the Lagrangian. Defining

$$C_0=p_0+\frac{1}{l}\int_0^1 d\sigma(\Pi^*\Phi'+\Pi\Phi'^*) \tag{2.40}$$

and

$$C_l=p_l+\frac{1}{l}\int d\sigma(\sigma-\tfrac{1}{2})(\Pi^*\Phi'+\Pi\Phi'^*), \tag{2.41}$$

the equations of constraint are

$$C_0=C_l=0, \tag{2.42}$$

and are consequences of the fact that no independent kinetic-energy term is associated with the motion of the boundary.

From the Lagrangian we obtain the Hamiltonian

$$H_0=\int_0^1 d\sigma\frac{\Pi^*(\sigma)\Pi(\sigma)+\Phi'^*(\sigma)\Phi'(\sigma)}{l}+Bl\,. \tag{2.43}$$

Because of the presence of equations of constraint, however, the infinitesimal generator of the motion will be an expression of the more general form[8,9]

$$H=H_0+v_0C_0+v_lC_l. \tag{2.44}$$

It is easy to see the meaning of the multipliers v_0 and v_l in Eq. (2.44). Taking the Poisson bracket of H with x_0 and l we find

$$\dot x_0=\{H,x_0\}=v_0 \tag{2.45}$$

and

$$\dot l=\{H,l\}=v_l\,. \tag{2.46}$$

v_0 and v_l are then the velocities of the quantities x_0 and l and are related through $v_1=v_0-\frac{1}{2}v_l$, $v_2=v_0+\frac{1}{2}v_l$ to the motion of the boundary. We cannot, however, insert into Eq. (2.44) the values of v_0 and v_l that we could deduce from the nonlinear boundary conditions, Eqs. (2.4); rather, we should find those equations as a consistency condition for the system.

It is easy to find further consistency conditions. One can check that $\{C_0,C_l\}=0$, and that

$$C_0'\equiv\{H_0,C_0\}\neq 0,\qquad C_l'\equiv\{H_0,C_l\}\neq 0. \tag{2.47}$$

The consistency requirements $\dot C_0=\dot C_l=0$ imply then the additional constraint equations

$$C_0'=C_l'=0. \tag{2.48}$$

At this point we indicate only formally how one should proceed to define the Hamiltonian, since the expansion for small boundary oscillations will introduce simplifications that will make the following passages not necessary. After having established the additional constraints $C_0'=C_l'=0$ one should check their consistency by taking their Poisson brackets with H. Now one has $\{C_0,C_0'\}\neq 0$, $\{C_l,C_l'\}\neq 0$, $\{H_0,C_0'\}\neq 0$, $\{H_0,C_l'\}\neq 0$, $\{C_0,C_l'\}=\{C_l,C_0'\}=0$, and the requirements $\dot C_0'=\dot C_l'=0$ give

$$v_0 = -\frac{\{H_0, C_0'\}}{\{C_0, C_0'\}}, \tag{2.49}$$

$$v_l = -\frac{\{H_0, C_l'\}}{\{C_l, C_l'\}}. \tag{2.50}$$

Equations (2.49) and (2.50) are only formal, because they both involve a ratio of infinite factors, of the type $\delta(0)/\delta(0)$. However, by a careful definition of those ratios, one shows that they give values of v_0 and v_l in agreement with the nonlinear boundary conditions. The equations $C_0 = C_0' = C_l = C_l' = 0$ can now be used in principle to eliminate all the boundary variables (coordinates and conjugate momenta) in favor of the field variables (but the Poisson brackets must be replaced with Dirac brackets) and Eqs. (2.44), (2.49), (2.50) define the Hamiltonian of the system. We shall see shortly that in the limit of small boundary oscillations the elimination of the boundary variables does not require the introduction of Dirac brackets, but can be performed by a linear canonical transformation.

The static-boundary solution is given by [see Eq. (2.8)]

$$\Phi(\sigma, t) = \sqrt{B}\,\frac{l}{\pi}\,e^{i\pi t/l + i\theta_0}\sin(\pi\sigma). \tag{2.51}$$

It is convenient to choose the phase and amplitude of this solution as dynamical coordinates. We do so by expanding

$$\Phi(\sigma,t) = \left[\sum_{n=1}^{\infty} \phi_n(t)\sqrt{2}\,\sin(\pi n\sigma)\right]e^{i\theta(t)}, \tag{2.52}$$

with $\phi_1(t)$ real, $\phi_n(t)$ $(n \geq 2)$ in general complex. In Eq. (2.52) we have expanded the field into a set of normal modes, and have factored out from all the modes the phase of the first one. The dynamical variables are now ϕ_1 (real), ϕ_n (generally complex, $n \geq 2$) and the angular variable θ. The conjugate momenta are given by

$$p_1 = \int_0^1 d\sigma\sqrt{2}\,\sin\pi\sigma\,[\Pi^*(\sigma)e^{i\theta} + \Pi(\sigma)e^{-i\theta}], \tag{2.53}$$

$$p_n = \int_0^1 d\sigma\sqrt{2}\,\sin\pi\sigma\,\Pi(\sigma)e^{-i\theta}, \tag{2.54}$$

and

$$p_\theta = i\,\phi_1\int_0^1 d\sigma\sqrt{2}\,\sin\pi\sigma[\Pi^*(\sigma)e^{i\theta} - \Pi(\sigma)e^{-i\theta}] + i\sum_{n>1}(\Pi_n^*\phi_n - \Pi_n\phi_n^*). \tag{2.55}$$

The static boundary solution is characterized by

$$\phi_1 = \left(\frac{B}{2}\right)^{1/2}\frac{l}{\pi}, \quad p_\theta = \frac{Bl^2}{\pi}, \quad \theta = \frac{\pi}{l}t + \theta_0,$$

and all other canonical variables equal to zero (for simplicity, we take the cyclic variable x_0 also equal to zero). Let us perform the canonical transformation

$$\tilde\phi_1 = \phi_1 - \left(\frac{p_\theta}{2\pi}\right)^{1/2},$$
$$\tilde l = l - \left(\frac{\pi p_\theta}{B}\right)^{1/2}, \tag{2.56}$$
$$\tilde\theta = \theta - \frac{p_a}{(2\pi p_\theta)^{1/2}} - \frac{p_l}{2}\left(\frac{\pi}{Bp_\theta}\right)^{1/2}$$

(all other canonical variables unchanged). Then the static-boundary solution becomes characterized by the set of "large" conjugate variables $\tilde\theta$ and p_θ, whereas all other canonical variables are zero. $\tilde\theta$ is a cyclic angular variable, p_θ is an action variable.

We can now describe the small oscillations of the system about the static-boundary solution by expanding the Hamiltonian of Eq. (2.44) up to second order in the small variables $\tilde\phi_1$, p_1, ϕ_n, p_n $(n \geq 2)$, $\tilde l$, p_l, and p_0 (x_0 is cyclic). Notice that v_0 and v_l, linearly related to the velocities of the end points of the bag, must turn out to be small variables if the system is consistent. This implies that we can limit the expansion of the constraint equations to the terms of first order in the small coordinates and momenta. This is the crucial point that will allow us to solve the boundary constraints and to obtain an explicit form for the approximate Hamiltonian.

Expanding H up to second order we find

$$H = 2(\pi B p_\theta)^{1/2} + \left(\frac{B}{\pi p_\theta}\right)^{1/2}\left\{B\tilde l^2 + \tfrac{1}{4}p_1^2 + 4\pi^2\tilde\phi_1^2 + \sum_{n>1}[p_n^* p_n + i\pi(\phi_n^* p_n - \phi_n p_n^*) + \pi^2 n^2\phi_n^*\phi_n]\right\}$$
$$+ v_0\left\{p_0 + \left(\frac{B}{2}\right)^{1/2}\left[p_1\langle S_1C_1\rangle + \sum_{n>1}(p_n + p_n^*)\langle C_1S_n\rangle + \sum_{n>1} i\pi n(\varphi_n^* - \phi_n)\langle S_1C_n\rangle\right]\right\}$$
$$+ v_l\left\{p_l + \left(\frac{B}{2}\right)^{1/2}\left[p_1\langle(\sigma - \tfrac{1}{2})S_1C_1\rangle + \sum_{n>1}(p_n + p_n^*)\langle(\sigma - \tfrac{1}{2})C_1S_n\rangle + \sum_{n>1} i\pi n(\phi_n^* - \phi_n)\langle(\sigma - \tfrac{1}{2})S_1C_n\rangle\right]\right\}, \tag{2.57}$$

where we have denoted by $\langle S_m C_n\rangle$ and $\langle(\sigma-\frac{1}{2})\,S_m C_n\rangle$ the integrals $\int_0^1 2\sin\pi m\sigma\cos\pi n\sigma\,d\sigma$ and $\int_0^1 2(\sigma-\frac{1}{2})\times\sin\pi m\sigma\cos\pi m\sigma\,d\sigma$.

This Hamiltonian can be brought to a more convenient form by a canonical transformation

$$p_n=\frac{\tilde p_{n+1,+}}{2}\left(\frac{n}{n+1}\right)^{1/2}+\frac{\tilde p_{n-1,-}}{2}\left(\frac{n}{n-1}\right)^{1/2}-\frac{i\pi}{2}\tilde\phi_{n+1,+}[n(n+1)]^{1/2}+\frac{i\pi}{2}\tilde\phi_{n-1,-}[n(n-1)]^{1/2},$$

$$\phi_n=\frac{\tilde\phi_{n+1,+}}{2}\left(\frac{n+1}{n}\right)^{1/2}+\frac{\tilde\phi_{n-1,-}}{2}\left(\frac{n-1}{n}\right)^{1/2}+\frac{i}{2\pi}\,\frac{\tilde p_{n+1,+}}{[n(n+1)]^{1/2}}-\frac{i}{2\pi}\,\frac{\tilde p_{n-1,-}}{[n(n-1)]^{1/2}},\quad n\geq 2, \tag{2.58}$$

where $\tilde p_{n,+}$, $\tilde\phi_{n,+}$, defined for $n\geq 3$, and $\tilde p_{n,-}$, $\tilde\phi_{n,-}$, defined for $n\geq 1$, are now real variables. It is useful to define also

$$p_1=\sqrt{2}\,\tilde p_{2,+},\quad \tilde\phi_1=\tilde\phi_{2,+}/\sqrt{2}. \tag{2.59}$$

In terms of these new variables H becomes

$$H=2(\pi Bp_\theta)^{1/2}+\left(\frac{B}{\pi p_\theta}\right)^{1/2}\left\{B\tilde l^2+\sum_{n,i}\left[\frac{\tilde p_{n,i}^{\,2}}{2}+\frac{(\pi n)^2\tilde\phi_{n,i}^{\,2}}{2}\right]\right\}$$
$$+v_0\left\{p_0+\sqrt{2B}\sum_{n\,\mathrm{odd}}\left[\frac{2\tilde p_{n,+}}{\pi n}\left(\frac{n-1}{n}\right)^{1/2}+\frac{2\tilde p_{n,-}}{\pi n}\left(\frac{n+1}{n}\right)^{1/2}\right]\right\}+v_l\left\{p_l-\sqrt{2B}\sum_{n\,\mathrm{even}}\left[\frac{\tilde p_{n,+}}{\pi n}\left(\frac{n-1}{n}\right)^{1/2}+\frac{\tilde p_{n,-}}{\pi n}\left(\frac{n+1}{n}\right)^{1/2}\right]\right\}, \tag{2.60}$$

where the sums start at $n=1$ for the terms with index $-$, at $n=2$ for those with index $+$.

The constraints involve now only momentum variables; by a canonical transformation of the form

$$P_0=p_0+\sqrt{2B}\sum_{n\,\mathrm{odd}}\left[\frac{2\tilde p_{n,+}}{\pi n}\left(\frac{n-1}{n}\right)^{1/2}+\frac{2\tilde p_{n,-}}{\pi n}\left(\frac{n+1}{n}\right)^{1/2}\right],$$

$$P_l=p_l-\sqrt{2B}\sum_{n\,\mathrm{even}}\left[\frac{\tilde p_{n,+}}{\pi n}\left(\frac{n-1}{n}\right)^{1/2}+\frac{\tilde p_{n,-}}{\pi n}\left(\frac{n+1}{n}\right)^{1/2}\right], \tag{2.61}$$

$$\psi_{n,\pm}=\tilde\phi_{n,\pm}+x_0\sqrt{2B}\,\frac{2}{\pi n}\left(\frac{n\mp 1}{n}\right)^{1/2},\quad n\ \text{odd}$$

$$\psi_{n,\pm}=\tilde\phi_{n,\pm}-\tilde l\sqrt{2B}\,\frac{2}{\pi n}\left(\frac{n\mp 1}{n}\right)^{1/2},\quad n\ \text{even},$$

the constraints are brought to the very simple form

$$P_0=0,\quad P_l=0, \tag{2.62}$$

and the Hamiltonian becomes

$$H=2(\pi Bp_\theta)^{1/2}+\left(\frac{B}{\pi p_\theta}\right)^{1/2}\left\{B\tilde l^2+\sum_{n,i}\left[\frac{\tilde p_{n,i}^{\,2}}{2}+\frac{(\pi n)^2\psi_{n,i}^{\,2}}{2}\right]\right\}$$
$$-x_0\sqrt{2B}\sum_{n\,\mathrm{odd}}\{2\pi\psi_{n,+}[n(n-1)]^{1/2}+2\pi\psi_{n,-}[n(n+1)]^{1/2}\}+\tilde l\sqrt{2B}\sum_{n\,\mathrm{even}}\{\pi\psi_{n,+}[n(n-1)]^{1/2}+\pi\psi_{n,-}[n(n+1)]^{1/2}\}$$
$$+2Bx_0^{\,2}\sum_{n\,\mathrm{odd}}2\left(\frac{n-1}{n}+\frac{n+1}{n}\right)+2B\tilde l^2\sum_{n\,\mathrm{even}}\frac{1}{2}\left(\frac{n-1}{n}+\frac{n+1}{n}\right)+v_0P_0+v_lP_l. \tag{2.63}$$

The coefficients of $x_0^{\,2}$ and $\tilde l^2$ in this last equation must be dealt with with care, since they are formal divergent expressions. These divergences are introduced by the change of independent variables of Eq. (2.61), and we expect that they cancel against other formal divergent terms originating from the asymptotic behavior of the $\psi_{n,\pm}$.

The correct way to treat the problem is to introduce a cutoff on the number of modes of the field, to eliminate the boundary variables, and then to let the cutoff go to infinity.

We introduce a cutoff by limiting the summations to $n_{\mathrm{max}}=2N-1$ and $2N$ for the odd and even terms. The coefficients of x^2 and $\tilde l^2$ in H become then $(B/\pi p_\theta)^{1/2}\times 8BN$ and $(B/\pi p_\theta)^{1/2}\times B(2N+1)$. By taking the Poisson brackets of H with P_0 and P_l we find the additional constraints

$$\begin{aligned}\sqrt{2B}\,x_0&=\frac{1}{4N}\sum_{n\,\text{odd}}^{2N-1}\{\pi\psi_{n,+}[n(n-1)]^{1/2}+\pi\psi_{n,-}[n(n+1)]^{1/2}\},\\ \sqrt{2B}\,\tilde{l}&=\frac{-1}{2N+1}\sum_{n\,\text{even}}^{2N}\{\pi\psi_{n,+}[n(n-1)]^{1/2}+\pi\psi_{n,+}[n(n+1)]^{1/2}\},\end{aligned}\tag{2.64}$$

which we can use to obtain a new Hamiltonian, expressed now entirely in terms of the field degrees of freedom:

$$\begin{aligned}H_0=2(\pi Bp_\theta)^{1/2}+\left(\frac{B}{\pi p_\theta}\right)^{1/2}\Bigg[&\sum_{n,i}\left(\frac{\tilde{p}_{n,i}{}^2}{2}+\frac{(\pi n)^2\psi_{n,i}{}^2}{2}\right)\\ &-\frac{1}{4N}\left(\sum_{n\,\text{odd}}^{2N-1}\{\pi\psi_{n,+}[n(n-1)]^{1/2}+\pi\psi_{n,-}[n(n+1)]^{1/2}\}\right)^2\\ &-\frac{1}{2(2N+1)}\left(\sum_{n\,\text{even}}^{2N}\{\pi\psi_{n,+}[n(n-1)]^{1/2}+\pi\psi_{n,-}[n(n+1)]^{1/2}\}\right)^2\Bigg].\end{aligned}\tag{2.65}$$

The structure of Eq. (2.65) is interesting. The term

$$H=\sum_{n,i}^{2N}\frac{\tilde{p}_{n,i}{}^2}{2}+\frac{(\pi n)^2\psi_{n,i}}{2}\tag{2.66}$$

is the Hamiltonian for a sequence of free oscillators, with frequencies π (appearing once because the sum starts at $n=2$ for $i=+$) and $2\pi, 3\pi, \ldots, 2N\pi$ (each appearing twice).

The other two terms in the Hamiltonian represent separate couplings of all the modes with odd and even frequencies, originated from the elimination of the boundary variables.

H_0 can be diagonalized by a rotation in the space of the $4N-1$ oscillators. This rotation can be done conveniently in two steps. The rotations

$$\begin{aligned}\tilde{\psi}_{n,+}&=\left(\frac{n+1}{2n}\right)^{1/2}\psi_{n,+}-\left(\frac{n-1}{2n}\right)^{1/2}\psi_{n,-},\\ \tilde{\psi}_{n,-}&=\left(\frac{n-1}{2n}\right)^{1/2}\psi_{n,+}+\left(\frac{n+1}{2n}\right)^{1/2}\psi_{n,-},\end{aligned}\tag{2.67}$$

($n=2, 3, \ldots$), defined within the subspaces of degenerate oscillators, decouple the modes $\tilde{\psi}_{n,+}$, which form then a sequence of free oscillators with frequencies $2\pi, 3\pi, \ldots, 2N\pi$, and the coupling terms take the form

$$\begin{aligned}H_3&=-\frac{1}{2N}\left(\sum_{n\,\text{odd}=1}^{2N-1}\pi n\tilde{\psi}_{n,-}\right)^2,\\ H_4&=-\frac{1}{2N+1}\left(\sum_{n\,\text{even}=2}^{2N}\pi n\tilde{\psi}_{n,-}\right)^2.\end{aligned}\tag{2.68}$$

To find the eigenmodes of $H_1+H_3+H_4$ we must now solve the equations

$$\omega^2\tilde{\psi}_{n,-}=(\pi n)^2\tilde{\psi}_{n,-}-\frac{\pi n}{N}\Psi_{\text{odd}}\quad(n=\text{odd})\tag{2.69}$$

and

$$\omega^2\tilde{\psi}_{n,-}=(\pi n)^2\tilde{\psi}_{n,-}-\frac{\pi n}{N+\frac{1}{2}}\Psi_{\text{even}}\quad(n=\text{even}),\tag{2.70}$$

where we have defined

$$\Psi_{\text{odd,even}}=\sum_{n\,\text{odd,even}}\pi n\tilde{\psi}_{n,-}.\tag{2.71}$$

From Eqs. (2.69) and (2.70) we find

$$\tilde{\psi}_{n,-}=\frac{\pi n\Psi_{\text{odd}}}{N[(\pi n)^2-\omega^2]}\quad(n\text{ odd}),\tag{2.72}$$

$$\tilde{\psi}_{n,-}=\frac{\pi n\Psi_{\text{even}}}{(N+\frac{1}{2})[(\pi n)^2-\omega^2]}\quad(n\text{ even}),\tag{2.73}$$

which, together with Eq. (2.71) give the eigenvalue equations

$$\sum_{n\,\text{odd}=1}^{2N-1}\frac{(\pi n)^2}{(\pi n)^2-\omega^2}=N,\tag{2.74}$$

$$\sum_{n\,\text{even}=2}^{2N}\frac{(\pi n)^2}{(\pi n)^2-\omega^2}=N+\tfrac{1}{2}.\tag{2.75}$$

These two equations can be written

$$\sum_{n\,\text{odd}}\left[\frac{(\pi n)^2}{(\pi n)^2-\omega^2}-1\right]=\omega\sum_{n\,\text{odd}}\frac{\omega}{(\pi n)^2-\omega^2}=0,\tag{2.76}$$

and

$$\sum_{n\,\text{even}}\left[\frac{(\pi n)^2}{(\pi n)^2-\omega^2}-1\right]-\tfrac{1}{2}=\omega\left(\sum_{n\,\text{even}}\frac{\omega}{(\pi n)^2-\omega^2}-\frac{1}{2\omega}\right)=0,\tag{2.77}$$

and we see that we can remove the cutoff. In the limit $N\to\infty$ Eqs. (2.76) and (2.77) become

$$\omega\tan\frac{\omega}{2}=0,\tag{2.78}$$

$$\omega \cot\frac{\omega}{2} = 0, \tag{2.79}$$

and generate the sequence of eigenvalues

$$\omega^{(n)} = \pi n, \quad n \geq 0. \tag{2.80}$$

Notice that the net effect of the elimination of the boundary variables has been to shift downward the frequencies of half of the eigenmodes of the field in the fixed cavity $0 \leq \sigma \leq 1$. In particular, the coupling introduced by the elimination of the variables x_0 and p_0 shifts one of the eigenfrequencies to zero. This zero-mode term is expected, and restores the translational invariance of the system. One can check that the corresponding normal momentum is indeed the total momentum of the system.

Summarizing, we have the following expression for the Hamiltonian:

$$H_0 = 2(\pi B p_\theta)^{1/2} + \left(\frac{B}{\pi p_\theta}\right)^{1/2} \left(\frac{p_0^2}{4B} + \sum_{n=2}^{\infty} \pi n a^*_{n,+} a_{n,+} + \sum_{n=1}^{\infty} \pi n a^*_{n,-} a_{n,-}\right), \tag{2.81}$$

where we have introduced the normal-mode variables $a_{n,i}$ and $a^*_{n,i}$ and have substituted for the momentum of the zero mode its expression in terms of p_0.

The right-hand side of Eq. (2.81) exhibits a divergence for $p_\theta \to 0$, which is, however, of kinematical orgin and is introduced by the expansion of the mass-squared operator in terms of the mass of the static-boundary solution. By taking the square of Eq. (2.81) and neglecting terms $O(1/p_\theta)$, we find

$$\begin{aligned} M^2 &= H_0^2 - p_0^2 \\ &= 4\pi B p_\theta + \sum_{n=2}^{\infty} 4\pi B n a^*_{n,+} a_{n,+} + \sum_{n=1}^{\infty} 4\pi B n a^*_{n,-} a_{n,-}, \end{aligned} \tag{2.82}$$

which we should consider an approximate expression, valid only for

$$p_\theta >> \sum_{n,i} 4 B n a^*_{n,i} a_{n,i}.$$

A semiclassical quantization of the system gives now

$$M_{\text{operator}}^2 = 4\pi B\left(N_\theta + \sum_{n=2}^{\infty} n a^\dagger_{n,+} a_{n,+} + \sum_{n=1}^{\infty} n a^\dagger_{n,-} a_{n,-} + k\right), \tag{2.83}$$

where N_θ stands for the possible integer values of the action variable p_θ; $a^\dagger_{n,i}$ and $a_{n,i}$ are creation and annihilation operators for an infinite set of harmonic oscillators, and we have allowed for the presence of an additive constant k.

This result should be compared with the exact form of the mass-squared operator of Eq. (2.32). We have derived Eq. (2.83) as an approximate expression which becomes exact only in the limit $N_\theta \to \infty$; but we see that it actually reproduces the correct result for the entirety of the spectrum.[10] This gives us some confidence that an analogous expression which we shall derive for the three-dimensional system can give useful information also on the masses of the low-lying states.

III. THREE-DIMENSIONAL SYSTEM

In three dimensions the Lagrangian for the bag containing a complex scalar field $\phi(\vec{x}, t)$ is

$$L = \int_{S(t)} d^3x \left[\left(\frac{\partial\phi^*}{\partial t}\frac{\partial\phi}{\partial t} - \vec{\nabla}_x\phi^* \cdot \vec{\nabla}_x\phi\right) - B\right]. \tag{3.1}$$

$S(t)$ is the spatial region occupied by the bag. We denote by $\partial S(t)$ the boundary of this region. ϕ is constrained to vanish on $\partial S(t)$:

$$\phi(\vec{x}, t) = 0 \text{ for } \vec{x} \in \partial S(t). \tag{3.2}$$

The equations of motion that follow from the variational principle $\delta\int dt L = 0$ are then

$$\frac{\partial^2}{\partial t^2}\phi(\vec{x}, t) - \nabla_x{}^2\phi(\vec{x}, t) = 0, \quad \vec{x} \in S(t) \tag{3.3}$$

and

$$[\vec{\nabla}_x\phi^*(\vec{x}, t) \cdot \vec{\nabla}_x\phi(\vec{x}, t)](1 - \vec{V}_\perp{}^2) = B, \quad \vec{x} \in \partial S(t), \tag{3.4}$$

where $\vec{V}_\perp$ is the velocity of displacement of the boundary in the direction of its normal.

Eqs. (3.2)–(3.4) admit solutions with a static spherical boundary of radius R_0. In particular, it can be checked easily that

$$\phi(\vec{x}, t) = \sqrt{B}\,\frac{R_0^2}{\pi}\,\frac{\sin\pi|x|}{|x|}\,e^{i\pi t/R_0} \tag{3.5}$$

solves the equations of motion.

A semiclassical method of quantization associates with this family of solutions the ground state of the system, and, for this reason, we shall study the small oscillations of the bag about this classical solution.

The formalism that we shall use is quite similar to the formalism used in Sec. II., and so we shall give only a concise description of the steps involved in the linearization of the constraints and the elimination of the boundary variables. We shall, however, treat in some detail the kinematics of the three-dimensional system.

It is convenient to map the region of the bag $S(t)$ into a constant region, which we take to be the unit sphere. We define the mapping by the equation

$$\vec{x}(\vec{z}) = R\vec{z} + \sum_{l,m} b_{l,m} \vec{\nabla}_z z^l Y^l_m(\theta, \phi), \tag{3.6}$$

where z, θ, ϕ are polar coordinates in the z space, l and m are indices that label a set of real orthonormal spherical functions, R and $b_{l,m}$ are parameters of the mapping.

This mapping defines a unique parametrization of the points in the interior and at the surface of the bag, once the shape of the surface is given (at least for not too large boundary deformations). The interior of the bag is represented in z space by points with $z \leq 1$, and the boundary by the surface of the unit sphere.

To understand better the properties of this mapping, notice that the expression

$$\vec{x}'(\vec{z}) = \sum_{l,m} b_{l,m} \vec{\nabla}_z z^l Y^l_m(\theta, \phi)$$

defines a field of vector displacements, over the unit sphere, with $\text{curl}\,\vec{x}(\vec{z}) = 0$, because $\vec{x}'$ is derived from a potential function

$$F(\vec{z}) = \sum_{l,m} b_{l,m} z^l Y^l_m(\theta, \phi),$$

and $\text{div}\,\vec{x}'(\vec{z}) = 0$, because $F(\vec{z})$ is a harmonic function. For small displacements of the boundary of the sphere, the shape of the new boundary is characterized by the value of the normal component of $\vec{x}'(\vec{z})$, $x'_n(\vec{z})$, whereas the tangential component of $\vec{x}'(\vec{z})$ just defines a mapping of the surface into itself. The problem of representing a small boundary deformation by the vector field $\vec{x}'(\vec{z})$ reduces therefore to solving the Neumann problem for a sphere, which admits one unique solution if and only if $\int d\Omega x'_n(\vec{z}) = 0$. Since the series $\sum_{l,m} b_{l,m} \times z^l Y^l_m(\theta, \phi)$ is a complete expansion for the functions harmonic within the unit sphere, we can represent by $\vec{x}'(\vec{z})$ an arbitrary small deformation of the bag, provided only that the volume remains unchanged. But an arbitrary dilation of the system can be represented by the first term $R\vec{z}$ in the right-hand side of Eq. (3.6), and we can see that Eq. (3.6) defines a unique representation of the small deformations of the bag.

In this article we shall be interested only in small deformations about a spherical shape, so that the discussion we have given of the mapping should be adequate. One can argue from continuity considerations that the mapping of Eq. (3.6) gives a unique representation also for a range of large deformations of the bag. Of course, for too large deformations the representation of Eq. (3.6) would not be adequate (it cannot be used, for instance, to describe the fission of a bag).

The vector representation of Eq. (3.6) has many computational advantages over other possible representations. In particular, the terms with $l = 1$ have the form

$$\sum_{m=1}^{3} b_{1,m} \left(\frac{3}{4\pi}\right)^{1/2} z_m. \tag{3.7}$$

A change in the coordinates $b_{l,m}$ with $l = 1$ generates then a uniform displacements of the whole bag, and the momentum conjugate to $b_{1,m}$ is $(3/4\pi)^{1/2} P_m$, $\vec{P}$ being the total momentum of the system.

Moreover, a quantity which often appears in the computations is the matrix

$$\frac{\partial x_i}{\partial z_j} = R\delta_{ij} + B_{ij}, \tag{3.8}$$

where we have defined

$$B_{ij} = \sum_{l,m} \frac{\partial^2}{\partial z_i \partial z_j} z^l Y^l_m(\theta, \phi). \tag{3.9}$$

We see that B_{ij} is a symmetric traceless matrix [traceless because $z^l Y^l_m(\theta, \phi)$ is a harmonic function]: This simplifies remarkably the steps leading to Eq. (3.31).

We define as M and M_{ij} the determinant and the inverse of the matrix $\partial x_i/\partial z_j = \partial x_j/\partial z_i$:

$$M = \det\left|\frac{\partial x_i}{\partial z_j}\right|, \tag{3.10}$$

$$\sum_j M_{ij} \frac{\partial x_k}{\partial z_j} = \delta_{ik}. \tag{3.11}$$

(Notice that $\det|M_{ij}| = M^{-1}$.)

We introduce the field

$$\Phi(\vec{z}, t) = \phi(\vec{x}(\vec{z}), t) \tag{3.12}$$

and express L in terms of Φ. We need formulas for $\partial\phi/\partial t|_{x\,\text{fixed}}$ and $\partial\phi/\partial x_i$. These are given by

$$\frac{\partial\phi}{\partial x_i} = M_{ij} \frac{\partial\Phi}{\partial z_j} \tag{3.13}$$

and

$$\frac{\partial\phi}{\partial t} = \dot{\Phi} - \dot{b}_\alpha \frac{\delta x_i}{\delta b_\alpha} M_{ij} \frac{\partial\Phi}{\partial z_j}, \tag{3.14}$$

where we have symbolically denoted by b_α all the boundary coordinates, a dot stands for a time derivative, and summations over repeated indices are implied. This convention is adopted also in the equations that follow. Explicitly,

$$\frac{\delta x_i}{\delta R} = z_i \tag{3.15}$$

and

$$\frac{\delta x_i}{\delta b_{l,m}} = \frac{\partial}{\partial z_i} z^l Y^l_m(\theta, \phi). \tag{3.16}$$

Inserting these expressions into Eq. (3.1) we find

$$L=\int_{z\leq 1} d^3z\, M\left[\left(\dot{\Phi}-\dot{b}_\alpha\frac{\delta x_i}{\delta b_\alpha}M_{ij}\frac{\partial\Phi}{\partial z_j}\right)^*\left(\dot{\Phi}-\dot{b}_\alpha\frac{\delta x_i}{\delta b_\alpha}M_{ij}\frac{\partial\Phi}{\partial z_j}\right)-\left(M_{ij}\frac{\partial\Phi}{\partial z_j}\right)^*\left(M_{ik}\frac{\partial\Phi}{\partial z_k}\right)-B\right]. \quad (3.17)$$

The canonical momenta conjugate to the variables $\Phi^*(z)$ and b_α are

$$\Pi(z)=\frac{\delta L}{\delta\dot{\Phi}^*(z)}=M\left(\dot{\Phi}-\dot{b}_\alpha\frac{\delta x_i}{\delta b_\alpha}M_{ij}\frac{\partial\Phi}{\partial z_j}\right) \quad (3.18)$$

and

$$P_\alpha=\frac{\delta L}{\delta\dot{b}_\alpha}=-\int d^3z\, M\left[\frac{\delta x_i}{\delta b_\alpha}M_{ij}\frac{\partial\Phi^*}{\partial z_j}\times\left(\dot{\Phi}-\dot{b}_\alpha\frac{\delta x_i}{\delta b_\alpha}M_{ij}\frac{\partial\Phi}{\partial z_j}\right)+\text{c.c.}\right] \quad (3.19)$$

The momenta are not independent dynamical variables, but are constrained by

$$C_\alpha\equiv P_\alpha-\int d^3z\, M\left[\frac{\delta x_i}{\delta b_\alpha}M_{ij}\frac{\partial\Phi^*}{\partial z_j}\Pi+\text{c.c.}\right]=0. \quad (3.20)$$

As a consequence the Hamiltonian is given by the expression (see Refs. 8, 9)

$$H=\int d^3z\left[\Pi^*(z)\Pi(z)M^{-1}+M_{ij}\frac{\partial\Phi^*}{\partial z_j}M_{ik}\frac{\partial\Phi}{\partial z_k}M+BM\right]+v_\alpha C_\alpha=H_0+v_\alpha C_\alpha, \quad (3.21)$$

where the multipliers v_α must be determined from consistency requirements. It is apparent from Eqs. (3.20) and (3.21) that v_α is the velocity of the boundary variable b_α.

The static-boundary solution is given by

$$\Phi(\vec{z},t)=\left(\frac{2BR_0^2}{\pi}\right)^{1/2}\frac{\sin\pi z}{\sqrt{2\pi}z}e^{i\pi t/R_0}. \quad (3.22)$$

The function

$$f(z)=\frac{\sin\pi z}{\sqrt{2\pi}z} \quad (3.23)$$

is normalized to unity.

It is convenient to factor a phase out of the field $\Phi(\vec{z},t)$ defining

$$\Phi(\vec{z})=\left(\frac{\tilde{\phi}(\vec{z})}{\sqrt{2}}+\frac{i\psi(\vec{z})}{\sqrt{2}}\right)e^{i\theta}, \quad (3.24)$$

where $\tilde{\phi}(\vec{z})$ and $\psi(\vec{z})$ are now two real fields, and $\psi(\vec{z})$ is subject to the constraint

$$\int d^3z\,\psi(\vec{z})f(z)=0. \quad (3.25)$$

θ, which is to be considered one of the independent coordinates, is the phase of the mode of the field appearing in the static-boundary solution.

The momenta conjugate to $\tilde{\phi}$, ψ, and θ are

$$p(\vec{z})=\frac{1}{\sqrt{2}}[e^{i\theta}\Pi^*(\vec{z})+e^{-i\theta}\Pi(\vec{z})] \quad (3.26)$$

$$q(\vec{z})=\frac{1}{\sqrt{2}}[ie^{i\theta}\Pi^*(\vec{z})-ie^{-i\theta}\Pi(\vec{z})]-\frac{f(z)}{\sqrt{2}}\int d^3z\, f(z)[ie^{i\theta}\Pi^*(\vec{z})-ie^{-i\theta}\Pi(\vec{z})], \quad (3.27)$$

and

$$p_\theta=\frac{1}{\sqrt{2}}\int d^3z\, f(z)\tilde{\phi}(\vec{z})\times\int d^3z\, f(z)[ie^{i\theta}\Pi^*(\vec{z})-ie^{-i\theta}\Pi(\vec{z})]+\int d^3z\,[\tilde{\phi}(\vec{z})q(\vec{z})-\psi(\vec{z})p(\vec{z})]. \quad (3.28)$$

In the static-boundary solution we have

$$\tilde{\phi}(\vec{z})=\sqrt{2}\,(2BR_0^2/\pi)^{1/2}f(z),\quad \theta(t)=\pi t/R_0,\quad P_\theta=4BR_0^4 \quad (3.29)$$

and $R=R_0$, whereas all remaining canonical variables [$\psi(\vec{z})$, $p(\vec{z})$, $q(\vec{z})$, $b_{l,m}$, p_R, and $p_{l,m}$] are zero.

We perform a canonical transformation by defining the new variables

$$r=R-(p_\theta/4B)^{1/4},$$

$$\phi(\vec{z})=\tilde{\phi}(\vec{z})-\left(\frac{2}{\pi}\right)^{1/2}(Bp_\theta)^{1/4}f(z),$$

and

$$\Theta=\theta-\frac{1}{4\sqrt{\pi}}\left(\frac{B}{p_\theta^3}\right)^{1/4}\int d^3z\, p(\vec{z})f(z)-(4Bp_\theta)^{-1/4}\frac{p_R}{4}. \quad (3.30)$$

[We call the new field variable ϕ for convenience of notation. This new $\phi(\vec{z})$ is not to be confused with the field $\phi(\vec{x})$ appearing in Eqs. (3.1)–(3.4).]

The static-boundary solution is now characterized by a pair of "large" conjugate variables Θ and p_θ (Θ is a cyclic angular variable), and by zero values of all remaining canonical variables.

We then expand the Hamiltonian of Eq. (3.21) up to second order in the "small" variables $\phi(\vec{z})$, $p(\vec{z})$, $\psi(\vec{z})$, $q(\vec{z})$, r, p_R, $b_{l,m}$, $p_{l,m}$, and v_α and find the Hamiltonian $H^{(2)}$ that describes the small oscillations of the system

$$H^{(2)}=\frac{16\pi BR_0^3}{3}+\int\frac{d^3z}{2}\Big\{R_0^{-3}[p(\vec{z})^2+q(\vec{z})^2]+R_0\{[\vec{\nabla}\phi(\vec{z})]^2+[\vec{\nabla}\psi(\vec{z})]^2\}$$
$$+3\pi^2R_0f(z)^2\left[\int d^3z'f(z')\phi(\vec{z}')\right]^2+2\pi R_0^{-1}[\psi(\vec{z})p(\vec{z})-\phi(\vec{z})q(\vec{z})]$$
$$+32BR_0\pi r^2f(z)^2+16R_0r(B\pi^3)^{1/2}f(z)\phi(\vec{z})+B_{ij}B_{ij}BR_0\left(2\pi f(z)^2-1-\frac{2}{\pi}[\vec{\nabla}f(z)]^2\right)$$
$$+\frac{12BR_0}{\pi}B_{ij}B_{ik}\frac{\partial f(z)}{\partial z_j}\frac{\partial f(z)}{\partial z_k}-\frac{8R_0}{\pi^2}(B\pi^3)^{1/2}B_{ij}\frac{\partial f(z)}{\partial z_i}\frac{\partial\phi(\vec{z})}{\partial z_k}\Big\}+v_\alpha C_\alpha^{(1)}, \qquad (3.31)$$

where $C_\alpha^{(1)}$ stands for the linearized form of the constraints and

$$R_0=(p_\theta/4B)^{1/4}. \qquad (3.32)$$

p_θ is the proper canonical variable that should be used in H, but the use of R_0 simplifies the notation. R_0 is the value of the radius of the bag when no oscillatory mode is present.

The explicit expression of the constraints $C_\alpha^{(1)}$ is

$$C_R^{(1)}=p_R+\int d^3z\left(\frac{2}{\pi}(B\pi)^{1/2}p(\vec{z})z_i\frac{\partial f(z)}{\partial z_i}+2R_0^2(B\pi)^{1/2}f(z)z_i\frac{\partial\psi(\vec{z})}{\partial z_i}\right) \qquad (3.33)$$

and

$$C_{l,m}^{(1)}=p_{l,m}+2(B\pi)^{1/2}\int d^3z\left(\frac{\partial}{\partial z_i}z^lY_m^l(\theta,\phi)\right)\left(\frac{1}{\pi}p(\vec{z})\frac{\partial f(z)}{\partial z_i}+R_0^2f(z)\frac{\partial\psi(\vec{z})}{\partial z_i}\right). \qquad (3.34)$$

This is all one needs to eliminate the boundary variables through a linear canonical transformation and to obtain a second-order Hamiltonian containing the field dynamical variables alone. However, it is convenient to perform first an expansion into partial waves of ϕ, ψ, p, and q, according to the general formula

$$F(\vec{z})=\sum_{l,m}\frac{F_{l,m}(z)}{z}Y_m^l(\theta,\phi). \qquad (3.35)$$

It is straightforward to check that after the expansion $H^{(2)}$ takes the form

$$H^{(2)}=\frac{16\pi BR_0^3}{3}+\sum_{l,m}H_{l,m}, \qquad (3.36)$$

where the partial-wave Hamiltonians $H_{l,m}$ as well as the constraints $C_{l,m}^{(1)}$ contain only the l,m components of the field and boundary variables (the $l=0$ components of the boundary variables are r and p_R). Thus, the motion of the system, always within the approximation of small oscillations, is resolved into a linear superposition of independent motions, each involving one definite angular momentum component of the fields and only one of the boundary degrees of freedom. The technique developed in Sec. II for the one-dimensional system can then be applied to the different l,m sectors to diagonalize the partial-wave Hamiltonians. In Sec. IV we shall consider in detail the sector with $l=1$, which, for reasons already mentioned in the Introduction, we find the most interesting.

IV. $l=1$ SECTOR

Inserting the expansion of Eq. (3.35) into Eq. (3.31) we find the partial-wave Hamiltonians $H_{l,m}$. The Hamiltonians $H_{1,m}$ which describe the P-wave small oscillations of the bag are given by

$$H_{1,m}=\int\frac{dz}{2}\Big\{R_0^{-3}[p_{1,m}^2(z)+q_{1,m}^2(z)]$$
$$+R_0\left[\left(\frac{d\phi_{1,m}(z)}{dz}\right)^2+\frac{2}{z^2}\phi_{1,m}^2(z)+\left(\frac{d\psi_{1,m}(z)}{dz}\right)^2+\frac{2}{z^2}\psi_{1,m}^2(z)\right]$$
$$+2\pi R_0^{-1}[\psi_{1,m}(z)p_{1,m}(z)-\varphi_{1,m}(z)q_{1,m}(z)]\Big\}+v_{1,m}C_{1,m}^{(1)}, \qquad (4.1)$$

and the constraint functions are

$$C^{(1)}_{1,m}=p_{1,m}+\sqrt{2B}\int dz\left(\frac{p_{1,m}(z)}{\pi}-R_0{}^2\psi_{1,m}(z)\right)g(z)\,, \tag{4.2}$$

with

$$g(z)=z\frac{\partial}{\partial z}\frac{\sin\pi z}{z}\,. \tag{4.3}$$

We recall from Sec. III that the momenta $p_{1,m}$ conjugate to the cyclic boundary variables $b_{1,m}$ are proportional to the components of the total momentum of the system. Precisely,

$$p_{1,m}=\left(\frac{3}{4\pi}\right)^{1/2}P_m^{\text{tot}}\,. \tag{4.4}$$

In the following we shall not write explicitly the indices $1,m$; we shall keep only the subscript 1 in $p_{1,m}$ to avoid confusion with the symbol of the field momentum $p(z)$.

The most convenient method to deal with the constraint $C^{(1)}=0$ is to introduce a linear canonical transformation such that the new momentum variable is proportional to the constraint itself. This leads us to the change of variables:

$$\begin{aligned}
\tilde{p}_1&=\frac{\pi}{\sqrt{2B}R_0}p_1+\frac{\sqrt{2B}\,R_0}{\pi}\int dz\left(\frac{p(z)}{R_0}-\pi R_0\psi(z)\right)g(z)\,,\\
\tilde{b}&=\frac{\sqrt{2B}\,R_0}{\pi}b\,,\\
\tilde{\phi}(z)&=R_0\phi(z)-\frac{\sqrt{2B}\,R_0{}^2}{\pi}bg(z)\,,\\
\tilde{q}(z)&=\frac{q(z)}{R_0}-\sqrt{2B}\,bg(z)\,,\\
\tilde{p}(z)&=\frac{p(z)}{R_0}\,,\\
\tilde{\psi}(z)&=R_0\psi(z)\,,
\end{aligned} \tag{4.5}$$

which gives

$$C^{(1)}=\frac{\sqrt{2B}\,R_0}{\pi}\tilde{p}_1\,.$$

However, this transformation has the unpleasant feature of transforming the field $\phi(z)$, which satisfies the boundary condition $\phi(1)=0$, into a field $\tilde{\phi}(z)$ which does not vanish at $z=1$. The nonvanishing of $\tilde{\phi}(1)$ entails divergences in some of the equations needed to eliminate the variable b, that make them ill defined.

As we have already seen at the end of Sec. II, the proper way to deal with the problem is to alter the constraint equations by the introduction of a cutoff, which will be removed at the end of the computation.

We introduce a cutoff in the following way. It is convenient to define the operator

$$K=-\frac{d^2}{dz^2}+\frac{2}{z^2}\,, \tag{4.6}$$

which is a positive-definite Hermitian operator in the space of functions vanishing at $z=0$ and $z=1$, with inner product

$$fg=\int_0^1 dz f^*(z)g(z)\,. \tag{4.7}$$

Throughout this section we shall frequently use the notation fg for $\int_0^1 dz\, f(z)g(z)$, or, more generally, fOg for $\int_0^1 dz f(z)Og(z)$, where O is an operator defined in the space of the functions $g(z)$. See Eq. (4.12) and the following ones.

The eigenvalues of K are the squares of the zeros of the spherical Bessel function

$$j_1(z)=\frac{\sin z}{z^2}-\frac{\cos z}{z}\,, \tag{4.8}$$

which we denote by κ_n ($\kappa_1<\kappa_2<\kappa_3\cdots$), and the corresponding eigenfunctions, normalized to unity, are given by

$$f_n(z)=\frac{\sqrt{2}\,\kappa_n}{\sin\kappa_n}zj_1(\kappa_n z)\,. \tag{4.9}$$

As a cutoff procedure, we shall replace $g(z)$ with its projection $g_N(z)$ over the subspace spanned by the first N eigenfunctions $f_n(z)$. One easily evaluates

$$\int_0^1 dzg(z)f_n(z)=-\frac{\sqrt{2}\,\pi\kappa_n}{\kappa_n{}^2-\pi^2}\,. \tag{4.10}$$

It follows that

$$\begin{aligned}
g_N(z)&=\sum_{n=1}^{N}(g_N f_n)f_n(z)\\
&=-\sum_{n=1}^{N}\frac{\sqrt{2}\,\pi\kappa_n}{\kappa_n{}^2-\pi^2}f_n(z)\,.
\end{aligned} \tag{4.11}$$

We perform now the change of variables of Eq. (4.5) with $g(z)$ replaced by $g_N(z)$, and the Hamiltonian becomes

$$\begin{aligned}
H=\frac{1}{2R_0}[&\tilde{p}^2+\tilde{q}^2+\tilde{\phi}K\tilde{\phi}+\tilde{\psi}K\tilde{\psi}+2\pi(\tilde{p}\tilde{\psi}-\tilde{\phi}\tilde{q})\\
&+2\tilde{b}(\tilde{\phi}Kg_N-\pi^2\tilde{\phi}g_N)+\tilde{b}^2g_N(K-\pi^2)g_N]\,,
\end{aligned} \tag{4.12}$$

where we have omitted the now trivial term vC.[1]

The consistency condition $\{H,\tilde{p}_1\}=0$ gives

$$\tilde{b}=\frac{\tilde{\phi}(K-\pi^2)g_N}{g_N(K-\pi^2)g_N}\,, \tag{4.13}$$

and we can use this relation to eliminate $\tilde{b}$ and to obtain the new Hamiltonian

$$\tilde{H} = \frac{1}{2R_0}\left(\tilde{p}^2 + \tilde{q}^2 + \tilde{\psi}K\tilde{\psi} + \tilde{\phi}K\tilde{\phi} + 2\pi(\tilde{p}\tilde{\psi} - \tilde{\phi}\tilde{q}) - \frac{[\tilde{\phi}(K-\pi^2)g_N]^2}{g_N(K-\pi^2)g_N}\right), \quad (4.14)$$

where only the field variables appear. [Notice that $(K-\pi^2)g(z)=0$. This demonstrates the necessity of using a regularized $g_N(z)$ to take into account correctly the effects of the motion of the boundary.]

To find the eigenvalues ω/R_0 of $\tilde{H}$, we look for solutions of the form $\tilde{\phi}(t,z) = e^{i\omega t/R_0}\tilde{\phi}(z)$ (and analogously $\tilde{\psi} = e^{i\omega t/R_0}\tilde{\psi}_0$ etc).

After some simple passages we find the equation

$$[K-(\omega+\pi)^2][K-(\omega-\pi)^2]\tilde{\phi}(z) = (K-\omega^2-\pi^2)(K-\pi^2)g_N(z)\frac{\tilde{\phi}(K-\pi^2)g_N}{g_N(K-\pi^2)g_N}. \quad (4.15)$$

Denoting by A the quantity

$$\frac{\tilde{\phi}(K-\pi^2)g_N}{g_N(K-\pi^2)g_N},$$

Eq. (4.15) gives

$$\tilde{\phi}(z) = \frac{(K-\omega^2-\pi^2)(K-\pi^2)}{[K-(\omega+\pi)^2][K-(\omega-\pi)^2]}g_N(z)A. \quad (4.16)$$

This in turn implies

$$g_N(K-\pi^2)g_NA = g_N(K-\pi^2)\tilde{\phi} = g_N\frac{(K-\omega^2-\pi^2)(K-\pi^2)^2}{[K-(\omega+\pi)^2][K-(\omega-\pi)^2]}g_NA, \quad (4.17)$$

or

$$g_N\left(\frac{(K-\omega^2-\pi^2)(K-\pi^2)^2}{[K-(\omega+\pi)^2][K-(\omega-\pi)^2]} - (K-\pi^2)\right)g_N = 0, \quad (4.18)$$

which is the eigenvalue equation for ω.

Expressed in terms of the components of g_N [see Eq. (4.10)], after some straightforward algebraic simplifications, Eq. (4.18) becomes

$$\omega^2\sum_n\left(\frac{\kappa_n}{(\kappa_n-\pi)[\omega^2-(\kappa_n-\pi)^2]} + \frac{\kappa_n}{(\kappa_n+\pi)[\omega^2-(\kappa_n+\pi)^2]}\right) = 0. \quad (4.19)$$

Notice that the series in the left-hand side of Eq. (4.19) is convergent; we can therefore remove the cutoff and let the sum range from 1 to ∞.

The main properties of the sequence of eigenvalues can be read off Eq. (4.19). As expected, there is a zero eigenvalue, which corresponds to no internal excitation, but to the translational degrees of freedom of the whole system. The zero-frequency eigenvalue implies the presence in H of a term of the form $\frac{1}{2}\beta p_0^2$, where p_0 is the normal momentum of the zero-frequency mode and β is a numerical factor. It can be shown that, as expected, p_0 is proportional to the boundary momentum p_1 and therefore to (one of the components of) the total momentum $\vec{P}$ of the system [see Eq. (4.4)] and that the corresponding term in H is given by

$$H_{\text{zero mode}} = \frac{3}{32\pi BR_0^3}\vec{P}_{\text{tot}}^{\,2}. \quad (4.20)$$

The left-hand side of Eq. (4.19) has simple poles for $|\omega| = (\kappa_n \mp \pi)$, is negative for $|\omega| < \kappa_1 - \pi$, and changes sign between two consecutive poles. We deduce that there is one eigenvalue in each of the intervals separated by the points $\kappa_1-\pi$, $\kappa_2-\pi$, $\kappa_1+\pi$, $\kappa_3-\pi$, $\kappa_2+\pi$, etc. In particular, since asymptotically $\kappa_n = (n+\frac{1}{2})\pi - O(1/n)$ we must have

$$|\omega_{2n+1}| = (n+\tfrac{3}{2})\pi - O(1/n). \quad (4.21)$$

Equation (4.19) is very convenient for a numerical computation of the eigenvalues, which gives

$$\begin{aligned}
\omega_1 &= 3.537\cdots = 1.126\cdots\pi,\\
\omega_2 &= 6.284\cdots = 2.000\cdots\pi,\\
\omega_3 &= 7.673\cdots = 2.442\cdots\pi,\\
\omega_4 &= 9.426\cdots = 3.000\cdots\pi,\\
\omega_5 &= 10.887\cdots = 3.466\cdots\pi.
\end{aligned} \quad (4.22)$$

Summerizing, we have found that the P-wave small oscillations of the bag are described by the Hamiltonian:

$$H = \frac{4\pi}{3}(4B)^{1/4}p_\theta^{3/4} + \frac{3}{8\pi(4B)^{1/4}p_\theta^{3/4}}\vec{P}_{\text{tot}}^{\,2} + \frac{(4B)^{1/4}}{p_\theta^{1/4}}\sum_{n=1}^{\infty}\sum_{m=1}^{3}\omega_n a_{n,m}^* a_{n,m}, \quad (4.23)$$

where p_θ is the action variable that characterizes the large static boundary mode around which the system oscillates, ω_n is the nth positive solution of the eigenvalue equation (4.19), $a_{n,m}$ and $a_{n,m}^*$ are the corresponding normal-mode variables, and we have included the contribution from the zero-frequency mode, even if it does not represent a proper oscillation of the system.

Equation (4.23) can be quantized semiclassically by replacing the action variable p_θ with an occupation-number operator N_θ, having integer eigenvalues, and the amplitudes $a_{n,m}$ and $a_{n,m}^*$ with annihilation and creation operators $a_{n,m}$ and $a^{\dagger}_{n,m}$. However, Eq. (4.23) is not suitable for an extrapolation to small values of N_θ, because of the divergences for $p_\theta \to 0$, and also has an unpleasant noncovariant aspect. We replace, therefore, Eq. (4.23) with the equation

$$M = (E^2 - \vec{P}_{tot}{}^2)^{1/2}$$
$$= \frac{4\pi}{3}(4B)^{1/4}\left(p_\theta + \frac{1}{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{3}\omega_n a^*_{n,m}a_{n,m}\right), \quad (4.24)$$

which is equivalent to Eq. (4.23) in the limit of large p_θ, as can be easily verified by solving Eq. (4.24) for E and expanding into descending powers of p_θ.

We use Eq. (4.24) to define the mass operator for the quantum system:

$$M = \frac{4\pi}{3}(4B)^{1/4}\left(N_\theta + \frac{1}{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{3}\omega_n a^{\dagger}_{n,m}a_{n,m} + c\right)^{3/4}. \quad (4.25)$$

(We allow for the presence of an additive constant that cannot be determined from the classical limit.)

Equation (4.25) is of course rigorously valid only in the limit of large N_θ [where Eq. (4.24) is no better than Eq. (4.23)], but, supported by the results obtained in Sec. II, we conjecture that Eq. (4.25) may give good approximate results also for the low-lying states of the spectrum.

Notice that, although we have considered only P waves in this section, Eq. (4.25) can be generalized to include all excitations. One must simply replace the sum over n and m with a sum over all possible values of l, m, and n, where the eigenvalues $\omega_{l,n}$ are to be determined from the diagonalizations of the separate partial-wave Hamiltonians, which can be performed following the method used in this section.

V. CONCLUSIONS

It is interesting to compare the results of Sec. IV, with what can be obtained from a simple variational computation.

If one quantizes the field $\phi(x,t)$, subject to the linear boundary condition Eq. (3.2), in a spherical cavity of radius R and assumes that only one excitation of the first S-wave mode is present (see Ref. 1), one finds for the energy of the system

$$E = \frac{\kappa_0}{R} + \frac{4\pi}{3}BR^3 \quad (5.1)$$

where $\kappa_0 = \pi$ is the first zero of the spherical Bessel function $j_0(z)$.

Minimizing the energy E, one obtains a radius

$$R_0 = \left(\frac{\kappa_0}{4\pi B}\right)^{1/4} = \frac{1}{(4B)^{1/4}} \quad (5.2)$$

with the corresponding energy

$$E = \frac{4\pi}{3}(4B)^{1/4}\left(\frac{\kappa_0}{\pi}\right)^{3/4} = \frac{4\pi}{3}(4B)^{1/4}. \quad (5.3)$$

These values agree with what one can find from the semiclassical quantization of the equations of motion, which, neglecting the additive constant in Eq. (4.25), gives precisely $E = \frac{4}{3}\pi(4B)^{1/4}$ for the lowest-lying S-wave state. Notice that in the variational ansatz one uses a field that obeys the D'Alembert equation of motion inside the bag and the linear boundary condition, but does not satisfy the nonlinear constraint of Eq. (3.4). Still, that constraint is compatible with the ansatz, and this is the reason why the variational computation reproduces the correct result.

One can perform the same minimization, assuming that a single P-wave excitation of the field is present. In this case one finds for the energy the expression

$$E = \frac{\kappa_1}{R} + \frac{4\pi}{3}BR^3, \quad (5.4)$$

where $\kappa_1 = 4.493\cdots = 1.430\cdots\pi$ is the first zero of the spherical Bessel function $j_1(z)$.

A minimization of E now gives

$$E = \frac{4\pi}{3}(4B)^{1/4}\left(\frac{\kappa_1}{\pi}\right)^{3/4}. \quad (5.5)$$

Our approximate treatment of the bag gives instead [see Eq. (4.25)] (always neglecting the additive constant)

$$E = \frac{4\pi}{3}(4B)^{1/4}\left(\frac{\omega_1}{\pi}\right)^{3/4}, \quad (5.6)$$

with $\omega_1 = 3.537\cdots = 1.126\cdots\pi$.

We can interpret these two results by saying that the inclusion of the motion of the boundary has the effect of lowering the eigenfrequency κ_1, corresponding to a P-wave oscialltion in a fixed cavity, replacing it with a new eigenfrequency ω_1, the first solution of the eigenvalue Eq. (4.19). It is understandable that the variational computation may originate too high an estimate for the value of the energy, since it does not allow for the boundary deformations that can lead to a lower energy of the system. On the other hand, but this is very speculative, it seems reasonable that our method, which becomes exact only in the limit of small boundary deformations, may overestimate the effect of the motion of the boundary when applied to the low-lying levels.

If this is true, then the correct value of the mass of the first P-wave state of the bag would be between the two estimates of Eqs. (5.5) and (5.6), which constitutes a rather good determination, since κ_1 and ω_1 differ by approximately 24%.

Finally, we would like to remark that we have considered a system containing a single charged scalar field in this article because we wanted to present our method of approximation without the algebraic complications that are introduced by the use of spinor or vector fields. It appears to us, however, that the method can be applied to a general system, and in particular to realistic models of hadrons, where, in the absence of a full theory, the expansion for small boundary deformations could be very useful to evaluate further predictions of the theory to be compared with the existing experimental data.

ACKNOWLEDGMENTS

I wish to thank my colleagues of the Center for Theoretical Physics of M. I. T., in particular, Dr. R. Jaffe and Professor K. Johnson, for many useful discussions, and Professor L. Faddeev for a very interesting conversation on constrained Lagrangian systems.

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*Work supported in part through funds provided by the Energy Research and Development Administration under Contract No. AT(11-1)-3069.

†On leave of absence from Istituto di Fisica Teorica, Università di Trieste, Trieste, Italy.

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