PHYSICAL REVIEW D VOLUME 12, NUMBER 8 15 OCTOBER 1975

Representations of Reggeon Green's functions: Their scaling form and the approach to scaling*

William R. Frazer and Moshe Moshe
Department of Physics, University of California, San Diego, La Jolla, California 92093
(Received 12 June 1975)

We derive representations for Green's functions in the Reggeon calculus. Our renormalization-group integration method gives the t dependence of the asymptotic scaling functions, as well as the correction terms which are present at nonasymptotic energies. The approach to the scaling form is governed by a critical exponent. The method can be used to calculate the scaling form and the approach to scaling of any n-to-m Reggeon Green's function. We discuss mainly the Pomeron propagator and the three-point function in the one-loop approximation.

I. INTRODUCTION

Using renormalization-group methods, Abarbanel and Bronzan[1] and Migdal, Polyakov, and Ter-Martirosyan[2] have been able to elucidate the nature of the high-energy limit in the Gribov Reggeon calculus. In this paper we gain more information about the scaling limit and the approach to that limit by utilizing a different method of integrating the renormalization-group equations. The method consists of deriving differential equations for the Green's functions of the theory, in which the differentiations are with respect to the dimensionless parameters of the theory. These equations can be integrated to yield integral representations for the Green's functions. Integration of these equations can be performed explicitly in some approximations. We shall refer to this method as the integral-representation method.

The integral-representation method has some advantages over the standard method. In the standard method one finds the critical exponents (e.g., the power of ln$s$ in the asymptotic total cross section), but is left with a "scaling function" of certain scaling variables. Some information can be obtained about the scaling function from comparison with perturbation theory, but the integral-representation method gives a more explicit answer. Moreover, corrections to the scaling limit can be investigated, the J-plane structure of amplitudes can be determined, and the explicit t dependence of the elastic scattering amplitude can be found.

For pedagogical simplicity we begin in Sec. II with an analysis of the simplest example, the derivation of a representation for the dressed Pomeron propagator at zero transverse momentum, $k^2=0$. The Sommerfeld-Watson transformation of the representation can be taken explicitly, which yields the asymptotic form of the total cross section. Some information is also gained about the approach to the asymptotic limit.

In Sec. III we present the general method at $k^2=0$; subsequently, we generalize in Sec. IV to $k^2\neq 0$. An immediate application is the derivation of the t dependence of elastic scattering amplitude in the scaling limit. This is presented in Sec. V. Another interesting application, which we shall present in the following paper[7], is the investigation of the triple-Pomeron process.

In Sec. VI we analyze the information given by the integral-representation method concerning the approach to the scaling limit. We find that the corrections to the scaling limit take the form of an expansion in powers of $(\ln s)^{-\lambda}$, where λ is a critical exponent which we discuss in detail. A discussion of our results is given in Sec. VII.

II. REPRESENTATION OF THE TOTAL CROSS SECTION

A. Representation of the single-Pomeron propagator

In this section we shall illustrate the general method of this paper by a specific simple example: the construction of a representation of the single-Pomeron Green's function $\Gamma^{1,1}(E,\vec{k}^2)$. We follow the standard notation of Ref. 1. For simplicity we shall first work out the case $\vec{k}^2=0$, and defer the treatment of the $\vec{k}^2$ dependence to Sec. IV. Our goal is to obtain a representation of $\Gamma^{1,1}(E,0)$ which is correct in two limits: It yields the correct perturbation expansion, and it yields the same singularity at $J=1$ as is given by the renormalization-group method in the $\epsilon\equiv 4-D$ expansion. Our method is a modification of that used by Sugar and White[3] in their discussion of the infrared behavior of the Reggeon field theory. We find it convenient, however, to use a different normalization condition on $\Gamma_R^{1,1}$, namely,

$$i\Gamma_R^{1,1}(-E_N,0)=-E_N, \qquad (2.1)$$

instead of the conventional condition on the first derivative.[4] Our condition implies that the unrenormalized Green's function is given by

$$i\Gamma^{1,1}(-E_N, 0) = -E_N/Z(-E_N), \tag{2.2}$$

and therefore, for any value of E,

$$i\Gamma^{1,1}(E, 0) = E/Z(E). \tag{2.3}$$

The next step, the evaluation of $Z \equiv Z(-E_N)$, follows Sugar and White. Since Z is dimensionless, it can be expressed as a function of the only dimensionless quantity one can form from the parameters of the theory, r_0, α_0', and E_N,

$$g_0(E_N) \equiv r_0 \alpha_0'^{-D/4} E_N^{-\epsilon/4}. \tag{2.4}$$

Alternatively, one can express Z as a function of renormalized parameters, that is, boundary values of the Green's functions at conveniently selected points. We follow the conventions of Abarbanel and Bronzan in defining

$$\alpha'(E_N) \equiv \frac{\partial}{\partial \vec{k}^2} i\Gamma_R^{1,1}(E, \vec{k}^2)\Big|_{E=-E_N,\ \vec{k}^2=0}, \tag{2.5}$$

$$r(E_N) \equiv (2\pi)^{(D+1)/2}\Gamma_R^{1,2}(E_1, \vec{k}_1, E_2, \vec{k}_2, E_3, \vec{k}_3)\big|_{E_1=2E_2=2E_3=-E_N;\ \vec{k}_i\cdot\vec{k}_j=0}. \tag{2.6}$$

In terms of these quantities one can form the dimensionless quantity

$$g \equiv r\alpha'^{-D/4} E_N^{-\epsilon/4}. \tag{2.7}$$

Moreover, we impose the usual condition that the renormalized Pomeron intercept lies at $J=1$,

$$\Gamma_R^{1,1}(0, 0) = 0. \tag{2.8}$$

In order to evaluate Z we proceed to derive equations for the dependence of Z on g, and in turn, of g on g_0. To this end we define the conventional renormalization-group functions,

$$\beta \equiv E_N \frac{\partial g}{\partial E_N}\Big|_{r_0, \alpha_0'}, \tag{2.9a}$$

$$\gamma \equiv E_N \frac{\partial \ln Z}{\partial E_N}\Big|_{r_0, \alpha_0'}, \tag{2.9b}$$

$$\zeta \equiv E_N \frac{\partial \alpha'}{\partial E_N}\Big|_{r_0, \alpha_0'}. \tag{2.9c}$$

Again, Z, γ, β, and ζ/α' are dimensionless. Therefore, when they are expressed in terms of renormalized parameters r, α', and E_N, it follows that they can depend only on the dimensionless combination g. Therefore, Eqs. (2.9a) and (2.9b) imply

$$\gamma = \beta \frac{d}{dg} \ln Z. \tag{2.10}$$

Since $Z=1$ at $g=0$ this equation can be integrated to yield

$$Z = \exp\left(\int_0^g dg' \gamma(g')\beta^{-1}(g')\right). \tag{2.11}$$

Now we require a relation between g and g_0. To this end we define

$$g = g_0 Z_g(g), \tag{2.12}$$

and proceed to derive a differential equation for $Z_g(g)$ by differentiation with respect to E_N,

$$\beta = E_N \frac{\partial g}{\partial E_N}\Big|_{r_0, \alpha_0'} = E_N \frac{dg_0}{dE_N} Z_g + E_N g_0 \frac{\partial Z_g}{\partial E_N}, \tag{2.13}$$

from which it follows that

$$\frac{d}{dg} \ln Z_g = \frac{1}{g} + \frac{\epsilon}{4\beta}. \tag{2.14}$$

This equation can be integrated with the boundary condition $Z_g(0)=1$ to yield the desired relation between g and g_0,

$$g = g_0 \exp\left\{\int_0^g dg'\left[\frac{1}{g'} + \frac{\epsilon}{4\beta(g')}\right]\right\}. \tag{2.15}$$

These equations [(2.12) and (2.15)] constitute the desired relationship between Z and g_0, but to proceed further we must have some knowledge of $\beta(g)$ and $\gamma(g)$. Following the usual line of argument, we evaluate these functions in perturbation theory. One then finds that if g_1 exists such that $\beta(g_1)=0$ and $\beta'(g_1)>0$, and if g_1 is sufficiently small to permit a valid perturbation expansion in powers of g_1, one is able to obtain information about the limit of $\Gamma^{n,m}(E)$ as $E \to 0$. The perturbation calculation in the one-loop approximation is very similar to that performed by Abarbanel and Bronzan, but differs slightly because of our normalization condition (2.1). The result is

$$\beta(g) = -\tfrac{1}{4}\epsilon g\left(1 - \frac{g^2}{g_1^2}\right), \tag{2.16}$$

$$\gamma(g) = \bar{\gamma} g^2/g_1^2, \tag{2.17}$$

$$\zeta/\alpha' = \bar{\zeta} g^2/g_1^2, \tag{2.18}$$

where $g_1, \bar{\gamma}, \bar{\zeta}$ depend only on D; namely,

$$g_1^2=\epsilon\,\frac{(8\pi)^{D/2}}{2J(D)\Gamma(3-D/2)}, \tag{2.19a}$$

$$\bar{\gamma}=-\epsilon x/4J(D), \tag{2.19b}$$

$$\bar{\xi}=-\frac{\epsilon}{8}\,\frac{2x-1}{J}: \tag{2.19c}$$

where $x^{-1}=D/2-1$ and

$$J(D)=8x(1-2^{1-D/2})-\tfrac{3}{2}x+\frac{D}{8}(2x-1). \tag{2.20}$$

Substituting (2.16) and (2.17) in (2.11) then yields

$$Z=(1-g^2/g_1^2)^{2\bar{\gamma}/\epsilon}. \tag{2.21}$$

Substituting similarly in (2.15) yields

$$g=g_0(1+g_0^2/g_1^2)^{-1/2}, \tag{2.22}$$

or

$$g_0=g(1-g^2/g_1^2)^{-1/2}, \tag{2.23}$$

and therefore

$$Z=(1+g_0^2/g_1^2)^{-2\bar{\gamma}/\epsilon}. \tag{2.24}$$

Returning now to Eqs. (2.3) and (2.4), we see that this implies that

$$i\Gamma^{1,1}(E,0)=E\left[1+\frac{r_0^2}{g_1^2(\alpha_0')^{D/2}}(-E)^{-\epsilon/2}\right]^{2\bar{\gamma}/\epsilon}. \tag{2.25}$$

The above equation is the representation we have been working toward. It is very similar to Eq. (63) of Sugar and White, but is improved in the following respect: No expansion in powers of ϵ has been made in its derivation. It therefore correctly reproduces perturbation theory in the one-loop approximation. We shall return to this point at the end of this section.

The representation of $\Gamma^{1,1}(E,0)$ also has the important property that if g_1 is sufficiently small it tells us about the behavior in the limit $E\to 0$. To see this, let us rewrite (2.25) in the form

$$i\Gamma^{1,1}(E,0)=E[1+g_0^2(-E)/g_1^2]^{2\bar{\gamma}/\epsilon}, \tag{2.26}$$

where $g_0(-E)$ is defined according to (2.4). At and in the vicinity of the physical point ($\epsilon=2$) ϵ is positive; therefore, as $E\to 0$ for fixed r_0 and α_0', $g_0(-E)\to\infty$. It then follows from (2.22) that as $g_0\to\infty$, $g\to g_1$. If g_1 is sufficiently small, we may then be justified *a posteriori* in our use of a perturbation expansion in powers of g in Eqs. (2.16)–(2.18).

We emphasize that we are considering the limit $E\to 0$ *for fixed* r_0 *and* α_0'. The experienced practitioner of field theory might question our reliance on unrenormalized parameters. We believe that in the physical problem under consideration the unrenormalized parameters are just as physical as the renormalized ones. We associate the unrenormalized parameters with a "bare" Pomeron, which we assume describes scattering at medium (Fermilab–CERN-ISR) energies. The renormalized parameters, and the renormalized Green's functions, are convenient auxiliary quantities which are of value in the formal development of the theory and which are somewhat more closely related to the renormalized Pomeron, which describes scattering at infinite energies.

It is doubtful whether g_1 is sufficiently small in the real world to permit accurate perturbation calculations. Abarbanel and Bronzan,[1] as well as Migdal, Polyakov, and Ter-Martirosyan,[2] noticed that $g_1=0$ at $D=4$ ($\epsilon=0$). They therefore proposed to expand in powers of ϵ. The one-loop perturbation calculation gives the leading behavior in ϵ; namely

$$g_1^2/(8\pi)^2=\epsilon/6+O(\epsilon^2), \tag{2.27}$$

$$\bar{\gamma}=-\epsilon/12+O(\epsilon^2). \tag{2.28}$$

These results are independent of the renormalization procedure; they are implied by our Eqs. (2.19a) and (2.19b) as well as by the method of Ref. 1. We shall often make use of the ϵ expansion in the following. Baker and Bronzan and Dash[5] have calculated higher-order terms in (2.27) and (2.28) and found them to be comparable to the lowest-order term at the physical point $\epsilon=2$. We therefore do not expect that those portions of our work which take only lowest-order terms in the ϵ expansion should yield quantitative results. We expect, however, that the method gives at least qualitative insight into the nature of the infinite-energy limit.

Next, we shall discuss the perturbation expansion of Eq. (2.25). Since it contains the perturbation expansion of $\Gamma^{1,1}$ correct to second order, Eq. (2.25) of course reproduces this result when expanded in powers of r_0^2,

$$i\Gamma^{1,1}(E,0)=E+\frac{2}{\epsilon}\bar{\gamma}\,\frac{r_0^2}{g_1^2(\alpha_0')^{D/2}}(-E)^{1-\epsilon/2}+O(r_0^4) \tag{2.29a}$$

$$=E-\frac{1}{2}\,\frac{r_0^2}{(\alpha_0')^{D/2}(8\pi)^{D/2}}\Gamma\left(1-\frac{D}{2}\right)(-E)^{1-\epsilon/2}+O(r_0^4). \tag{2.29b}$$

Note the infinity at $D=2$. At this point our theory has both infrared and ultraviolet difficulties. In principle the ultraviolet difficulties are not serious, because we know that the physics of high-energy scattering suggests that the momentum-transfer integrals which cause the divergence should really have exponential cutoffs. We have neglected the cutoff in the interest of simplicity, because the introduction of a cutoff would allow us to form a second dimensionless parameter, thereby invalidating the simple representation used in Eq. (2.25).

It would be highly desirable to repeat our entire treatment in the presence of a cutoff. One can formally generalize the representations of the Green's functions, but at a cost of greatly increased complexity. We therefore defer this program, and instead give only a description of the sort of result one finds in cutoff theories.

We have been investigating several cutoff theories of the type which make the integrals finite by means of a factor $\exp(-b\alpha_0' k^2)$, which will be discussed in detail in a subsequent publication. Such theories involve incomplete gamma functions $\Gamma(1-D/2, -Eb)$ rather than $\Gamma(1-D/2)$. That is, Eq. (2.29) can be interpreted through the replacement

$$(-E)^{1-\epsilon/2}\Gamma\left(1-\frac{D}{2}\right)\to\int_b^\infty dt\, e^{Et}t^{-D/2}, \qquad (2.30)$$

which then yields the result

$$i\Gamma^{1,1}(E,0)=E-\frac{\frac{1}{2}r_0^2}{(\alpha_0')^{D/2}(8\pi)^{D/2}}\int_b^\infty dt\, e^{Et}t^{-D/2} + O(r_0^4). \qquad (2.31)$$

Although the cutoff has removed the ultraviolet divergence at $D=2$, the perturbation series in Eq. (2.31) still has infrared difficulties, which have been analyzed by Sugar and White.[3] In particular, the second-order perturbation term fails to satisfy Eq. (2.8), which implies $\Gamma^{1,1}(0,0)=0$. It is, in fact, impossible to enforce this condition order-by-order in perturbation theory.

Our representation (2.25) gives the correct $E\to 0$ behavior only in an ϵ expansion. To obtain the representation one must at each order in perturbation theory take ϵ small enough so that no infinites appear at $E=0$.[3,4] In second order one sees from (2.30) that this means $\epsilon<2$. In order r_0^{2n} the infinity is encountered[3] at $\epsilon=2/n$, so that one is required to work arbitrarily close to $\epsilon=0$. Finally, one continues the resulting representation back to the physical point $\epsilon=2$.

B. The Pomeron propagator in the s plane

Having arrived in Eq. (2.25) at an explicit representation of $\Gamma^{1,1}(E,0)$ which incorporates the perturbation expansion correct to order r_0^2 as well as the correct limit at $E=0$ as implied by the renormalization group, we shall in this section take the Sommerfeld-Watson transformation of Eq. (2.25) and discuss its properties. Its contribution to the total cross section is given by

$$\sigma_T(s)=-N_1^2\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}[i\Gamma^{1,1}(E,0)]^{-1}e^{-EY}dE, \qquad (2.32)$$

where the contour runs to the left of all singularities in the E plane, N_1 is a Pomeron-particle vertex factor, and where $Y=\ln s$. At the physical point $D=2$ the integral simplifies. Deforming the contour to encircle the branch points of $\Gamma^{1,1}(E,0)$ one finds

$$\sigma_T(s)=-N_1^2\frac{\sin\pi\eta}{\pi}\int_0^1 d\xi\, e^{-\kappa Y\xi}\xi^{\eta-1}(1-\xi)^{-\eta}, \qquad (2.33)$$

where

$$\eta=-\tfrac{1}{6}, \qquad (2.34)$$

and where

$$\kappa\equiv r_0^2/(\alpha_0')^{D/2}g_1^2=r_0^2/\alpha_0' g_1^2. \qquad (2.35)$$

The integral in Eq. (2.33) is an integral representation of the confluent hypergeometric function $\Phi(\eta,1;-\kappa Y)$, for which the notations ${}_1F_1$ and M are also used,[6] so that we can write

$$\sigma_T(s)=N_1^2\Phi(\eta,1;-\kappa Y). \qquad (2.36)$$

Notice that the variable on which σ_T depends is $\kappa Y\equiv(r_0^2/\alpha_0' g_1^2)Y$. The dependence on $(r_0^2/\alpha_0')Y$ has been observed in Ref. 2, but it is interesting that the factor g_1^{-2} is present also. For small g_1—that is, for theories which are nearly asymptotically free—the renormalization-group limit is achieved at smaller Y. We shall discuss the approach to scaling in more detail in Sec. VI.

The asymptotic expansion of the cross section for large κY can be obtained by expansion of the last factor in the integral representation (2.33). The leading term,[6] evaluated using the ϵ expansion of Eqs. (2.27) and (2.28), is

$$\sigma_T(s)\sim N_1^2\left[\frac{3r_0^2}{\alpha_0'(8\pi)}\right]^{1/6}\frac{Y^{1/6}}{\Gamma(\frac{7}{6})}. \qquad (2.37)$$

The power of Y agrees, of course, with the results of Abarbanel and Bronzan.[1] The coefficient may also be of interest, although it is model-dependent—that is, it does not have the universality property of being independent of the Pomeron interaction chosen. We shall return to this point after discussing the perturbation expansion.

The perturbation expansion of Eq. (2.33) can be found by expanding the exponential to obtain the usual confluent hypergeometric series. The first few terms are

$$\sigma_T(s)=N_1^2\left[1-\eta\kappa Y+\eta(\eta+1)\frac{(\kappa Y)^2}{(2!)^2}+\cdots\right]. \qquad (2.38)$$

Thus the zeroth order term is $\sigma_T^{(0)}(s)=N_1^2$, the contribution of bare Pomeron exchange.

The first two terms in (2.38) are just the Y-plane version of (2.29). Again, the series is meaningless at $D=2$, because $\kappa=\infty$ at that point. If, however, we introduce a cutoff and thereby replace (2.29) by (2.31), we can transform to the Y plane and obtain for $Y \gg b$

$$\sigma_T(s) = N_1^2\left[1+\frac{r_0^2}{16\pi\alpha_0'}\left(Y\ln\frac{Y}{b}-Y\right)+\cdots\right]. \qquad (2.39)$$

This gives us an interpretation of our representation, Eq. (2.25), in the region where a low-energy perturbation series is valid. We have already exhibited above expressions which are valid for large Y. Although divergence difficulties prevent us from exhibiting a representation which interpolates smoothly between small and large Y, we have found a relation between the magnitude of the cross section at the two extremes; namely,

$$\lim_{s\to\infty}\sigma_T(s)=\sigma_T^{(0)}\left(\frac{3r_0^2}{8\pi\alpha_0'}\right)^{1/6}\frac{Y^{1/6}}{\Gamma(\frac{7}{6})}, \qquad (2.40)$$

where $\sigma_T^{(0)}$ is the contribution to the total cross section arising from bare Pomeron exchange. If the bare Pomeron can be isolated by phenomenological studies at Fermilab–CERN-ISR energies (as well as the magnitude r_0 of the bare triple-Pomeron coupling) then (2.40) can be used to predict the asymptotic cross section. Such an isolation would, however, be model-dependent.

III. GENERAL METHOD, $\vec{k}^2=0$

In Sec. II we have illustrated the method for obtaining a representation of the two-point Green's function at $\vec{k}^2=0$. We shall present in this section the general method for obtaining the representation of any N-point Green's function at $\vec{k}^2=0$. In particular the representation of the inclusive cross section in the triple-Pomeron region will be discussed in the following paper.[7]

The renormalization-group analysis exploits the fact that the energy scale is set in the theory by the arbitrarily chosen renormalization point E_N. Since the variation of the location of the renormalization point $-E_N\to\eta E_N$ is a symmetry transformation, the variation of the parameters of the theory due to this transformation are correlated in such a way that no change is introduced in the physical content. The detailed functional variations of the renormalization constant Z, the renormalized slope, and the coupling constant can be found in this way. Using this information, one can investigate the infrared behavior of the renormalized and unrenormalized Green's functions of the theory, which in turn determines the asymptotic behavior of scattering amplitudes at infinite energy.

The renormalization conditions which we will use are (2.1) and (2.5), but we shall generalize (2.6) to[8]

$$r(E_N,E_M,E_L)=(2\pi)^{(D+1)/2}\,\Gamma_R^{1,2}(E_ik_i)\big|_{E_1=-E_N,\,E_2=-E_M,\,E_3=-E_L;\;k_i\cdot k_j=0}. \qquad (3.1)$$

Since g is no longer the only dimensionless quantity we must look now more carefully at the rest of the results in Sec. II. In second-order perturbation theory we now find

$$g=g_0\left[1-\tfrac{1}{2}g_0^2\frac{\Gamma_{(2-D/2)}}{(8\pi)^{D/2}}J(D,l_2,l_3)\right], \qquad (3.2)$$

where

$$J(D,l_2,l_3)=4I(D,l_2,l_3)-3x/2+(2x-1)D/8, \qquad (3.3a)$$

$$I(D,l_2,l_3)=x\left[\frac{1-l_2^{D/2-1}}{2(1-l_2)}+\frac{1-l_3^{D/2-1}}{2(1-l_3)}\right], \qquad (3.3b)$$

and where

$$l_2=\frac{E_M}{E_N},\quad l_3=\frac{E_L}{E_N}, \qquad (3.3c)$$

In calculating γ we have to take into account the fact that not only g but also l_2 and l_3 are dimensionless. The generalization of (2.9b) is as follows:

$$\begin{aligned}\gamma&=E_N\frac{\partial}{\partial E_N}\ln Z\bigg|_{E_M,E_L}\\&=E_N\frac{\partial g}{\partial E_N}\bigg|_{E_M,E_L}\frac{\partial}{\partial g}\ln Z+E_N\frac{\partial l_2}{\partial E_N}\bigg|_{E_M,E_L}\frac{\partial\ln Z}{\partial l_2}\\&\quad+E_N\frac{\partial l_3}{\partial E_N}\bigg|_{E_M,E_L}\frac{\partial\ln Z}{\partial l_3},\end{aligned} \qquad (3.4)$$

where all derivatives are taken with r_0, α_0', and D held fixed. We can simplify this expression by observing that

$$\begin{aligned}E_M\frac{\partial}{\partial E_M}\ln Z\bigg|_{E_N,E_L}&=0\\&=E_M\frac{\partial g}{\partial E_M}\bigg|_{E_N,E_L}\frac{\partial}{\partial g}\ln Z\\&\quad+l_2\frac{\partial}{\partial l_2}\ln Z,\end{aligned} \qquad (3.5)$$

and similarly for E_L to obtain

$$\gamma=\left(E_N\frac{\partial g}{\partial E_N}\bigg|_{E_M,E_L}+E_M\frac{\partial g}{\partial E_M}\bigg|_{E_N,E_L}+E_L\frac{\partial g}{\partial E_L}\bigg|_{E_N,E_M}\right)\times\frac{\partial\ln Z}{\partial g}. \qquad (3.6a)$$

Introducing the notation

$$\beta=E_N\frac{\partial g}{\partial E_N}\bigg|_{E_M,E_L}+E_M\frac{\partial g}{\partial E_M}\bigg|_{E_N,E_L}+E_L\frac{\partial g}{\partial E_L}\bigg|_{E_N,E_M}, \qquad (3.7a)$$

we can write (3.6a) in the same form as (2.10),

$$\gamma=\beta\frac{\partial}{\partial g}\ln Z. \qquad (3.6b)$$

The quantity β is just the total derivative $E_N dg/dE_N$ with l_2 and l_3 held fixed,

$$\beta=E_N\frac{d}{dE_N}g(E_N, E_M=l_2E_N, E_L=l_3E_N)\bigg|_{l_2,l_3}. \qquad (3.7b)$$

Using this formula one easily finds from (3.2) that

$$\beta=-\frac{\epsilon}{4}g+\frac{g^3}{2}\frac{\Gamma(3-D/2)}{(8\pi)^{D/2}}J(D,l_2,l_3), \qquad (3.8)$$

and γ remains unchanged.

Note that as $\epsilon\to 0$, $J(D,l_2,l_3)\to 3+O(\epsilon)$. The general procedure works as follows: Take Z_x to be any renormalization constant in the theory, either the wave-function renormalization Z or a constant Z_x defined by

$$x=Z_x x_0, \qquad (3.9)$$

where x is r, α', etc., and x_0 is the unrenormalized x.

We find

$$\gamma_x=\beta(g)\frac{\partial\ln Z_x}{\partial g}, \qquad (3.10)$$

where

$$\gamma_x=E_N\frac{d\ln Z_x}{dE_N}(E_N, E_i=l_iE_N)=\sum_i E_i\frac{\partial\ln Z_x}{dE_i}\bigg|_{E_{j\neq i}}=\sum_i\gamma_{xE_i}. \qquad (3.11)$$

Integrating Eq. (3.10) one finds

$$x=x_0\exp\left[\int_0^g\gamma_x(g')\beta^{-1}(g')dg'\right]. \qquad (3.12)$$

The interesting case occurs of course when $\beta(g)$ has a zero. We will continue to use Eq. (3.8) which was calculated to second order in perturbation theory.

Define $\bar{\gamma}_x$ by

$$\gamma_x=\bar{\gamma}_x\frac{g^2}{g_1^{\,2}}, \qquad (3.13)$$

where g_1 is the zero of $\beta(g)$, and is given by Eq. (2.19a) with the change $J(D)\to J(D,l_2,l_3)$. Then $g(g_0l_2l_3D)$ is evaluated as in Sec. II and is given by Eq. (2.22) with $g_1(D)\to g_1(D,l_2,l_3)$ [and $\to g_1(D)$ as $D\to 4$]. From Eq. (3.12) we see that the rest of the parameters of the theory are proportional to powers of $(1+g_0^{\,2}/g_1^{\,2})^{-1/2}$:

$$Z_x=\left[\left(1+\frac{g_0^{\,2}}{g_1^{\,2}}\right)^{-1/2}\right]^{4\bar{\gamma}_x/\epsilon}, \qquad (3.14a)$$

$$x=x_0\left[1+\frac{r_0^{\,2}}{(\alpha_0')^{D/2}}\frac{E_N^{-\epsilon/2}}{g_1^{\,2}}\right]^{-2\bar{\gamma}_x/\epsilon}. \qquad (3.14b)$$

In particular, we are interested in α' and r. If we define

$$\alpha'\gamma_{\alpha'}=E_N\frac{\partial\alpha'}{\partial E_N}\bigg|_{r_0,\alpha_0',E_M,E_L}\equiv\zeta=\alpha'\bar{\zeta}\frac{g^2}{g_1^{\,2}}, \qquad (3.15)$$

then from (3.14) it follows that

$$\alpha'=\alpha_0'\left(1+\frac{g_0^{\,2}}{g_1^{\,2}}\right)^{-2\bar{\zeta}/\epsilon}. \qquad (3.16)$$

Moreover, one finds

$$r=r_0\left(1+\frac{g_0^{\,2}}{g_1^{\,2}}\right)^{-2\bar{\gamma}_r/\epsilon}=r_0\left(1+\frac{g_0^{\,2}}{g_1^{\,2}}\right)^{-1/2-\bar{\zeta}D/2\epsilon}. \qquad (3.17)$$

From (3.14a) Z is found to be

$$Z=\left[\left(1+\frac{g_0^{\,2}}{g_1^{\,2}}\right)^{-1/2}\right]^{4\bar{\gamma}/\epsilon}=\left(1+\frac{g_0^{\,2}}{g_1^{\,2}}\right)^{-2\bar{\gamma}/\epsilon}. \qquad (3.18)$$

Returning now to our previously mentioned program, we vary $-E_N\to\eta E_N$ in order to investigate the response of r, α', g, and Z to this change. Define $X(\eta)$ as the value taken on by the parameter X after the transformation $-E_N\to\eta E_N$. It follows that $\alpha'(\eta=-1)=\alpha'$, $r(-1)=r$, $g(-1)=g$, and $Z(-1)=Z$.

Using Eqs. (2.22), (3.16), (3.17), and (3.18) we find

$$g(\eta)=g_1\left[1+(-\eta)^{\epsilon/2}\left(\frac{g_1^{\,2}}{g^2}-1\right)\right], \qquad (3.19a)$$

$$Z(\eta)=Z(-\eta)^{\bar{\gamma}}\left(\frac{g^2}{g(\eta)^2}\right)^{-2\bar{\gamma}/\epsilon}, \qquad (3.19b)$$

$$\alpha'(\eta)=\alpha'(-\eta)^{\bar{\zeta}}\left(\frac{g^2}{g(\eta)^2}\right)^{-2\bar{\zeta}/\epsilon}, \qquad (3.19c)$$

$$r(\eta)=r(-\eta)^{\bar{\gamma}_r}\left(\frac{g^2}{g(\eta)^2}\right)^{-2\bar{\gamma}_r/\epsilon}$$

$$[\text{where } \bar{\gamma}_r=\epsilon/4+(D/4)\bar{\xi}]. \tag{3.19d}$$

The existence of a zero in $\beta(g)$ at $g=g_1$ is responsible for the limit $\lim_{\eta\to 0}g(\eta)=g_1$ in (3.19a). We see therefore that Eqs. (3.19a)–(3.19d) give the exponents of Z, α', r as the scale of the renormalization point is varied, in the limit $\eta\to 0$:

$$\begin{aligned} Z(\eta) &\underset{\eta\to 0}{\sim} (-\eta)^{\bar{\gamma}} Z f_Z\left(\frac{g}{g_1}\right), \\ \alpha'(\eta) &\underset{\eta\to 0}{\sim} (-\eta)^{\bar{\xi}}\alpha' f_{\alpha'}\left(\frac{g}{g_1}\right), \\ r(\eta) &\underset{\eta\to 0}{\sim} (-\eta)^{\epsilon/4+(D/4)\bar{\xi}_r} r f_r\left(\frac{g}{g_1}\right). \end{aligned} \tag{3.20}$$

Equations (3.19a)–(3.19d) can be used to calculate the infrared asymptotic behavior of any renormalized Green's function. For simplicity we first consider $k_i=0$. From dimensional considerations, we have

$$i\Gamma_R^{n,m}(E_i=-l_iE_N,\, k_i=0,\, \alpha',\, r,\, -E_N) = r^{n+m-2}(-E_N)^{3-m-n}\,\phi_{n,m}(g,\, l_i). \tag{3.21}$$

The $-E_N$ in $\Gamma_R^{n,m}$ denotes the fact that the theory was renormalized in the sense of Eqs. (2.1), (2.5), and (3.1). The powers of r in Eq. (3.21) were chosen by power counting in any zero-loop diagram which contributes to $\Gamma_R^{n,m}$, but any other form is also appropriate for the discussion.

The following relation enables us to find the infrared asymptotic behavior of any Green's function by using the exponential behavior of the parameters as $-E_N\to\eta E_N$ is varied:

$$\begin{aligned} i\Gamma_R^{n,m}(E_i=l_iE,\, k_i=0,\, \alpha',\, r,\, -E_N) &= Z(g)^{(m+n)/2}\, i\Gamma^{n,m}(E_i=l_iE,\, k_i=0,\, \alpha_0',\, r_0) \\ &= Z(g)^{(m+n)/2}\, Z(g(\eta))^{-(m+n)/2}\, i\Gamma_R^{n,m}(E_i=l_i\eta E_N,\, k_i=0,\, \alpha',\, r,\, \eta E_N)\big|_{\eta=E/E_N}. \end{aligned} \tag{3.22}$$

Using Eq. (3.21) one then finds

$$i\Gamma_R^{n,m}(E_i=l_iE,\, k_i=0,\, \alpha',\, r,\, -E_N)=\left(\frac{Z}{Z(\eta)}\right)^{(m+n)/2} r(\eta)^{m+n-2}\,(\eta E_N)^{3-m-n}\phi_{n,m}(g,\, l_i)\Big|_{\eta=E/E_N}. \tag{3.23}$$

Using (3.19a)–(3.19d) one finally obtains

$$\begin{aligned} i\Gamma_R^{n,m}(E_i=l_iE,\, k_i=0,\, \alpha',\, r,\, -E_N) &= E\left(-\frac{E}{E_N}\right)^{-(m+n)\bar{\gamma}/2-Dz(m+n-2)/4}\left(\frac{E_N}{\alpha'}\right)^{D(2-m-n)/4} \\ &\times\left\{\frac{g^2}{g_1^2}\left[1+\left(-\frac{E}{E_N}\right)^{\epsilon/2}\left(\frac{g_1^2}{g^2}-1\right)\right]\right\}^{\bar{\gamma}(m+n)/\epsilon-[D(1-z)/2\epsilon+1/2](m+n-2)} \\ &\times(-g)^{m+n-2}\phi_{m,n}(g,\, l_i), \end{aligned} \tag{3.24}$$

where $z=1-\zeta$. This result gives the $E\to 0$ asymptotic behavior in agreement with that found by Abarbanel and Bronzan.[1] In addition, it gives for $\Gamma^{1,1}$ and $\Gamma^{1,2}$ representations which contain the perturbation expansion in powers of r_0 (correct to second order) as well as the asymptotic behavior.

It is possible, of course, to obtain from Eq. (3.24) the representations for any unrenormalized Green's functions $\Gamma^{n,m}(E_ik_i)$ as well.

The functions $\phi_{n,m}$ are, at this point, unknown. For the special cases $\Gamma_R^{1,1}$ and $\Gamma_R^{1,2}$ these functions are determined by the renormalization conditions to be $\phi_{1,1}(g,\, l_i)=1$ and $\phi_{1,2}(g,\, l_i)=(2\pi)^{-(D+1)/2}$. If we were interested in higher Green's functions we could choose appropriate renormalization conditions to determine their ϕ functions, thus changing our definitions of the parameters.

Turning now to our simple special cases, we find from (3.24) that the unrenormalized $\Gamma^{1,1}$ is given by

$$i\Gamma^{1,1}(E,\, 0,\, \alpha_0',\, r_0)=E\left[1+\frac{r_0^2}{(\alpha_0')^{D/2}}\,\frac{(-E)^{-\epsilon/2}}{g_1^2}\right]^{2\bar{\gamma}/\epsilon}, \tag{3.25}$$

which was discussed in Sec. II. We also find

$$i\Gamma^{1,2}(E,\, l_2E,\, l_3E,\, \vec{0},\, \alpha_0',\, r_0)=\frac{r_0}{(2\pi)^{(D+1)1/2}}\left[1+\frac{r_0^2}{(\alpha_0')^{D/2}}\,\frac{(-E)^{-\epsilon/2}}{g_1^2}\right]^{3\bar{\gamma}/\epsilon+zD/2\epsilon-2/\epsilon}. \tag{3.26}$$

Equation (3.26) [as shown for (3.25)] when expanded in power series gives the correct second-order perturbation expression for $\Gamma^{1,2}$. It also gives the asymptotic infrared behavior of $\Gamma^{1,2}$ calculated to the one-loop approximation through β, γ, ζ.

IV. GENERAL METHOD, $\vec{k}^2 \neq 0$

The method that we have used in the preceding sections gave us representations for the various Green's functions at zero momentum transfer. It will be extended now to include also nonforward amplitudes. The only additional step required is the introduction of a momentum scale k_N^2 into the analysis, which is accomplished by the use of renormalization points at $k^2 \neq 0$.

We define the parameters of the theory by the following conditions:

$$i\Gamma_R^{1,1}\big|_{E=-E_N,\ \vec{k}=\vec{k}_N} = -E_N - \alpha' k_N^2, \tag{4.1a}$$

$$i\frac{\partial\Gamma_R^{1,1}}{\partial k^2}\bigg|_{E=-E_N,\ \vec{k}=\vec{k}_N} = -\alpha', \tag{4.1b}$$

$$\Gamma_R^{1,2}\bigg|_{\substack{E_1=-E_N,\ E_2=-E_M,\ E_3=-E_L;\\ \vec{k}_1=\vec{k}_N,\ \vec{k}_2=\vec{v}_2|\vec{k}_N|,\ \vec{k}_3=\vec{v}_3|\vec{k}_N|}} = \frac{r}{(2\pi)^{(D+1)/2}}. \tag{4.1c}$$

The set of dimensionless parameters now contains the v_i^2, $l_2 \equiv E_M/E_N$, $l_3 \equiv E_L/E_N$,

$$g = \frac{rE_N^{-\epsilon/4}}{(\alpha')^{D/4}} \text{ and also } h = \frac{\alpha' k_N^2}{E_N}.$$

From (4.1a)–(4.1c) one finds the following expressions to second order in g_0:

$$Z = 1 + \tfrac{1}{2}g_0^2\frac{\Gamma(2-D/2)}{(8\pi)^{D/2}}\left(1+\frac{h_0}{2}\right)^{D/2-2} \times\left[1-\left(\frac{D/2-2}{D/2-1}\right)\left(1+\frac{h_0}{2}\right)\right], \tag{4.2a}$$

$$\alpha' = \alpha_0'\left\{1+\tfrac{1}{2}g_0^2\frac{\Gamma(2-D/2)}{(8\pi)^{D/2}}\left(1+\frac{h_0}{2}\right)^{D/2-2} \times\left[\frac{1}{2}-\left(\frac{D/2-2}{D/2-1}\right)\left(1+\frac{h_0}{2}\right)\right]\right\}. \tag{4.2b}$$

$$g = g_0\left[1-\tfrac{1}{2}g_0^2\frac{\Gamma(2-D/2)}{(8\pi)^{D/2}}J\right], \tag{4.2c}$$

where

$$J = 4F + \left(\frac{D}{8}-\frac{3}{2}\right)\left(1+\frac{h_0}{2}\right)^{D/2-2} - \frac{D-4}{D-2}\left(\frac{3}{2}-\frac{D}{4}\right)\left(1+\frac{h_0}{2}\right)^{D/2-1}, \tag{4.2d}$$

$$F = \tfrac{1}{2}\int_0^1 d\zeta\left\{\left[\zeta+(1-\zeta)l_2 + h_0\left(v_2^2+\zeta v_3^2-\tfrac{1}{2}(v_2^2-\zeta v_3^2)\right)\right]^{D/2-2} + [2\leftrightarrow 3]^{D/2-2}\right\}, \tag{4.2e}$$

and where

$$h_0 = \alpha_0' k_N^2/E_N.$$

Since a new momentum scale has been introduced into the discussion, the derivation which led to (3.14) has to be slightly modified.

We shall generalize the discussion here to include in the theory any number of parameters in addition to r and α'. In general the new parameters can be chosen to be dimensionless. We denote the set of dimensionless parameters (excluding l_i and v_i, which we treat separately) in the theory by $\{y\} = \{g, h, y_i\}$.

For each parameter $x = x_0 Z_x$ in the set ($x = g$, h, or y_i) define ζ_{xE_i}:

$$\zeta_{xE_i} = E_i\frac{\partial x}{\partial E_i}\bigg|_{B,\ E_{j\neq i},\ k_j}, \tag{4.3}$$

where B denotes r_0, α_0', and any additional dimensional or dimensionless unrenormalized parameters. Let us also define

$$\gamma_{xE} \equiv \sum_i E_i\frac{\partial \ln Z_x}{\partial E_i}\bigg|_{B,\ E_{j\neq i},\ k_j} \tag{4.4a}$$

$$= E_N\frac{d\ln Z_x}{dE_N}\bigg|_{B,\ l_j,\ k_j}. \tag{4.4b}$$

We shall find that it is only the total derivative in Eq. (4.4b) which we require in forming integral representations. This is fortunate in that the total derivative turns out to be a much simpler quantity to calculate than the individual terms in Eq. (4.4a).

Now it follows from the fact that Z_x is dimensionless, and is therefore a function of dimensionless quantities only, that

$$\gamma_{xE} = \sum_y \zeta_{yE}\frac{\partial \ln Z_x}{\partial y}, \tag{4.5}$$

where

$$\zeta_{yE} = \sum_i \zeta_{yE_i} = E_N\frac{dy}{dE_N}\bigg|_{B,\ l_j,\ k_j}. \tag{4.6}$$

The same can be done now by using derivatives with respect to the k_i's. One finds

$$\gamma_{x_k} = \sum_y \zeta_{y_k} \frac{\partial \ln Z_x}{\partial y}, \tag{4.7}$$

where

$$\gamma_{x_k} = \sum_i k_i^2 \frac{\partial \ln Z_x}{\partial k_i^2}\bigg|_{B,\,E_j,\,k_j\neq i} = k_N^2 \frac{d\ln Z_x}{dk_N^2}\bigg|_{B,\,E_j,\,v_j} \tag{4.8}$$

and

$$\zeta_{y_k} = \sum_i k_i^2 \frac{\partial y}{\partial k_i^2}\bigg|_{B,\,E_j,\,k_j\neq i} = k_N^2 \frac{dy}{dk_N^2}\bigg|_{B,\,E_j,\,v_j}. \tag{4.9}$$

In our nonrelativistic theory, Eqs. (4.5) and (4.7) are independent and therefore we can get rid of one of the derivatives with respect to any one of the dimensionless parameters of $\{y\}=\{g,h,y_i\}$. Eliminating h, for example, one gets from (4.5) and (4.7)

$$[\gamma_x, \zeta_h]_{Ek} = [\beta, \zeta_h]_{Ek} \frac{\partial \ln Z_x}{\partial g} + \sum_y [\zeta_y, \zeta_h]_{Ek} \frac{\partial \ln Z_x}{\partial y}. \tag{4.10}$$

Here we use the notation $\beta=\zeta_g$ and

$$[\zeta_a \zeta_b]_{Ek} = \zeta_{a_E}\zeta_{b_k} - \zeta_{a_k}\zeta_{b_E}. \tag{4.11}$$

In a theory with the dimensionless parameters g and h only we have

$$Z_x = \exp\left\{\int_0^g \frac{[\gamma_x, \zeta_h]_{Ek}}{[\beta, \zeta_h]_{Ek}} dg'\right\}. \tag{4.12}$$

Using

$$\zeta_{h_E} = h\left(-1 + \frac{1}{\alpha'} \zeta_{\alpha'_E}\right) \tag{4.13}$$

and

$$\zeta_{h_k} = h\left(1 + \frac{1}{\alpha'} \zeta_{\alpha'_k}\right), \tag{4.14}$$

one finds

$$\gamma_x + \left[\gamma_x, \frac{1}{\alpha'}\zeta_{\alpha'}\right]_{Ek} = \left(\beta + \left[\beta, \frac{1}{\alpha'}\zeta_{\alpha'}\right]_{Ek}\right)\frac{\partial \ln Z_x}{\partial g}, \tag{4.15}$$

where

$$\gamma_x = \gamma_{x_E} + \gamma_{x_k} = E_N \frac{d\ln Z_x}{dE_N}\bigg|_{B,\,l_j,\,k_j} + k_N^2 \frac{d\ln Z_x}{dk_N^2}\bigg|_{B,\,E_j,\,v_j} = E_N \frac{d\ln Z_x}{dE_N}\bigg|_{B,\,l_i,\,v_i,\,h_0} \tag{4.16a}$$

and

$$\beta = \beta_E + \beta_k = E_N \frac{dg}{dE_N}\bigg|_{B,\,l_i,\,v_i,\,h_0}. \tag{4.16b}$$

We will return now to our specific problem. From (4.2c) one finds

$$\beta(g) = -\frac{\epsilon}{4}g + \tfrac{1}{2}g^3 \frac{\Gamma_{(3-D/2)}}{(8\pi)^{D/2}} J(D, l_i, v_i, h). \tag{4.17}$$

[The only difference between Eqs. (4.17) and (3.8) is that now J is given by (4.2d).]

In the use of Eq. (4.10) with renormalization-group functions calculated in perturbation theory, only terms up to order g^3 and ϵg^2 are kept in the lowest order in the ϵ expansion. To this order Eq. (4.15) simplifies to the form

$$\gamma_x = \beta \frac{\partial \ln Z_x}{\partial g}. \tag{4.18}$$

We also see that to lowest order in the ϵ expansion β has no dependence on l, v_i, h and we have

$$\beta(g) = -\frac{\epsilon}{4}g\left(1 - \frac{g^2}{g_1^2}\right), \tag{4.19}$$

and, again, as in the preceding section we find

$$Z_x = \left(1 - \frac{g^2}{g_1^2}\right)^{2\bar{\gamma}_x/\epsilon}, \tag{4.20}$$

where $\bar{\gamma}_x$ is defined from $\gamma_x = \bar{\gamma}_x g^2/g_1^2$. Note that if x_0 is independent of E_i and k_i, we have $\gamma_x = \zeta_x / x$.

To lowest order in ϵ one finds

$$\bar{\gamma}_g = \frac{\epsilon}{4}, \tag{4.21a}$$

$$\bar{\gamma}_{\alpha'} \equiv \bar{\zeta} = -\frac{\epsilon}{24}, \tag{4.21b}$$

$$\bar{\gamma} = -\frac{\epsilon}{12}. \tag{4.21c}$$

Note that $\gamma_g = (1/g)\beta + \epsilon/4$, thus

$$g = g_0\left(1 + \frac{g_0^2}{g_1^2}\right)^{-1/2}, \tag{4.22}$$

and in terms of the renormalized g we have

$$\alpha' = \alpha_0'\left(1 + \frac{g_0^2}{g_1^2}\right)^{-\bar{\zeta}2/\epsilon} = \alpha_0'\left(1 - \frac{g^2}{g_1^2}\right)^{\bar{\zeta}2/\epsilon}, \tag{4.23a}$$

$$Z = \left(1 + \frac{g_0^2}{g_1^2}\right)^{-\bar{\gamma}2/\epsilon} = \left(1 - \frac{g^2}{g_1^2}\right)^{\bar{\gamma}2/\epsilon}. \tag{4.23b}$$

These simple expressions, which embody the results of our renormalization-group analysis, enable one to write down representations for any Reggeon-calculus Green's function. In the next section we write down and analyze the representation for $\Gamma^{1,1}$, thereby obtaining the t dependence of asymptotic elastic scattering.

V. THE SHAPE OF THE DIFFRACTION PEAK

The simplest, and one of the most interesting, applications of the formalism we have developed is the calculation of the shape of the diffraction peak in elastic scattering at asymptotic energies. The leading contribution comes from $G^{1,1}(E,k^2)$. Combining the same reasoning which led to Eq. (2.3) with our renormalization condition given in Eq. (4.1a), one finds

$$iG^{1,1}(E,k^2)=\frac{-Z(-E,k)}{E-\alpha'(-E,k)k^2}\,. \tag{5.1}$$

Now Z and α' can be evaluated by means of Eq. (4.20) to yield the result

$$iG^{1,1}(E,k^2)=\{-E+\alpha_0'k^2[1+\kappa(-E)^{-\epsilon/2}]^{-2\bar{\zeta}/\epsilon}\}^{-1}\times[1+\kappa(-E)^{-\epsilon/2}]^{-2\bar{\gamma}/\epsilon}, \tag{5.2}$$

where $\kappa\equiv r_0^2/(\alpha_0')^{D/2}g_1^2$, and where $\bar{\gamma}$ and $\bar{\zeta}$ are defined in Eqs. (2.17) and (2.18). Note that to this order in the ϵ expansion the k^2 dependence is relatively simple, and there is no dependence on l_i or v_i. Moreover, to this order

$$\bar{\gamma}=-\frac{\epsilon}{12}\text{ and }\bar{\zeta}=-\frac{\epsilon}{24}\,. \tag{5.3}$$

Our next task is to take the Mellin transform of Eq. (5.2) to obtain the scattering amplitude in the s plane. An approximate result for small $t=-k^2$ can be obtained by first replacing Eq. (5.2) by an approximate form valid for small E (and therefore large energy),

$$iG^{1,1}(E,k^2)\approx[-E+\alpha_0'k^2\kappa^{1/12}(-E)^{-\epsilon/24}]^{-1}\times\kappa^{1/6}(-E)^{-\epsilon/12}\,. \tag{5.4}$$

Then expanding the first factor one obtains

$$iG^{1,1}(E,k^2)\approx\kappa^{1/6}(-E)^{-1-\epsilon/12}\times\sum_{n=0}^{\infty}[\alpha_0't\kappa^{1/12}(-E)^{-1-\epsilon/24}]^n. \tag{5.5}$$

Performing the Mellin transform one then finds for the absorptive part $A(s,t)$ of the elastic scattering amplitude

$$A(s,t)=N_a(t)N_b(t)\kappa^{1/6}(\ln s)^{\epsilon/12}\times\sum_{n=0}^{\infty}\frac{[\alpha_0't\kappa^{1/12}(\ln s)^{1+\epsilon/24}]^n}{\Gamma(1+n+\epsilon(1/12+n/24))}, \tag{5.6}$$

where the N's are the usual Pomeron-particle vertex functions. Now if we make the further approximation of neglecting terms of order ϵ in the argument of the gamma function we find[9]

$$A(s,t)\approx N_a(t)N_b(t)\kappa^{1/6}s(\ln s)^{\epsilon/12}\times\exp[\alpha_0't\kappa^{1/12}(\ln s)^{1+\epsilon/24}]. \tag{5.7}$$

This simple result indicates an approximately exponential diffraction peak, with a rate of shrinkage faster than that of a Regge pole.

The above approximate result is useful only for small t, such that the argument of the exponential is not large. To be more quantitative, we return to Eq. (5.2) and perform the Mellin transform directly, after expanding the first factor as before. At the physical point $\epsilon=2$ the transform can be expressed in terms of confluent hypergeometric functions, as in Eq. (2.33),

$$A(s,t)=sN_a(t)N_b(t)\times\sum_{n=0}\frac{(\alpha_0't\ln s)^n}{n!}\Phi\left(-\frac{1}{6}-\frac{n}{12},n+1,-\kappa\ln s\right). \tag{5.8}$$

Using the asymptotic approximation for Φ leads again to Eq. (5.6).[6] We have, moreover, evaluated Eq. (5.8) numerically, and in Fig. 1 we compare the results with the approximate form (5.7).

The contribution of the forward diffractive peak, in the ϵ expansion, to the elastic cross section is given, according to Eq. (5.7), by

$$\sigma_{\text{el}}=\frac{1}{16\pi^2s^2}\int|T|^2d^Dk=\frac{1}{16\pi^2}(\kappa\ln s)^{\epsilon/6}\times\int N_a^2(t)N_b^2(t)\exp[2\kappa^{\epsilon/24}\alpha_0't(\ln s)^{1+\epsilon/24}]d^Dk. \tag{5.9}$$

If we take $N(t)=N(0)e^{Bt}$ we have $\sigma_T=N_a(0)N_b(0)(\kappa\ln s)^{\epsilon/12}$ and

$$\sigma_{\text{el}}=\frac{\sigma_{\text{tot}}^2}{16\pi}[4B+2\kappa^{\epsilon/24}\alpha_0'(\ln s)^{1+\epsilon/24}]^{-D/2}\underset{s\to\infty}{\sim}\frac{N_{(0)}^4}{64\pi}\kappa^{1/6}(\ln s)^{-5/6}. \tag{5.10}$$

It is interesting to examine the J-plane singularity structure of the amplitudes we have derived, although one should regard the results with proper caution. An approximation to a scattering amplitude could give a useful approximation to the t dependence over a range of t and yet have a very different singularity structure from the exact amplitude. After that word of caution, we proceed to examine Eq. (5.2). One sees that the singularities of $G^{1,1}$ in the complex J plane consist of a branch point at $J=1$ and a pole whose position is a function of t. Turning to Eq. (5.4), which is valid near $J=1$, one sees that for positive t there is one pole on the physical sheet, which occurs for real $J>1$. For t negative, there is a pair of complex conjugate poles on the physical sheet at

$$J_{\text{pole}}=1-(\alpha_0'|t|\kappa^{1/12})^{1-\epsilon/24}(1\pm i\pi\epsilon/24). \tag{5.11}$$

These poles make the following contribution to the imaginary part of the elastic amplitude:

$$A_{\text{pole}}(s,t)=\frac{2N_a(t)N_b(t)\kappa^{1/6}}{1+\epsilon/24}\,\frac{e^{-\tau\ln s}}{\tau^{\epsilon/12}}\cos\left[\frac{\pi\epsilon}{24}(\tau\ln s-2)\right], \tag{5.12}$$

where

$$\tau=(\alpha_0'|t|\kappa^{1/12})^{1-\epsilon/24}. \tag{5.13}$$

Note, however, that the oscillatory cosine term is equal to unity to first order in ϵ, and is therefore not significant in our order of approximation.

In addition to the pole contribution, the cut at

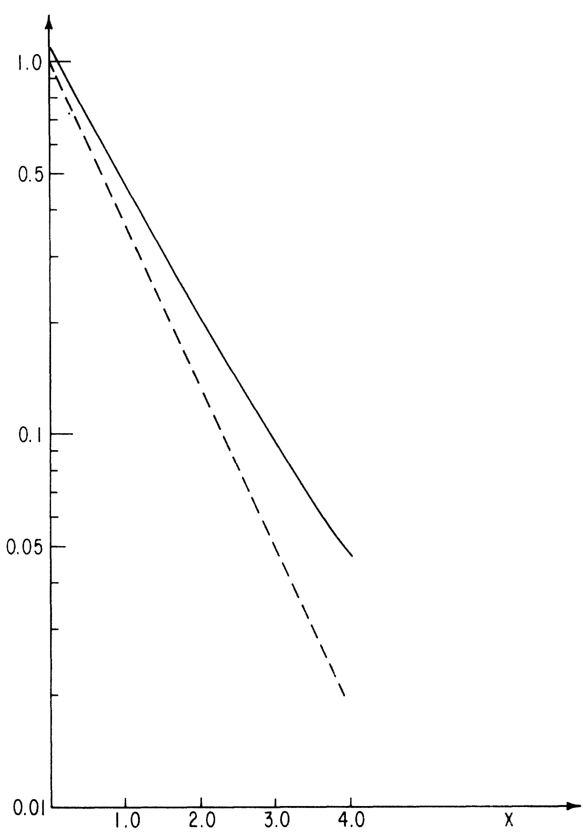

FIG. 1. Numerical comparison of the expression of the diffractive peak in the elastic amplitude calculated to the one-loop approximation in Eq. (5.8) with a pure exponential e^{-x} in the scaling variable (the graph was drawn taking $\kappa\ln s=10$). The solid line represents

$$(\kappa\ln s)^{-1/6}\sum_{n=0}\frac{(\alpha_0' t\ln s)^n}{n!}\,\Phi\left(-\frac{1}{6}-\frac{n}{12},\,n+1,\,-\kappa\ln s\right),$$

and the dashed line represents e^{-x}, $x=-\alpha_0' t\kappa^{1/12}(\ln s)^{1+1/12}$

$E=0$ contributes

$$A_{\text{cut}}(s,t)=-N_a(t)N_b(t)\kappa^{1/6}\frac{\epsilon}{24}\times\int_0^\infty dx\,\frac{e^{-x\ln s}(T+2x^{1+\epsilon/24})}{x^{\epsilon/24}(T+x^{1+\epsilon/24})^2}, \tag{5.14}$$

where

$$T\equiv\alpha_0'|t|\kappa^{1/12}. \tag{5.15}$$

Note that as $t\to0$ both pole and cut contributions are singular. It is easy to see that the singular part of the cut contribution cancels the pole in this limit.

Note that when $G^{1,1}(E,k)$ in Eq. (5.2) is expanded in powers of κ, without expanding in ϵ, it correctly reproduces the first two terms in perturbation theory. This happens in the same way as was discussed in Sec. II for the case of $G^{1,1}(E,0)$. Therefore, $G^{1,1}(E,k)$ will contain explicitly the input moving cut. The absence of a moving cut in the first-order ϵ expansion [namely, using Eq. (5.3) for $\bar{\gamma}$ and $\bar{\xi}$ and $g_1^2=(\epsilon/6)(8\pi)^2$] may be seen therefore as a special property of this order of the calculation and may change when higher-order terms in ϵ are included.[10]

VI. THE APPROACH TO THE SCALING LIMIT

In the preceding sections we have derived representations of Reggeon Green's functions which exhibit the asymptotic, or scaling, limit implied by the renormalization-group approach. Moreover, these representations [such as given in Eqs. (2.25) and (5.2)] imply nonasymptotic corrections to the scaling limit. In this section we investigate the generality and reliability of these nonscaling corrections.

A. Approach to scaling at $k^2=0$

Let us first consider the simpler case of $k^2=0$. Our representation, given by Eqs. (2.25) and (2.29), is

$$\sigma_T(s)=-N_1^2\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}E^{-1}[1+\kappa(-E)^{-\epsilon/2}]^{-2\bar{\gamma}/\epsilon}\times e^{-EY}dE, \tag{6.1}$$

where

$$\kappa=r_0^2/(\alpha_0')^{D/2}g_1^2. \tag{6.2}$$

Expanding the term in brackets in Eq. (6.1) for small E and performing the Mellin transform one obtains the asymptotic expansion

$$\sigma_T(s)=N_1^2\kappa^{-2\bar{\gamma}/\epsilon}(\ln s)^{-\bar{\gamma}}\left[\frac{1}{\Gamma(1+\bar{\gamma})}+\frac{(-2\bar{\gamma}/\epsilon)\kappa^{-1}(\ln s)^{-\epsilon/2}}{\Gamma(1+\bar{\gamma}-\epsilon/2)}+\cdots+\left(-\frac{2}{\epsilon}\bar{\gamma}\right)\left(-\frac{2}{\epsilon}\bar{\gamma}-1\right)\cdots\left(-\frac{2}{\epsilon}\bar{\gamma}-k+1\right)\frac{\kappa^{-k}(\ln s)^{-k\epsilon/2}}{k!\Gamma(1-\bar{\gamma}-k\epsilon/2)}+\cdots\right]. \tag{6.3}$$

As we discussed at the end of Sec. IIA, the divergences of the simple Reggeon calculus prevent us from applying our results directly at the physical point $D=2$. Instead one performs an ϵ expansion, which is then continued to $\epsilon=2$. The calculations to this point were within the one-loop approximation for the renormalization-group functions β, γ, ζ; therefore, the critical exponents as well as the scaling functions can be calculated in an ϵ expansion to order ϵ. Expanding Eq. (6.3) one finds for $\sigma_T(s)$

$$\sigma_T(s)=N_1^2\kappa^{1/6}(\ln s)^{\epsilon/12}\left[1+\frac{\epsilon}{12}\ln\gamma_e+\kappa^{-1}(\ln s)^{-\epsilon/2}\left(\tfrac{1}{6}-\tfrac{5}{72}\epsilon\ln\gamma_e\right)+O(\kappa^{-2}(\ln s)^{-\epsilon})\right], \tag{6.4}$$

where γ_e is the Euler-Mascheroni constant.

We kept in Eq. (6.4) only the leading correction to the asymptotic form, although the rest of these terms of order $\kappa^{-k}\ln s^{-\epsilon/2k}$ are easily calculated to the desired order from Eq. (6.1).

Note that for fixed r_0 and α_0', κ^{-1} is proportional to ϵ. To get better insight into the significance of this expansion, it is interesting to reformulate it in terms of renormalized quantities. Making use of Eq. (2.23) one finds that

$$\kappa^{-1}=g_1^2(\alpha_0')^{D/2}/r_0^2=g_1^2E_N^{-\epsilon/2}/g_0^2=E_N^{-\epsilon/2}(g_1^2-g^2)/g^2. \tag{6.5}$$

Expressing the total cross section in terms of renormalized quantities according to the relations

$$G_R^{1,1}=Z^{-1}G^{1,1}, \tag{6.6a}$$

$$N_{1R}=Z^{1/2}N_1, \tag{6.6b}$$

we can then write

$$\sigma_T(s)=N_{1R}^2(E_N\ln s)^{\epsilon/12}\left[1+\frac{\epsilon}{12}\ln\gamma_e+\frac{g_1^2-g^2}{g^2}(E_N\ln s)^{-\epsilon/2}\left(\tfrac{1}{6}-\tfrac{5}{72}\epsilon\ln\gamma_e\right)+\cdots\right]. \tag{6.7}$$

Both Eqs. (6.3) and (6.7) are valid forms of the asymptotic expansion of the cross section, which exhibit the lowest-order corrections to the scaling limit. Note that the asymptotic expansion is an expansion in powers of $(\ln s)^{\lambda}$, where $\lambda=\epsilon/2$. We shall discuss the significance of this exponent[11] in detail in Sec. VIB.

Note also that the expansion parameter is proportional to κ^{-1}. In terms of bare parameters, κ^{-1} is given in Eq. (6.2). If one holds r_0 and α_0' fixed, then it is the size of g_1 (which depends on the particular model and method of renormalization chosen) which determines the scale of the nonscaling terms. On the other hand, one can focus on the renormalized parameters g and E_N. Then we see from Eqs. (6.4) and (6.7) that it is the proximity of g to g_1 which determines the scale of the nonscaling terms.

B. General discussion of the approach to scaling

The approach to the scaling limit presented in Eqs. (6.3) and (6.4) was calculated in the one-loop approximation. From Eqs. (3.6b) and (3.10) one can see the general form of the term which governs the approach to the scaling limit. Consider a theory in which the dimensionless coupling constant has the form $g_0=\phi E_N{}^P$, where ϕ is a function of the bare parameters and P is some power of E_N (in our case $P=-\epsilon/4$). Defining $g=g_0Z_g$ we have

$$\gamma_g(g)=\beta(g)\frac{\partial}{\partial g}\ln Z_g, \tag{6.8a}$$

where

$$\gamma_g(g)=\frac{1}{g}\beta(g)-P. \tag{6.8b}$$

If $\beta(g)$ has a zero at g_1 and if the renormalized g is close to g_1, we have

$$Z_g=\exp\left[\int_0^g\frac{\gamma_g(g)}{\beta(g)}dg\right]\simeq(1-g/g_1)^{\gamma_g(g_1)/\lambda}, \tag{6.9}$$

where

$$\lambda=\left.\frac{d\beta(g)}{dg}\right|_{g=g_1},\quad \gamma_g(g_1)=-P.$$

For the wave-function renormalization constant one finds

$$Z \simeq (1-g/g_1)^{\gamma(g_1)/\lambda}$$
$$\simeq (g_0/g_1)^{\gamma(g_1)/P}\left[1+\frac{\gamma(g_1)}{P}(g_0/g_1)^{\lambda/P}+\cdots\right]. \tag{6.10}$$

Therefore, we find for the case in which g is close to g_1

$$\Gamma^{1,1}(E,0) \simeq (-E)^{1-\gamma(g_1)}(\phi/g_1)^{-\gamma(g_1)/P} \times\left[1-\left(\frac{\phi}{g_1}\right)^{\lambda/P}\frac{\gamma(g_1)}{P}(-E)^{\lambda}+\cdots\right], \tag{6.11a}$$

$$\Gamma_R^{1,1}(E,0) \simeq E_N(-E/E_N)^{1-\gamma(g_1)} \times\left[1-\left(\frac{g_1-g}{g_1}\right)(g/g_1)^{\lambda/P}\frac{\gamma(g_1)}{P}(-E/E_N)^{\lambda}+\cdots\right]. \tag{6.11b}$$

Comparing Eqs. (6.11) and (6.3)–(6.7) one sees the origin of the various expressions containing an ϵ dependence in Eqs. (6.3)–(6.7). The infrared-stability condition ($\lambda>0$) will ensure the decrease of the nonleading terms as we approach the scaling limit as $\ln s\to\infty$. The region in which the nonleading terms may be neglected depends on the exponent λ and the "nonuniversal" constant in front of $(-E)^{\lambda}$ in Eqs. (6.11).

The new critical exponent λ can be calculated within the ϵ expansion, and to first order in ϵ we find $\lambda=\epsilon/2$. The higher orders in ϵ can be found by using the calculations of Refs. 5. One finds

$$\lambda=\frac{\epsilon}{2}-\epsilon^2\left(\tfrac{157}{576}+\tfrac{149}{288}\ln\tfrac{4}{3}\right). \tag{6.12}$$

This is a poorly convergent expression. It is interesting to note that in solid-state physics[12] the analogous critical exponent (which was calculated there to order ϵ^3) is also one of the worst-converging critical exponents and is evaluated by using Padé approximants. Here we have $\lambda(\epsilon=2)<0$, and Padé approximants suggest $\lambda(\epsilon=2)=0.37$. Higher-order terms in ϵ might change this value considerably.

The corrections to the scaling contribution of $G^{1,1}$ to the elastic amplitude are in principle competing with the contributions of graphs in which several Pomerons couple to the external particles.[13] The determination of the leading corrections to the scaling contribution depends on the power behavior in $\ln s$ and in practice, at finite s, it depends as well on the so-called "nonuniversal" coefficients which stand in front of the $\ln s$ power. The power of the leading correction to scaling which comes from $G^{1,1}$ is $\ln s^{-\bar{\gamma}-\lambda}$ (to second order in ϵ one finds[5] $-\bar{\gamma}\simeq\epsilon/12+0.05\epsilon^2\simeq 0.38$); this term is competing with the leading contribution of $G^{1,2}\sim\ln s^{-1}$. The bad convergence properties of $\lambda(\epsilon)$ preclude making any definite conclusions as to the dominant corrections. It is highly desirable at this point to calculate the exponent $\lambda(\epsilon)$ using other methods in the Reggeon field theory.

C. Approach to scaling at $k^2\neq 0$

The representation of $G^{1,1}(E,t)$ given in Eq. (5.2) results in the form (5.8) for $A(s,t)$. Equation (5.2) can be written in the form

$$iG^{1,1}=\sum_{n=0}^{\infty}(t\alpha_0')^n\kappa^{-(2/\epsilon)\bar{\gamma}-(2/\epsilon)\bar{\zeta}n}(-E)^{-1-n+\bar{\gamma}+\bar{\zeta}n}\left[1+\kappa^{-1}(-E)^{\epsilon/2}\right]^{-(2/\epsilon)\bar{\gamma}-(2/\epsilon)\bar{\zeta}n}. \tag{6.13}$$

Again the asymptotic expansion in powers of κ^{-1} introduces the nonleading terms which govern the approach to scaling:

$$iG^{1,1}(s,t)=\sum_{n=0}^{\infty}(\alpha_0't)^n\kappa^{-(2/\epsilon)\bar{\gamma}-(2/\epsilon)\bar{\zeta}n}s(\ln s)^{-\bar{\gamma}} \times\left[\frac{(\ln s)^{n-\bar{\zeta}n}}{\Gamma(1+n-\bar{\zeta}n-\bar{\gamma})}-\frac{2}{\epsilon}(\bar{\gamma}+\bar{\zeta}n)\kappa^{-1}\frac{(\ln s)^{n-\bar{\zeta}n-\epsilon/2}}{\Gamma(1+n-\bar{\gamma}-\bar{\zeta}n-\epsilon/2)}+O(\kappa^{-2}(\ln s)^{n-\bar{\zeta}n-\epsilon})\right]. \tag{6.14}$$

The ϵ expansion to order ϵ gives

$$iG^{1,1}(s,t)=s\kappa^{1/6}(\ln s)^{\epsilon/12}e^x\left\{1-\frac{\epsilon}{12}\phi(x)+\tfrac{1}{6}\kappa^{-1}(\ln s)^{-\epsilon/2}\left[1+\frac{x}{2}-\frac{\epsilon}{24}\left(x\frac{d}{dx}\phi(x)+x\phi(x)-10\phi(x)\right)\right]\right\}, \tag{6.15}$$

where

$$x=\alpha_0' t\kappa^{-(2/\epsilon)\bar{\zeta}}(\ln s)^{1-\bar{\zeta}}=\alpha_0' t\kappa^{1/12}(\ln s)^{1+\epsilon/24}$$

and

$$\phi(x)=\left(1+\frac{x}{2}\right)\left[-\ln\gamma_e+\int_0^x\frac{dz}{z}(1-e^{-z})\right]+\frac{1-e^{-x}}{2}. \tag{6.16}$$

The function $\phi(x)$ is "universal" and will not be changed when higher orders in ϵ will be computed. The approach to the scaling limit is governed by $\ln s^{-\lambda}$ and a "nonuniversal" coupling κ^{-1}. Again we may write (6.15) in terms of renormalized parameters using Eqs. (6.4)–(6.6); in addition we have

$$\alpha_0'\kappa^{-(2/\epsilon)\bar{\zeta}}=\alpha' E_N^{-\bar{\zeta}}(g_1^2/g^2)^{(2/\epsilon)\bar{\zeta}}\xrightarrow[g\to g_1]{}\alpha' E_N^{-\bar{\zeta}} \tag{6.17a}$$

$$x=\alpha_0' t\kappa^{1/12}(\ln s)^{1+\epsilon/24}=\alpha' t E_N^{\epsilon/24}(\ln s)^{1+\epsilon/24}. \tag{6.17b}$$

Finally

$$iG_R^{1,1}(s,t)=s(E_N\ln s)^{\epsilon/12}e^x\left\{1-\frac{\epsilon}{12}\phi(x)+\frac{1}{6}\frac{g_1^2-g^2}{g^2}(E_N\ln s)^{-\epsilon/2}\left[1+\frac{x}{2}-\frac{\epsilon}{24}\left(x\frac{d}{dx}\phi(x)+x\phi(x)-10\phi(x)\right)\right]\right\}. \tag{6.18}$$

VII. DISCUSSION

As the onset of scaling in Reggeon field theory is probably beyond present laboratory energies, it seems highly desirable to study the transition from the perturbative scheme applicable at present energies to the critical phenomena approach presented in Refs. 1 and 2. In this study of Reggeon field theory we have presented a method of integration of the renormalization-group equations and then used it to discuss the problem of the approach to the scaling limit and the detailed functional form of this limit. As the energy increases, a transition region is reached in which the perturbative scheme is not very useful in the sense that one must add many terms, while the energy is not high enough for the asymptotic form to emerge. At these energies it seems very useful to have a description of the structure of the terms which govern the approach to the scaling limit. For example, according to Eq. (2.36) the total cross section is given by

$$\sigma_T(s)=N^2\Phi\left(-\tfrac{1}{6},1,\frac{r_0^2}{(\alpha_0')^{D/2}g_1^2}\ln s\right)$$
$$=N_R^2(E_N\ln s)^{1/6}\left\{1+\tfrac{1}{12}\ln\gamma_e+\frac{g_1^2-g^2}{g^2}(E_N\ln s)^{-1}(\tfrac{1}{6}-\tfrac{5}{36}\ln\gamma_e)+O((g-g_1)^2(\ln s)^{-2})\right\}. \tag{7.1}$$

If one expands Φ in powers of $r_0^2\ln s$ one obtains the perturbative scheme for the Pomeron propagator contribution to the total cross section. At high energies one would prefer the asymptotic expansion written out in Eq. (7.1); namely, a leading $(\ln s)^{\epsilon/12}$ term multiplied by an expansion in decreasing powers of $(\ln s)^{-\epsilon/2}$. Written in terms of renormalized parameters, the above expression presents us with a very appealing description of the total cross section in the preasymptotic region.

Similar expressions to (7.1) can be easily obtained from the analysis described in Sec. III for any $\Gamma^{n,m}$. We further extend our method of integration of the renormalization-group equation to include the k_i^2 dependence. Including any number of additional dimensionless parameters, a differential equation (4.10) has been obtained for each of the renormalization constants Z_x in the theory. We integrate this general equation in the one-loop approximation for the ϕ^3 Reggeon field theory under consideration. We choose the renormalization condition

$$i\Gamma_R^{1,1}(-E_N,k_N)=-E_N-\alpha' k_N^2 \tag{7.2}$$

[a natural generalization for $k_i^2\neq 0$ of the condition of Eq. (2.1)], which proves to be a great technical improvement since the need for a two-dimensional integration is avoided. The shape of the diffractive peak and the scaling functions at $t\neq 0$ were easily obtained in our method. The absorptive elastic amplitude is given by Eqs. (5.7), (5.8), and (6.15). Its J-plane structure is discussed in Sec. V.

We present in Sec. VI B the general form of the approach to scaling (up to this point it was only discussed in the one-loop approximation for the renormalization-group functions). The general form is characterized by an exponent λ governing the approach to the scaling limit. Since the exponent $\lambda(\epsilon)$ has a poorly convergent ϵ expansion [Eq. (6.12)] it precludes drawing any definite conclusion, at second order in the ϵ expansion, as to the dominance of the nonleading terms in $G^{1,1}$ on contributions from the leading terms in the "nonenhanced" graphs. It would be desirable to calculate the exponent λ and to study the approach to scaling by using methods in the Reggeon field theory which avoid the need of an ϵ expansion.

ACKNOWLEDGMENTS

We wish to thank John Bronzan for helpful conversations concerning his related work in progress in collaboration with H. Abarbanel and J. Bartels. We also thank R. Sugar for several helpful conversations.

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*Work supported in part by the U. S. Energy Research and Development Administration.

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