Soliton "dictionary" for massive quantum electrodynamics *

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The Thirring model and the sine-Gordon theory in two dimensions have been shown to be equivalent. We extend this result by demonstrating the equivalence between massive QED (with self-interacting fermions) and a vector generalization of the sine-Gordon theory. We demonstrate this equivalence by perturbation theory and by constructing the appropriate transformation between the two sets of fields. Because the sine-Gordon equation is known to have classical "kink" or soliton solutions, we are led to suspect that hadrons correspond to extended solutions to the equations of motion in the strong-coupling region.

I. INTRODUCTION

Recent work in the theory of strings[1] and bags[2] leads us to suspect that theories based on extended objects, rather than point systems, yield many qualitative features of hadron theory, and the work of Nielsen and Olesen[3] suggests to us that these extended objects emerge naturally out of classical solutions of spontaneously broken local field theory in the strong-coupling region. These qualitative remarks have been partially realized in recent work on the quantum dynamics of strong-coupling theories,[4] which lends further support to the conjecture that extended systems emerge as bound states of field theories in the region of strong coupling.

In two dimensions, the classical "kink" solutions (solitons[5]) of the sine-Gordon equation are very suggestive of strong-coupling solutions of a quantum field theory. The realization of this notion comes from the work of Coleman,[6] who used perturbation theory to prove the equivalence of the quantized sine-Gordon equation and the massive Thirring model. Mandelstam[7] has explicitly constructed the operator equivalence relations from which a "dictionary" can be constructed for translating back and forth from either language.

In this work, we extend the results of Coleman and Mandelstam to include massive QED. By using perturbation theory and by constructing explicit solutions, we show that massive QED is equivalent to a vector generalization of the sine-Gordon equation. We suspect, therefore, that this vector sine-Gordon equation possesses classical "kink" solutions (solitons).

We are currently investigating the corresponding SU(n) generalization of this model, from which information concerning bound states of confined quarks may be derived.

II. GREEN'S FUNCTIONS

We first start with massive QED (with fermion self-coupling), which is exactly solvable[8]:

$$\begin{aligned}\mathcal{L} &= -i\,\bar{\psi}(\not\partial\psi) + \tfrac{1}{4}F_{\mu\nu}{}^2 - \tfrac{1}{2}\mu^2 A_\mu{}^2 \\ &\quad + \tfrac{1}{2}\sigma\bar{\psi}\gamma_\mu\psi\bar{\psi}\gamma^\mu\psi + g\bar{\psi}\gamma^\mu\psi A_\mu\,, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu\,, \quad g_{00} = 1 = -g_{11}\end{aligned} \tag{2.1}$$

Since the theory is exactly solvable, we can use these exact solutions in order to calculate Green's functions corresponding to an interaction term:

$$\mathcal{L}_I = -m\bar{\psi}\psi\,. \tag{2.2}$$

We wish to investigate the relationship between this theory and the free field theory of a massless scalar particle and a massive vector particle:

$$\begin{aligned}\mathcal{L}' &= \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \tfrac{1}{4}f_{\mu\nu}{}^2 - \tfrac{1}{2}\mu'^2 a_\mu{}^2\,, \\ f_{\mu\nu} &= \partial_\mu a_\nu - \partial_\nu a_\mu = \epsilon_{\mu\nu} f\,, \quad \epsilon_{01} = 1.\end{aligned} \tag{2.3}$$

We wish to perturb this free field theory by adding a modified vector generalization of the sine-Gordon interaction:

$$\begin{aligned}\mathcal{L}_I' &= \frac{\alpha_0}{\beta^2}\cos(\beta\phi + \gamma f) - \frac{\alpha_0}{\beta^2}\,, \\ f &= \partial_0 a_1 - \partial_1 a_0\,.\end{aligned} \tag{2.4}$$

Our goal is to prove that the Green's functions generated by the perturbation term (2.2), when expanded around the exact solutions of (2.1), are equivalent to the Green's functions generated by (2.4) when expanded around the solution of (2.3). [We notice that there are infrared divergences associated with the massless fermion. This is remedied by multiplying (2.2) by a suitable support function, which we will later set to be 1. Also, since a massless scalar free field does not exist in two dimensions, we will have to implicitly add in a small mass term to the scalar particle in (2.3).] Assuming, of course, that these theories actually exist, we may prove their equivalence perturbatively by comparing Green's functions for both theories.

It is a simple manner to first derive the equivalence results for the current and vector matrix elements and commutators. Following Sommerfield,[8] we find the Green's functions for massive QED with self-interacting fermions:

$$i\langle 0|Tj_\mu(x)j_\nu(y)|0\rangle = \frac{-\partial_\mu\partial_\nu D(x-y)}{\pi[1-(\alpha+\lambda)^2]} + \lambda\,[2\pi(1+\alpha)(1+\alpha+\lambda)]^{-1}(\mu'^2 g_{\mu\nu} - \partial_\mu\partial_\nu)\Delta(x-y;\mu')$$
$$+\delta(x^1-y^1)g_\mu{}^0 g_\nu{}^0(2-\lambda)[2\pi(1+\alpha)(1-\alpha-\lambda)]^{-1}, \tag{2.5}$$

$$i\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \frac{-2\lambda\partial_\mu\partial_\nu D(x-y)}{\mu^2[1-(\alpha+\lambda)^2]} + (1+\alpha)\mu^{-2}(1+\alpha+\lambda)^{-1}(\mu'^2 g_{\mu\nu} - \partial_\mu\partial_\nu)\Delta(x-y;\mu')$$
$$+g_\mu{}^0 g_\nu{}^0\delta(x^1-y^1)(1-\alpha)\mu^{-2}(1-\alpha-\lambda)^{-1}, \tag{2.6}$$

$$i\langle 0|TA_\mu(x)j_\nu(y)|0\rangle = \frac{g}{2\pi\mu^2}\left[\frac{-2\partial_\mu\partial_\nu D(x-y)}{1+(\alpha+\lambda)^2} - (1+\alpha+\lambda)^{-1}(\mu'^2 g_{\mu\nu} - \partial_\mu\partial_\nu)\Delta(x-y;\mu')\right.$$
$$\left.+g_\mu{}^0 g_\nu{}^0\delta(x^1-y^1)(1-\alpha-\lambda)^{-1}\right], \tag{2.7}$$

where

$$\alpha = \sigma/2\pi,$$
$$\lambda = g^2/2\pi\mu^2,$$
$$\mu'^2 = \mu^2(1+\alpha+\lambda)/(1+\alpha),$$
$$\Box_x D(x-y) = i\,\delta(x-y),$$
$$(\Box_x + \mu'^2)\Delta(x-y;\mu') = i\,\delta(x-y),$$
$$j_\mu \equiv -a\epsilon_{\mu\nu}\partial^\nu\phi + ba_\mu, \tag{2.8}$$
$$A_\mu \equiv -c\epsilon_{\mu\nu}\partial^\nu\phi + da_\nu. \tag{2.9}$$

The form of these matrix elements for massive QED with self-interacting fermions leads one to suspect that, with a suitable combination of free fields [one massless scalar particle and one massive vector particle of mass $\mu' = \mu(1+\alpha+\lambda)^{1/2} \times (1+\alpha)^{-1/2}$], we might be able to reproduce these relations. We are then led to try the free-field representation (2.8) and (2.9).

On the left-hand sides of (2.8) and (2.9), we have the exact operators of massive QED with self-interacting fermions, and on the right we have operators of a free scalar and vector theory. By inserting (2.8) and (2.9) into (2.5)–(2.7) we can show that these Green's functions can be reproduced if we set

$$a = \{\pi[1-(\alpha+\lambda)^2]\}^{-1/2},$$
$$b = -\mu'(\lambda)^{1/2}[2\pi(1+\alpha)(1+\alpha+\lambda)]^{-1/2}$$
$$= \frac{-g}{2\pi(1+\alpha)}, \tag{2.10}$$
$$c = \mu^{-1}(2\lambda)^{1/2}[1-(\alpha+\lambda)^2]^{-1/2},$$
$$d = 1.$$

Furthermore, we also easily verify that the solution given by (2.10), (2.8), and (2.9) satisfies Sommerfield's equal-time commutation relations:

$$[A_0(x), A_1(y)]_{x^0=y^0}$$
$$= i(1-\alpha)(1+\alpha+\lambda)[1-(\alpha+\lambda)^2]^{-1}\mu^{-2}\partial_x\delta(x^1-y^1),$$
$$[A_0(x), j_1(y)]_{x^0=y^0} = \frac{ig}{2\pi\mu^2}(1-\alpha-\lambda)^{-1}\partial_x\delta(x^1-y^1),$$
$$[j_0(x), j_1(y)]_{x^0=y^0} = i(2\pi)^{-1}(2-\lambda)(1-\alpha)^{-1}(1-\alpha-\lambda)^{-1}$$
$$\times\partial_x\delta(x^1-y^1)$$

if (2.11)

$$[a_0(x), a_1(y)]_{x^0=y^0} = i\partial_x\delta(x^1-y^1)\mu'^{-2},$$
$$[\dot\phi(x), \phi(y)]_{x^0=y^0} = -i\partial_x\delta(x^1-y^1).$$

Now that we have established the representation of the exact operators ψ and A_μ in terms of free-field operators ϕ and a_μ, our next step is to demonstrate the equivalence of the interaction terms (2.2) and (2.4). We first need the commutator between vector fields and the spinor:

$$[A_\mu(x), \psi(y)]_{x^0=y^0}$$
$$= \frac{-g}{\mu^2}g_{\mu 0}(1-\alpha-\lambda)^{-1}\delta(x^1-y^1)\psi(y), \tag{2.12}$$

$$[j_\mu(x), \psi(y)]_{x^0=y^0} = \mp[g_{\mu 0}(1-\alpha-\lambda)^{-1} + \epsilon_{\mu 0}\gamma_5(1+\alpha)^{-1}]$$
$$\times\delta(x^1-y^1)\psi(y). \tag{2.13}$$

If we take

$$\sigma_\pm = \tfrac{1}{2}Z\bar\psi(1\pm\gamma_5)\psi,$$
$$\sigma_+ = Z\psi_1^*\psi_2, \tag{2.14}$$
$$\sigma_- = Z\psi_2^*\psi_1$$

then we find

$$[A_\mu(x), \sigma_\pm(y)]_{x^0=y^0} = 0,$$
$$[j_\mu(x), \sigma_\pm(y)]_{x^0=y^0} = \mp 2\epsilon_{\mu 0}(1+\alpha)^{-1}\delta(x^1-y^1)\sigma_\pm(y). \tag{2.15}$$

Z is a multiplicative renormalization constant needed to define the limiting process of when two spinor fields are defined at the same point. Now we construct, out of free fields, the quantity $A_\pm$:

$$A_\pm = N \exp[\pm i(\beta\phi + \gamma f)] . \tag{2.16}$$

We easily find

$$\begin{aligned} [-a\epsilon_{\mu\nu}\partial^\nu \phi(x) + ba_\mu(x), A_\pm(y)]_{x^0=y^0} \\ = \mp(a\beta - b\gamma)\epsilon_{\mu 0}\delta(x^1 - y^1)A_\pm(y) , \end{aligned} \tag{2.17}$$

$$\begin{aligned} [-c\epsilon_{\mu\nu}\partial^\nu \varphi(x) + da_\mu(x), A_\pm(y)]_{x^0=y^0} \\ = \pm(c\beta + d\gamma)A_\pm(y)\epsilon_{\mu 0}\delta(x^1 - y^1) \\ = 0 . \end{aligned} \tag{2.18}$$

Comparing (2.17) and (2.18) with (2.12) and (2.13), we arrive at

$$\begin{aligned} \beta &= 2[\pi(1-\alpha-\lambda)]^{1/2}(1+\alpha+\lambda)^{-1/2} , \\ \gamma &= 2\mu'^{-1}(2\pi\lambda)^{1/2}[(1+\alpha)(1+\alpha+\lambda)]^{-1/2} \end{aligned} \tag{2.19}$$

[the cutoff parameter N is needed because $D(0)$ and $\Delta(0, \mu')$ need to be regularized].

We have now shown that the QED commutators of (2.14) and (2.15) are satisfied if we make the correspondence

$$\begin{aligned} \sigma_\pm &\cong A_\pm/2 , \\ m &\cong \frac{\alpha_0}{\beta^2} . \end{aligned} \tag{2.20}$$

The last thing we would like to demonstrate is the equivalence of the Green's functions generated by products of $\sigma_\pm$ and by $A_\pm$. Following Coleman and Sommerfield, we know that

$$\left\langle 0 \left| T\left(\prod_{i=1}^n \sigma_+(x_i)\sigma_-(y_i)\right) \right| 0 \right\rangle \sim \frac{\prod_{i>j}^n [(x_i - x_j)^2(y_i - y_j)^2]^{1-2(\alpha+\lambda)/(1+\alpha+\lambda)}}{\prod_{i,j}^n [(x_i - y_j)^2]^{1-2(\alpha+\lambda)/(1+\alpha+\lambda)}} \times \frac{\prod_{i>j}^n \exp\left\{ \frac{i8\pi\lambda}{(1+\alpha)(1+\alpha+\lambda)} [\Delta(x_i - x_j; \mu') - \Delta(y_i - y_j; \mu')] \right\}}{\prod_{i,j}^n \exp\left[\frac{i8\pi\lambda}{(1+\alpha)(1+\alpha+\lambda)} \Delta(x_i - y_j; \mu') \right]} . \tag{2.21}$$

On the other hand, it is a simple matter to construct the free-field Green's functions for $A_\pm$

$$\left\langle 0 \left| T\left(\prod_{i=1}^n A_+(x_i)A_-(y_i)\right) \right| 0 \right\rangle \sim \frac{\prod_{i>j}^n [(x_i - x_j)^2(y_i - y_j)^2]^{\beta^2/4\pi}}{\prod_{i,j}^n [(x_i - y_j)^2]^{\beta^2/4\pi}} \frac{\prod_{i>j}^n \exp\{(\mu'^2\gamma^2)[i\Delta(x_i - x_j; \mu') - i\Delta(y_i - y_j; \mu')]\}}{\prod_{i,j}^n \exp[(\mu'^2\gamma^2)i\Delta(x_i - y_j; \mu')]} . \tag{2.22}$$

if

$$i\langle 0|T(f(x), f(y))|0\rangle = \mu'^2\Delta(x - y; \mu') . \tag{2.23}$$

Again, on comparing these two expressions, we rederive the correspondences given in (2.19) and (2.20).

Since all Green's functions for one theory can be written in terms of the Green's functions for the other, we have shown that if these theories indeed exist then they must be the same.

From the relations between coupling constants of the two theories, we again establish that the weak coupling of one theory is the strong coupling of the other, and hence the bound states of one may be looked at as the "free" states of the other.

III. SOLITON DICTIONARY

Though the classical solutions of our vector sine-Gordon equation are not known, there is a strong possibility that they will be "kink" solitons. At the quantum level, it is possible to make a dictionary between operator functions in one language in terms of operators in the other.

Following the work of Mandelstam, we have been able to construct this dictionary in the case of massless fermions and free scalar particles. Unfortunately, we have *not* been able to complete this dictionary for the case of the fully interacting scalar particles and the massive fermions, because the commutators involved in the calculation are not well defined.

Our task is to find solutions of the following equations for ψ and A_μ:

$$-i\partial\!\!\!/\psi - m\psi + gA\!\!\!/\psi + \sigma j\!\!\!/\psi = 0\,, \tag{3.1}$$

$$\partial_\mu F^{\mu\nu} + \mu^2 A^\nu - gj^\nu = 0\,, \tag{3.2}$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\quad \gamma_5 = \gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

These solutions must be in terms of the fields ϕ and a_μ, which obey

$$\Box\phi + \frac{\alpha_0}{\beta}\sin(\beta\phi + \gamma f) = 0\,, \tag{3.3}$$

$$\partial_\mu\left(f^{\mu\nu} + \frac{\alpha_0}{\beta}\epsilon^{\mu\nu}\sin(\beta\phi + \gamma f)\right) + \mu'^2 a^\nu = 0\,. \tag{3.4}$$

The canonical quantization relations are given by

$$\frac{\delta(\mathcal{L}' + \mathcal{L}_I')}{\delta\partial_0 a_1} = \Pi_1 = -\left[f + \frac{\alpha_0}{\beta}\sin(\beta\phi + \gamma f)\right]\,, \tag{3.5}$$

$$[a_1(x), \Pi_1(y)]_{x^0 = y^0} = -i\,\delta(x^1 - y^1)\,, \tag{3.6}$$

$$\partial_1\Pi_1 = +\mu'^2 a_0, \quad \partial_0\Pi_1 = +\mu'^2 a_1\,, \tag{3.7}$$

$$[a_1(x), a_0(y)]_{x^0 = y^0} = -i\,\delta(x^1 - y^1)\mu'^{-2}\,. \tag{3.8}$$

Notice that the complexity of (3.6) prevents us from evaluating all commutators needed in the calculation of currents. Because of this difficulty, we will *not* try to solve the model in terms of the fully interacting case, but instead will construct the dictionary between massless fermions and free scalars.

We begin by postulating the form of the spinor in terms of *free* scalar and vector fields (k_1 and k_2 are mass-dependent factors which simply normalize the anticommutation relations):

$$\psi_1(x) = k_1 {:}\exp\left[-\frac{2\pi i}{\beta}\int_{-\infty}^{x^1}\dot\phi(x^0, \xi)d\xi - \frac{i\beta}{2}\phi(x^0, x^1) - \frac{i\gamma}{2}f(x^0, x^1)\right]{:}\,, \tag{3.9}$$

$$\psi_2(x) = -k_2 {:}\exp\left[-\frac{2\pi i}{\beta}\int_{-\infty}^{x^1}\dot\phi(x^0, \xi)d\xi + \frac{i\beta}{2}\phi(x^0, x^1) + \frac{i\gamma}{2}f(x^0, x^1)\right]{:}\,. \tag{3.10}$$

In the case where ϕ and a_μ are free fields, there is no ambiguity in forming the exponential function.

Our first task will be to calculate the combination $\bar\psi\psi$ and show that we rederive the identities of the previous section. Then we will construct the current $j_\mu \sim \bar\psi\gamma_\mu\psi$ with a careful limiting process. With the form of the current, we will verify that the correct commutation relations between all vector quantities with spinor fields are reestablished. Finally, we will show that the proper equations of motion (for massless fermions and free scalar particles) are satisfied by (3.9) and (3.10). We begin by defining the quantities

$$\begin{aligned} \sigma_+ &= \lim_{x\to y}|x - y|^\delta \psi_2^*(x)\psi_1(y)\,, \\ \sigma_- &= \lim_{x\to y}|x - y|^\delta \psi_1^*(x)\psi_2(y)\,, \end{aligned} \tag{3.11}$$

where the path $x \to y$ is taken in a space-like direction. We will use the useful identities

$$\begin{aligned} e^A e^B &= e^{[A,B]} e^B e^A\,, \\ {:}e^A{:}\,{:}e^B{:} &= e^{[A_+, B_-]}{:}e^{A+B}{:}\,, \end{aligned} \tag{3.12}$$

where the commutator between A and B is a c number, and the + and − refer to creation and annihilation operators. First, we notice that the presence of the f fields in (3.9) and (3.10) does not change the basic anticommutation relations. Second, we notice that combinations like $\psi_2^*\psi_1$ contain terms like $\dot\phi$ which contribute in $O(x-y)$, while the terms ϕ and f contribute in $O(1)$. Therefore, the only terms in the exponent which survive the reconstitution process are the ϕ and f fields. We find

$$\sigma_\pm \cong -\tfrac{1}{2}\exp[\pm i(\beta\phi + \gamma f)] \tag{3.13}$$

so that

$$\begin{aligned} \bar\psi\psi &\cong \cos(\beta\phi + \gamma f)\,, \\ \bar\psi\gamma_5\psi &\cong \sin(\beta\phi + \gamma f)\,. \end{aligned} \tag{3.14}$$

Now, our task is to construct the current. We must be particularly cautious because, as Johnson has noted, the naive definition of the current yields nonrelativistic or nonconserved results. The proper relativistic version of the current must be the average over different, perpendicular space-time directions. If ξ and $\bar\xi$ are two orthogonal vectors,

$$\begin{aligned} &\xi\cdot\bar\xi = 0\,, \\ &\xi_\alpha\xi_\beta/\xi^2 + \bar\xi_\alpha\bar\xi_\beta/\bar\xi^2 = g_{\alpha\beta}\,, \end{aligned} \tag{3.15}$$

then we will adopt the definition of the current as

$$\begin{aligned} j_\mu(x) = \tfrac{1}{4}\lim_{a\to 0}\{&\bar\psi(x)\gamma_\mu\psi(x + a\xi) \\ &- \gamma_\mu\psi(x + a\xi)\bar\psi(x) + (\xi \to \bar\xi)\}\,. \end{aligned} \tag{3.16}$$

If we average over two points separated by a spacelike separation then the commutators yield no problem, but if we average over timelike separations we cannot use equal-time commutation relations and must use the approximation

$$\phi(x^0+\delta x^0, x^1) \sim \phi(x^0, x^1) + \delta x^0 \dot{\phi}(x^0, x^1) + \cdots . \tag{3.17}$$

If we were investigating the fully interacting theory then the formula (3.13) would have to be modified, and instead the full Baker-Hausdorff theorem would have to be invoked. (Because of the complexity of the problem, we will only treat the case of massless fermions.) Upon averaging in two separate directions, we easily find

$$-4\pi j_\mu = \left(\frac{4\pi}{\beta}+\beta\right)\epsilon_{\mu\nu}\partial^\nu\phi + \gamma\mu'^2 a_\mu . \tag{3.18}$$

Is easy to show that this is equal to the expression for the current found in the previous section, (2.8) and (2.10).

Using this result, we can now also calculate A_μ in terms of ϕ and a_μ. Putting this result in the wave equation (3.2), we find

$$+A_\mu = -\frac{ga}{\mu^2}\epsilon_{\mu\nu}\partial^\nu\phi + \frac{\mu'^2+gb}{\mu^2}a_\mu . \tag{3.19}$$

Again, this expression can be shown to be identical to the expression found in the previous section, (2.9) and (2.10).

As a further check on our results, we can now calculate the commutator between vector quantities and the spinor field. It is straightforward to evaluate all commutators, and we find

$$[j_\mu(x), \psi(y)]_{x^0=y^0} = -\left[\frac{2\pi a}{\beta}g_{\mu 0} + \epsilon_{\mu 0}\gamma_5\left(\frac{a\beta}{2}-\frac{b\gamma}{2}\right)\right] \times \psi(y)\delta(x^1-y^1), \tag{3.20}$$

$$[A_\mu(x), \psi(y)]_{x^0=y^0} = -\left[\frac{2\pi c}{\beta}g_{\mu 0} + \epsilon_{\mu 0}\gamma_5\left(\frac{c\beta}{2}-\frac{d\gamma}{2}\right)\right] \times \psi(y)\delta(x^1-y^1). \tag{3.21}$$

Again, we find exact correspondence with results of the previous section, (2.12) and (2.13).

At this point, we wish to show that the wave equations for ψ and A_μ are satisfied when they are expressed in terms of ϕ and a_μ, which satisfy free field equations. In particular, we wish to show that the quantized version of the following equation is satisfied:

$$-\dot{\psi}_1 + \psi_1' + im\psi_2 + i(-gA_0 + gA_1 - \sigma j_0 + \sigma j_1)\psi_1 = 0 . \tag{3.22}$$

By taking the derivatives of the spinor field, we find

$$-\dot{\psi}_1 + \psi_1' = i\tfrac{1}{2}\left\{\tfrac{1}{2}\left(\frac{4\pi}{\beta}-\beta\right)(\dot{\phi}-\phi') + \frac{\gamma\mu'^2}{2}(a_0-a_1), \psi_1\right\}, \tag{3.23}$$

$$\frac{\gamma\mu'^2}{2} = \sigma b + gd , \tag{3.24}$$

$$\frac{2\pi}{\beta} - \frac{\beta}{2} = gc + \sigma a . \tag{3.25}$$

Rearranging terms, we find

$$-\dot{\psi}_1 + \psi_1' + im\psi_2 = i\{[-g(c(\dot{\phi}-\phi') + d(a_1-a_0)(-1)) - \sigma(a(\dot{\phi}-\phi') + b(a_1-a_0))]\psi_1\} .$$

This last expression is easily regrouped to give the original wave equation. Thus, we find that all the results of the previous section are satisfied, and that we can construct the following dictionary between the two sets of fields:

$$j_\mu = \frac{-\epsilon_{\mu\nu}\partial^\nu\phi}{\{\pi[1-(\alpha+\lambda)^2]\}^{1/2}} + \frac{-g}{2\pi(1+\alpha)}a_\mu ,$$

$$A_\mu = \frac{-\epsilon_{\mu\nu}\partial^\nu\phi}{\mu\{[1-(\alpha+\lambda)^2]/2\lambda\}^{1/2}} + a_\mu ,$$

$$\bar{\psi}\gamma_5\psi \cong \sin(\beta\phi + \gamma f),$$

$$\bar{\psi}\psi \cong \cos(\beta\phi + \gamma f),$$

$$\psi_{1,2} = k_{1,2}{:}\exp\left\{-i\left[\frac{2\pi}{\beta}\int_{-\infty}^{x^1}\dot{\phi}(x^0,\xi)d\xi \pm \frac{\beta}{2}\phi(x^0,x^1) \pm \frac{\gamma}{2}f(x^0,x^1)\right]\right\}{:},$$

$$\alpha = \frac{\sigma}{2\pi},$$

$$\lambda = \frac{g^2}{2\pi\mu^2},$$

$$\mu' = \mu\left(\frac{1+\alpha+\lambda}{1+\alpha}\right)^{1/2},$$

$$\beta = 2\sqrt{\pi}\left(\frac{1-\alpha-\lambda}{1+\alpha+\lambda}\right)^{1/2},$$

$$\gamma = \frac{2g}{\mu^2(1+\alpha+\lambda)} .$$

Notice that this dictionary is constructed between *massless* fermions and *free* scalar and vector fields. The complete dictionary, with massive fermions and interacting scalar and vector fields, has not yet been found because of the complexity of the commutator (3.6). But does this mean that there does not exist a correspondence between massive QED (with massive fermions) and some interacting theory? Probably not. For example, it is known that a vector field can be represented by two spinless fields in the following way:

$$a_\mu = \epsilon_{\mu\nu}\partial^\nu\sigma + \partial_\mu\lambda, \quad \Box\lambda = 0$$

where the σ field has two poles, one at the mass of the vector field and the other a massless pole (which is canceled by the zero-mass pole coming from the λ field). By analogy, we are tempted to represent the f field by a spinless field $f = \mu'^{-1}\sigma$ because of Eq. (2.23). If we now replace the free Lagrangian for the a_μ field by the corresponding term for a spinless field with the same mass, then the canonical quantization relations (3.5)–(3.8) lose their dependence on the sine term. Thus, by replacing the vector field a_μ by a spinless field, we greatly simplify the canonical commutation relations, which in turn make it possible to construct simple expressions for the current. The net effect of this is to replace the f appearing in (3.9) and (3.10) with a spinless field with the canonical relations of a free field.

IV. CONCLUSION

In much the same way that the weak coupling of the Thirring model corresponds to the strong coupling of the sine-Gordon equation, and vice versa, we found that the weak and strong couplings of the two models studied here also have the same relationship. In particular, the Green's functions found when perturbing both models are exactly equivalent. We were also able to show that, in the massless fermion and free scalar case, we could give the explicit construction of one set of fields in terms of the other. In the case of the fully interacting model (massive fermions) we were *not* able to construct the dictionary, given the complexity of the commutators arising from the limiting process of defining two spinors at the same space-time point.

If we set $\lambda = 0$, then we reproduce the results of Coleman, as the vector fields decouple from the system. (If we set $\alpha = 0$, then we find that all four fields still remain, so that pure massive QED still has a representation in terms of vectors and scalars.)

Notice that the combination $\alpha + \lambda$ occurs in the coupling constant relations, so the net effect of introducing a vector interaction is to make $\alpha \to \alpha + \lambda$ in the expression for β. In particular, if $\sigma/2\pi = g^2/2\pi\mu^2 = 0$ then the *free* fermion field corresponds to the interacting sine-Gordon equation with $\beta^2 = 4\pi$. As $\alpha + \lambda$ gets larger β^2 gets smaller until, at $\alpha + \lambda = 1$, $\beta = 0$ (Sommerfield finds one more restriction on the coupling constants arising from considerations of the definition of the cutoff: $1 > |\alpha + \lambda|$, $1 + \alpha > 0$). (Notice also that the coupling-constant convention chosen by Coleman and Mandelstam differs from that chosen by Sommerfield.)

It seems like that the vector sine-Gordon equation possesses solitonlike static solutions, which correspond to the fermion states in the other language. This is currently being investigated.

Also, though we have not been able to represent the massive fermion theory in terms of interacting scalar fields, there is the possibility that the massive fermion theory may be represented by two spinless fields if we make the replacement for $f = \mu'^{-1}\sigma$ in (3.9) and (3.10). Because the canonical commutation relations no longer are dependent on the sine term, there is the possibility that all currents and products of spinor fields can be rigorously defined and that a new equivalence to massive QED (with massive fermions) may be found.

The power of establishing the equivalence between the weak coupling of one field theory and the strong coupling of another is, of course, that we may gain further insight into the strong-coupling bound states of four-dimensional field theories, and that conjectures of quark confinement (such as those advanced by Kogut and Susskind[9]) may be realized in a more complete theory. One physically relevant procedure is to extend the Abelian model studied here to the case of SU(n), where properties such as quark confinement may emerge. This is currently under investigation.

Yet another approach is to examine the field theory of strings[10] and see whether a correspondence exists to a four-dimensional Higgs theory.

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